Communications in Mathematics and Applications
Vol. 7, No. 3, pp. 227–240, 2016
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Special Issue on
Advances in Fixed Point Theory towards Real World Optimization Problems
Proceedings of The 9th Asian Conference on Fixed Point Theory and Optimization
May 18 – 20, 2016, King Mongkut's University of Technology Thonburi, Bangkok, Thailand

Fixed Point Theorems for Generalized (α,ψ)-Expansive Mappings in Generalized Metric Spaces

Research Article

Manoj Kumar[1], Serkan Araci[2] and Poom Kumam[3,4,*]

[1] *Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India*
[2] *Faculty of Economics, Administrative and Social Sciences, Department of Economics, Hasan Kalyoncu University, TR-27410 Gaziantep, Turkey*
[3] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand*
[4] *Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
***Corresponding author:** poom.kum@kmutt.ac.th

Abstract. The aim of this paper is to introduce new notion of generalized (α,ψ)-expansive mappings in generalized metric spaces and to study the existence of a fixed point for the mappings in this space. Our new notion complements the concept of generalized (α,ψ)-contractions on generalized metric spaces introduced recently by Aydi *et al.* (*Journal of Inequalities and Applications* **2014**, 229 (2014)). The presented theorems extend, generalize and improve many existing results in the literature

Keywords. Generalized metric spaces; Expansive mapping; Fixed point

MSC. 47H09; 47H10

Received: June 26, 2016 **Revised:** July 5, 2016 **Accepted:** July 12, 2016

1. Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Moreover, it is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach [3] in 1920 was published in 1922 is one of the most important theorems in classical functional analysis. Moreover, being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive mapping. In 1984, Wang *et al.* [16] presented some interesting work on expansion mappings in metric spaces. In 2000, Branciari [4] introduced a concept of generalized metric space by replacing the triangle inequality by a more general inequality. As such, any metric space is a generalized metric space but the converse is not true. Later many authors worked on this interesting space. For more, the reader can refer to [2,5,10,11,13,14]. Recently, Samet *et al.* [12] introduced a new concept of α-ψ-contractive type mappings and established some fixed point theorems for such mappings in complete metric spaces. Shahi *et al.* [15] proved some interesting results for (ξ,α)-expansive mappings in complete metric spaces and generalized the results of Samet *et al.* [12].

In this paper, we introduce a new notion of generalized (α,ψ)-expansive mappings and establish various fixed point theorems for such mappings in complete generalized metric spaces. The presented theorems extend, generalize and improve many existing results in the literature.

First, we recall some fundamental definitions and basic results that will be used throughout this paper. Wang *et al.* [16] defined expansion mappings in the form of following theorem:

Theorem 1.1. *Let (X,d) be a complete metric space. If $f: X \to X$ is an onto mapping and there exists a constant $k > 1$ such that*

$$d(fx, fy) \geq k d(x,y),$$

for each $x, y \in X$. Then f has a unique fixed point in X.

In the following, we recall the notion of a generalized metric space introduced by Branciari [4].

Definition 1.2. Let X be a non-empty set and $d: X \times X \to [0,\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$, each of them different from x and y, one has

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ (the rectangular inequality).

Then (X,d) is called a generalized metric space (or for short g.m.s.).

Definition 1.3. Let (X,d) be a generalized metric space. A sequence $\{x_n\}$ in X is said to be

(i) g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$;

(ii) g.m.s. Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number n (ε) such that for all $n > m > n(\varepsilon)$, $d(x_n, x_m) \prec \varepsilon$;

(iii) completeg.m.s. if every g.m.s. Cