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Research Article

Common Fixed Point of Mappings Satisfying Rational Inequality in Complex Valued *b*-Metric Spaces

Anil Kumar Dubey[1,*], Urmila Mishra[2] and Manjula Tripathi[3]

[1] *Department of Mathematics, Bhilai Institute of Technology, Bhilai House, Durg, Chhattisgarh, India*
[2] *Department of Mathematics, Vishwavidyalaya Engineering College, Lakhanpur-Ambikapur, Chhattisgarh, India*
[3] *Department of Mathematics, U.P.U. Government Polytechnic, Durg, Chhattisgarh, India*
***Corresponding author:** anilkumardby@rediffmail.com

Abstract. In this paper, we prove some common fixed point theorems satisfying rational inequality in complex valued b-metric spaces. The presented theorems extend, generalize and improve the corresponding result of Rouzkard and Imdad [12]. Also, an example is given to illustrate our obtained result.

Keywords. Common fixed point; Complex valued b-metric spaces

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1. Introduction

Most recently, Azam *et al.* [2] first introduced the concept of complex valued metric spaces which is generalized form of metric spaces and also proved some common fixed point theorems for mappings satisfying generalized contraction condition. This new idea can be utilized to define complex valued normed spaces and complex valued inner product spaces and also to define rational expressions which are not meaningful in cone metric spaces. Subsequently, several authors studied many common fixed point results on complex valued metric spaces (see [5, 7, 8, 12, 13]).

In 2013, Rao *et al.* [10] introduced the concept of complex valued b-metric spaces, which was more general than the well known complex valued metric spaces defined by Azam *et al.* [2]. Afterwards, Mukheimer [9] established the results of common fixed point satisfying a rational inequality on complex valued b-metric spaces.

The purpose of this work is to obtain common fixed point results for two self mappings satisfying a rational inequality on complex valued b-metric spaces which generalizes the results of [2, 9] and [12].

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2), \mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$. Thus $z_1 \precsim z_2$ if one of the following holds:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(iii) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(iv) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied, also we write $z_1 \prec z_2$ if only (iv) is satisfied. Notice that $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$ and $z_1 \precsim z_2$, $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

The following definition is introduced by Azam *et al.* [2].

Definition 1. Let X be a nonempty set.A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$.

The pair (X,d) is called a complex valued metric space.

Example 2 ([13])**.** Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x,y) = e^{ik}|x-y|$, where $k \in \mathbb{R}$ and for all $x, y \in X$. Then (X,d) is a complex valued metric space.

Definition 3 ([10])**.** Let X be a non empty set and let $s \ge 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \precsim s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a complex valued b-metric space.

Example 4 ([10]). *Let $X = [0,1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$. Then (X,d) is a complex valued b-metric space with $s = 2$.*

Definition 5 ([10]). Let (X,d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) := \{y \in X : d(x,y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x,r) \bigcap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x,r) : x \in X \quad \text{and} \quad 0 \prec r\}$.

Definition 6 ([10]). Let (X,d) be a complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$ then there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 7 ([10]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 8 ([10]). *Let (X,d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence in X if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section, we prove common fixed point theorems for contraction conditions described by rational expressions in complex valued b-metric space. Our main result runs as follows:

Theorem 9. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$ are mappings satisfying:*

$$d(Sx,Ty) \precsim \alpha d(x,y) + \beta \frac{[1+d(x,Sx)]d(y,Ty)}{[1+d(x,y)]} + \gamma[d(x,Sx)+d(y,Ty)] + \delta[d(x,Ty)+d(y,Sx)] \tag{3.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha + \beta + 2\gamma + 2s\delta < 1$. Then the maps S and T have a unique common fixed point in X.

Proof. For any arbitrary point $x_0 \in X$, define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}, \quad \text{for } n = 0,1,2,\dots. \tag{3.2}$$

Now, we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{[1 + d(x_{2n}, Sx_{2n})] d(x_{2n+1}, Tx_{2n+1})}{[1 + d(x_{2n}, x_{2n+1})]} \\ &\quad + \gamma [d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] \\ &\quad + \delta [d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})] \\ &= \alpha d(x_{2n}, x_{2n+1}) + \beta \frac{[1 + d(x_{2n}, x_{2n+1})] d(x_{2n+1}, x_{2n+2})}{[1 + d(x_{2n}, x_{2n+1})]} \\ &\quad + \gamma [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &\quad + \delta [d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n+1}, x_{2n+2}) + \gamma [d(x_{2n}, x_{2n+1}) \\ &\quad + d(x_{2n+1}, x_{2n+2})] + s\delta [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \end{aligned}$$

so that,

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\le \alpha |d(x_{2n}, x_{2n+1})| + \beta |d(x_{2n+1}, x_{2n+2})| + \gamma |[d(x_{2n}, x_{2n+1}) \\ &\qquad + d(x_{2n+1}, x_{2n+2})]| + s\delta |[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]| \end{aligned}$$

and hence

$$|d(x_{2n+1}, x_{2n+2})| \le \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta} |d(x_{2n}, x_{2n+1})|. \tag{3.3}$$

Similarly we obtain,

$$|d(x_{2n+2}, x_{2n+3})| \le \frac{\alpha + \gamma + s\delta}{1 - \beta - \gamma - s\delta} |d(x_{2n+1}, x_{2n+2})|. \tag{3.4}$$

Since $\alpha + \beta + 2\gamma + 2s\delta < 1$ and $s \ge 1$, we have $h = \frac{\alpha+\gamma+s\delta}{1-\beta-\gamma-s\delta} < 1$ and for all $n \ge 0$ and consequently, we get

$$|d(x_{2n+1}, x_{2n+2})| \le h |d(x_{2n}, x_{2n+1})| \le h^2 |d(x_{2n-1}, x_{2n})| \le \dots \le h^{2n+1} |d(x_0, x_1)|$$

i.e.

$$|d(x_{n+1}, x_{n+2})| \le h |d(x_n, x_{n+1})| \le h^2 |d(x_{n-1}, x_n)| \le \dots \le h^{n+1} |d(x_0, x_1)|. \tag{3.5}$$

Thus for any $m > n$ and $m, n \in \mathbb{N}$,

$$\begin{aligned} |d(x_n, x_m)| &\le s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^2 |d(x_{n+2}, x_m)| \\ &\le s|d(x_n, x_{n+1})| + s^2 |d(x_{n+1}, x_{n+2})| + s^2 |d(x_{n+2}, x_{n+3})| + s^3 |d(x_{n+3}, x_m)| \\ &\vdots \end{aligned}$$

$$\leq s|d(x_n,x_{n+1})| + s^2|d(x_{n+1},x_{n+2})| + s^3|d(x_{n+2},x_{n+3})| + \dots$$
$$+ s^{m-n-1}|d(x_{m-2},x_{m-1})| + s^{m-n}|d(x_{m-1},x_m)|.$$

By (3.5), we get

$$\begin{aligned}|d(x_n,x_m)| &\leq sh^n|d(x_0,x_1)| + s^2h^{n+1}|d(x_0,x_1)| + s^3h^{n+2}|d(x_0,x_1)| + \dots \\ &\quad + s^{m-n-1}h^{m-2}|d(x_0,x_1)| + s^{m-n}h^{m-1}|d(x_0,x_1)| \\ &= \sum_{i=1}^{m-n} s^i h^{i+n-1}|d(x_0,x_1)|.\end{aligned}$$

Therefore

$$\begin{aligned}|d(x_n,x_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}h^{i+n-1}|d(x_0,x_1)| \\ &= \sum_{t=n}^{m-1} s^t h^t|d(x_0,x_1)| \\ &\leq \sum_{t=n}^{\infty}(sh)^t|d(x_0,x_1)| \\ &= \frac{(sh)^n}{1-sh}|d(x_0,x_1)|\end{aligned}$$

and hence

$$|d(x_n,x_m)| \leq \frac{(sh)^n}{1-sh}|d(x_0,x_1)| \to 0 \text{ as } m,\ n \to \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Assume not, then there exists $z \in X$ such that

$$|d(u,Su)| = |z| > 0. \tag{3.6}$$

So by using the triangular inequality and (3.1), we get

$$\begin{aligned}z = d(u,Su) &\precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Su) \\ &= sd(u,x_{2n+2}) + sd(Tx_{2n+1},Su) \\ &\precsim sd(u,x_{2n+2}) + s\alpha d(u,x_{2n+1}) + s\beta\frac{[1+d(u,Su)]d(x_{2n+1},Tx_{2n+1})}{[1+d(u,x_{2n+1})]} \\ &\qquad + s\gamma[d(u,Su)+d(x_{2n+1},Tx_{2n+1})] + s\delta[d(u,Tx_{2n+1})+d(x_{2n+1},Su)] \\ &= sd(u,x_{2n+2}) + s\alpha d(u,x_{2n+1}) + s\beta\frac{[1+d(u,Su)]d(x_{2n+1},x_{2n+2})}{[1+d(u,x_{2n+1})]} \\ &\qquad + s\gamma[d(u,Su)+d(x_{2n+1},x_{2n+2})] + s\delta[d(u,x_{2n+2})+d(x_{2n+1},Su)]\end{aligned}$$

which implies that,

$$\begin{aligned}|z| = |d(u,Su)| &\leq s|d(u,x_{2n+2})| + s\alpha|d(u,x_{2n+1})| + s\beta\frac{|[1+d(u,Su)]||d(x_{2n+1},x_{2n+2})|}{|[1+d(u,x_{2n+1})]|} \\ &\quad + s\gamma|[d(u,Su)+d(x_{2n+1},x_{2n+2})]| + s\delta|[d(u,x_{2n+2})+d(x_{2n+1},Su)]|.\end{aligned} \tag{3.7}$$

Taking the limit on (3.7) as $n \to \infty$, we get $|z| = |d(u,Su)| \leq 0$, a contradiction with (3.6). So $|z| = 0$. Hence $Su = u$. Similarly, one can also show that $Tu = u$.

To prove the uniqueness of common fixed point, let u^* (in X) be another common fixed point of S and T i.e. $u^* = Su^* = Tu^*$. Then

$$\begin{aligned} d(u,u^*) = d(Su,Tu^*) &\precsim \alpha d(u,u^*) + \beta \frac{[1+d(u,Su)]d(u^*,Tu^*)}{[1+d(u,u^*)]} \\ &\quad + \gamma[d(u,Su)+d(u^*,Tu^*)] + \delta[d(u,Tu^*)+d(u^*,Su)] \\ &= \alpha d(u,u^*) + \delta[d(u,u^*)+d(u,u^*)] \end{aligned}$$

so that, $|d(u,u^*)| \le (\alpha+2\delta)|d(u,u^*)|$, a contradiction, so that $u = u^*$. This completes the proof. □

Corollary 10. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying:*

$$d(Tx,Ty) \precsim \alpha d(x,y) + \frac{\beta[1+d(x,Tx)]d(y,Ty)}{[1+d(x,y)]} + \gamma[d(x,Tx)+d(y,Ty)] + \delta[d(x,Ty)+d(y,Tx)] \tag{3.8}$$

for all $x, y \in X$, where $\alpha,\beta,\gamma,\delta$ are nonnegative reals with $\alpha+\beta+2\gamma+2s\delta < 1$. Then T has a unique fixed point in X.

Proof. We can prove this corollary by applying Theorem 9 with $S = T$. □

Remark 11. By equating α,β,γ and δ to 0 in all possible combinations, one can derive a host of corollaries which include Banach fixed point theorem on complex valued b-metric spaces.

Corollary 12. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \ge 1$ and let $T : X \to X$ be a mapping satisfying (for some fixed n):*

$$\begin{aligned} d(T^n x,T^n y) &\precsim \alpha d(x,y) + \frac{\beta[1+d(x,T^n x)]d(y,T^n y)}{[1+d(x,y)]} \\ &\quad + \gamma[d(x,T^n x)+d(y,T^n y)] + \delta[d(x,T^n y)+d(y,T^n x)] \end{aligned} \tag{3.9}$$

for all $x, y \in X$, where $\alpha,\beta,\gamma,\delta$ are nonnegative reals with $\alpha+\beta+2\gamma+2s\delta < 1$. Then the maps T has a unique fixed point in X.

Proof. From Corollary 10, we obtain that $u \in X$ such that $T^n u = u$. The uniqueness follows from

$$\begin{aligned} d(Tu,u) &= d(TT^n u,T^n u) \\ &= d(T^n Tu,T^n u) \\ &\precsim \alpha d(Tu,u) + \beta \frac{[1+d(Tu,T^n Tu)]d(u,T^n u)}{[1+d(Tu,u)]} \\ &\quad + \gamma[d(Tu,T^n Tu)+d(u,T^n u)] + \delta[d(Tu,T^n u)+d(u,T^n Tu)] \\ &\precsim \alpha d(Tu,u) + \beta \frac{[1+d(Tu,TT^n u)]d(u,u)}{[1+d(Tu,u)]} \\ &\quad + \gamma[d(Tu,TT^n u)+d(u,u)] + \delta[d(Tu,u)+d(u,TT^n u)] \\ &= (\alpha+2\delta)d(Tu,u). \end{aligned} \tag{3.10}$$

Taking modulus of above equation, we get $|d(Tu,u)| \leq (\alpha+2\delta)|d(Tu,u)|$. Since $|1+d(Tu,u)| > |d(Tu,u)|$, therefore $|d(Tu,u)| \leq (\alpha+2\delta)|d(Tu,u)|$, a contradiction. So $Tu = u$. Hence $Tu = T^n u = u$. Therefore, the fixed point of T is unique. This completes the proof. □

Example 13. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that $d(z_1,z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2$ where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $(\mathbb{C},d)$ is a complex valued b-metric space with $s = 2$. Now, define two self mappings $S,T : X \to X$ as follows:

$$T(x+iy) = \begin{cases} 0; & x,y \in \mathbb{Q} \\ i; & x,y \in \mathbb{Q}^c \\ 1; & x \in \mathbb{Q}^c,\ y \in \mathbb{Q} \\ 1+i; & x \in \mathbb{Q},\ y \in \mathbb{Q}^c. \end{cases}$$

such that $S = T$ and $z = x+iy$. Let $x = \frac{1}{\sqrt{3}}$ and $y = 0$ and since $\alpha \in [0,1)$ we have

$$d(Tx,Ty) = d\left(T\left(\frac{1}{\sqrt{3}}\right),T(0)\right) = d(1,0) = 2 \precsim \alpha d\left(\frac{1}{\sqrt{3}},0\right) = \alpha\left(\frac{2}{\sqrt{3}}\right).$$

Thus $\alpha \geq \sqrt{3}$ which is a contradiction as $0 \leq \alpha < 1$. However, notice that $T^n z = 0$ for $n > 1$, so

$$\begin{aligned} d(T^n x, T^n y) = 0 \precsim \alpha d(x,y) &+ \frac{\beta[1+d(x,T^n x)]d(y,T^n y)}{[1+d(x,y)]} \\ &+ \gamma[d(x,T^n x)+d(y,T^n y)] + \delta[d(x,T^n y)+d(y,T^n x)] \end{aligned}$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta \geq 0$ with $\alpha+\beta+2\gamma+2s\delta < 1$. So, all conditions of Corollary 12 are satisfied to get a unique fixed point 0 of T.

Theorem 14. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $S,T : X \to X$ are mappings satisfying:*

$$d(Sx,Ty) \precsim \begin{cases} \lambda d(x,y) + \mu \dfrac{d(x,Sx)d(y,Ty)+d(y,Sx)d(x,Ty)}{d(Sx,x)+d(Ty,y)} & \\ \quad +\gamma \dfrac{d(x,Sx)d(x,Ty)+d(y,Sx)d(y,Ty)}{d(Sx,y)+d(Ty,x)}, & \text{if } D \neq 0,\ D_1 \neq 0 \\ 0, & \text{if } D = 0 \text{ or } D_1 = 0. \end{cases} \tag{3.11}$$

for all $x,y \in X$, where $D = d(Sx,x)+d(Ty,y)$ and $D_1 = d(Sx,y)+d(Ty,x)$ and λ,μ,γ are nonnegative reals with $s\lambda+\mu+\gamma < 1$. Then maps S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for } n = 0,1,2,3,\ldots. \tag{3.12}$$

Now, we distinguish two cases: First, if (for $n = 0,1,2,3\ldots$)

$$d(Sx_{2n},x_{2n}) + d(Tx_{2n+1},x_{2n+1}) \neq 0 \quad \text{and} \quad d(Sx_{2n},x_{2n+1}) + d(Tx_{2n+1},x_{2n}) \neq 0,$$

then

$$d(x_{2n+1},x_{2n+2}) = d(Sx_{2n},Tx_{2n+1})$$

$$
\begin{aligned}
&\precsim \lambda d(x_{2n},x_{2n+1})+\mu\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})d(x_{2n},Tx_{2n+1})}{d(Sx_{2n},x_{2n})+d(Tx_{2n+1},x_{2n+1})}\\
&\quad+\gamma\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{d(Sx_{2n},x_{2n+1})+d(Tx_{2n+1},x_{2n})}\\
&=\lambda d(x_{2n},x_{2n+1})+\mu\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})+d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+2})}{d(x_{2n+1},x_{2n})+d(x_{2n+2},x_{2n+1})}\\
&\quad+\gamma\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n+1},x_{2n+1})+d(x_{2n+2},x_{2n})}
\end{aligned}
$$

or

$$
\begin{aligned}
d(x_{2n+1},x_{2n+2})&\precsim \lambda d(x_{2n},x_{2n+1})+\mu\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{d(x_{2n+1},x_{2n})+d(x_{2n+2},x_{2n+1})}\\
&\quad+\gamma\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})}{d(x_{2n+2},x_{2n})}
\end{aligned}
$$

so that,

$$|d(x_{2n+1},x_{2n+2})|\le\lambda|d(x_{2n},x_{2n+1})|+\mu\frac{|d(x_{2n},x_{2n+1})||d(x_{2n+1},x_{2n+2})|}{|d(x_{2n+1},x_{2n})|+|d(x_{2n+2},x_{2n+1})|}+\gamma|d(x_{2n},x_{2n+1})|.$$

Since, $|d(x_{2n+1},x_{2n})+d(x_{2n+2},x_{2n+1})|>|d(x_{2n+1},x_{2n})|$, therefore,

$$|d(x_{2n+1},x_{2n+2})|\le\lambda|d(x_{2n},x_{2n+1})|+\mu|d(x_{2n+1},x_{2n+2})|+\gamma|d(x_{2n},x_{2n+1})|$$

so that,

$$|d(x_{2n+1},x_{2n+2})|\le\frac{\lambda+\gamma}{1-\mu}|d(x_{2n},x_{2n+1})|. \tag{3.13}$$

Similarly, we can obtain

$$|d(x_{2n+2},x_{2n+3})|\le\frac{\lambda+\gamma}{1-\mu}|d(x_{2n+1},x_{2n+2})|. \tag{3.14}$$

Since $s\lambda+\mu+\gamma<1$ and $s\ge1$, we get $\lambda+\mu+\gamma<1$. Now, with $h=\frac{\lambda+\gamma}{1-\mu}$, we have (for all n)

$$|d(x_{2n+1},x_{2n+2})|\le h|d(x_{2n},x_{2n+1})|\le h^2|d(x_{2n-1},x_{2n})|\le\ldots\le h^{2n+1}|d(x_0,x_1)|$$

i.e.

$$|d(x_{n+1},x_{n+2})|\le h|d(x_n,x_{n+1})|\le h^2|d(x_{n-1},x_n)|\le\ldots\le h^{n+1}|d(x_0,x_1)|. \tag{3.15}$$

Thus for any $m>n$, and $m,n\in\mathbb{N}$,

$$
\begin{aligned}
|d(x_n,x_m)|&\le s|d(x_n,x_{n+1})|+s|d(x_{n+1},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+s^2|d(x_{n+2},x_m)|\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+s^2|d(x_{n+2},x_{n+3})|+s^3|d(x_{n+3},x_m)|\\
&\;\;\vdots\\
&\le s|d(x_n,x_{n+1})|+s^2|d(x_{n+1},x_{n+2})|+s^3|d(x_{n+2},x_{n+3})|+\ldots\\
&\qquad+s^{m-n-1}|d(x_{m-2},x_{m-1})|+s^{m-n}|d(x_{m-1},x_m)|.
\end{aligned}
$$

By (3.15), we get

$$\begin{aligned}
|d(x_n,x_m)| &\le sh^n|d(x_0,x_1)| + s^2h^{n+1}|d(x_0,x_1)| + s^3h^{n+2}|d(x_0,x_1)| + \dots \\
&\qquad + s^{m-n-1}h^{m-2}|d(x_0,x_1)| + s^{m-n}h^{m-1}|d(x_0,x_1)| \\
&= \sum_{i=1}^{m-n} s^i h^{i+n-1}|d(x_0,x_1)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|d(x_n,x_m)| &\le \sum_{i=1}^{m-n} s^{i+n-1}h^{i+n-1}|d(x_0,x_1)| \\
&= \sum_{t=n}^{m-1} s^t h^t |d(x_0,x_1)| \\
&\le \sum_{t=n}^{\infty}(sh)^t|d(x_0,x_1)| = \frac{(sh)^n}{1-sh}|d(x_0,x_1)|
\end{aligned}$$

and hence

$$|d(x_n,x_m)| \le \frac{(sh)^n}{1-sh}|d(x_0,x_1)| \to 0 \quad \text{as } m,\ n\to\infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u\in X$ such that $x_n\to u$ as $n\to\infty$. Assume not, then there exists $z\in X$ such that

$$|d(u,Su)| = |z| > 0. \tag{3.16}$$

So by using the triangular inequality and (3.11), we get

$$\begin{aligned}
z = d(u,Su) &\precsim sd(u,x_{2n+2}) + sd(x_{2n+2},Su) \\
&= sd(u,x_{2n+2}) + sd(Tx_{2n+1},Su) \\
&\precsim sd(u,x_{2n+2}) + s\lambda d(u,x_{2n+1}) \\
&\qquad + s\mu\frac{d(u,Su)d(x_{2n+1},Tx_{2n+1}) + d(x_{2n+1},Su)d(u,Tx_{2n+1})}{d(Su,u)+d(Tx_{2n+1},x_{2n+1})} \\
&\qquad + s\gamma\frac{d(u,Su)d(u,Tx_{2n+1}) + d(x_{2n+1},Su)d(x_{2n+1},Tx_{2n+1})}{d(Su,x_{2n+1})+d(Tx_{2n+1},u)}
\end{aligned}$$

which implies that,

$$\begin{aligned}
|z| = |d(u,Su)| &\le s|d(u,x_{2n+2})| + s\lambda|d(u,x_{2n+1})| \\
&\qquad + s\mu|z|\frac{|d(x_{2n+1},x_{2n+2})| + |d(x_{2n+1},Su)||d(u,x_{2n+2})|}{|d(Su,u)|+|d(x_{2n+2},x_{2n+1})|} \\
&\qquad + s\gamma|z|\frac{|d(u,x_{2n+2})| + |d(x_{2n+1},Su)||d(x_{2n+1},x_{2n+2})|}{|d(Su,x_{2n+1})|+|d(x_{2n+2},u)|}
\end{aligned} \tag{3.17}$$

a contradiction, so that $|z| = |d(u,Su)| = 0$, i.e. $u = Su$. It follows, similarly that $u = Tu$. We now prove that S and T have a unique common fixed point. For this, assume that $u^*\in X$ is an another common fixed point of S and T. Then we have $u^* = Su^* = Tu^*$. Now,

$$d(u,u^*) = d(Su,Tu^*) \precsim \lambda d(u,u^*) + \mu\frac{d(u,Su)d(u^*,Tu^*) + d(u^*,Su)d(u,Tu^*)}{d(Su,u)+d(Tu^*,u^*)}$$

$$+\gamma\frac{d(u,Su)d(u,Tu^*)+d(u^*,Su)d(u^*,Tu^*)}{d(Su,u^*)+d(Tu^*,u)}$$
$$\precsim \lambda d(u,u^*)+\mu\frac{d(u^*,u)d(u,u^*)}{d(Su,u)+d(Tu^*,u^*)}.$$

Since $D = d(Su,u)+d(Tu^*,u^*) = 0$, therefore by definition of contraction condition, $d(u,u^*) = d(Su,Tu^*) = 0$, so that $u = u^*$, which proves the uniqueness of common fixed point. Now consider the case when,

$$(d(Sx_{2n},x_{2n})+d(Tx_{2n+1},x_{2n+1}))(d(Sx_{2n},x_{2n+1})+d(Tx_{2n+1},x_{2n})) = 0$$

(for any n) implies $d(Sx_{2n},Tx_{2n+1}) = 0$. Now, if $d(Sx_{2n},x_{2n})+d(Tx_{2n+1},x_{2n+1}) = 0$, then $x_{2n} = Sx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2}$. Thus, we have $x_{2n+1} = Sx_{2n} = x_{2n}$, so there exists k_1 and l_1 such that $k_1 = Sl_1 = l_1$. Using forgoing arguments, one can also show that there exists k_2 and l_2 such that $k_2 = Tl_2 = l_2$. As $d(Sl_1,l_1)+d(Tl_2,l_2) = 0$ (due to definition), implies $d(Sl_1,Tl_2) = 0$, so that $k_1 = Sl_1 = Tl_2 = k_2$ which yields that $k_1 = Sl_1 = Sk_1$. Similarly, one can also have $k_2 = Tk_2$. As $k_1 = k_2$, implies $Sk_1 = Tk_1 = k_1$, therefore $k_1 = k_2$, is a common fixed point of S and T.

We now prove that S and T have unique common fixed point. For this, assume that $k_1^* \in X$ is an another common fixed point of S and T. Then we have $Sk_1^* = Tk_1^* = k_1^*$. Since $D = d(Sk_1,k_1)+ d(Tk_1^*,k_1^*) = 0$, therefore by definition of contraction condition $d(k_1,k_1^*) = d(Sk_1,Tk_1^*) = 0$, so that $k_1 = k_1^*$. If $d(Sx_{2n},x_{2n+1})+d(Tx_{2n+1},x_{2n}) = 0$, implies that $d(Sx_{2n},Tx_{2n+1}) = 0$, then also proof can be completed on the preceding lines. This completes the proof. □

Corollary 15. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying:*

$$d(Tx,Ty) \precsim \begin{cases} \lambda d(x,y)+\mu\dfrac{d(x,Tx)d(y,Ty)+d(y,Tx)d(x,Ty)}{d(Tx,x)+d(Ty,y)} \\ \quad +\gamma\dfrac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(Tx,y)+d(Ty,x)}; & \text{if } D \neq 0,\ D_1 \neq 0 \\ 0; & \text{if } D = 0 \text{ or } D_1 = 0. \end{cases} \tag{3.18}$$

for all $x,y \in X$, where $D = d(Tx,x)+d(Ty,y)$ and $D_1 = d(Tx,y)+d(Ty,x)$ and λ,μ,γ are nonnegative reals with $s\lambda+\mu+\gamma < 1$. Then maps T has a unique fixed point in X.

Proof. We can prove this result by applying Theorem 14 with $S = T$. □

Corollary 16. *Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying (for some fixed n):*

$$d(T^nx,T^ny) \precsim \begin{cases} \lambda d(x,y)+\mu\dfrac{d(x,T^nx)d(y,T^ny)+d(y,T^nx)d(x,T^ny)}{d(T^nx,x)+d(T^ny,y)} \\ \quad +\gamma\dfrac{d(x,T^nx)d(x,T^ny)+d(y,T^nx)d(y,T^ny)}{d(T^nx,y)+d(T^ny,x)}; & \text{if } D \neq 0,\ D_1 \neq 0 \\ 0; & \text{if } D = 0 \text{ or } D_1 = 0. \end{cases} \tag{3.19}$$

for all $x,y \in X$, where $D = d(T^nx,x)+d(T^ny,y)$ and $D_1 = d(T^nx,y)+d(T^ny,x)$ and λ,μ,γ are nonnegative reals with $s\lambda+\mu+\gamma < 1$. Then maps T has a unique fixed point in X.

Proof. From Corollary 15, we obtain that $u \in X$ such that $T^n u = u$. The proof of uniqueness is similar as Corollary 12. □

4. Conclusion

In this paper we have established common fixed point results for rational type contraction mappings in the context of complex valued b-metric spaces.

The results established by us may lead to other applications; at the same time, this concept may provide new interpretation for the concept of complex valued b-metric spaces. For the usability of our results we also present one example.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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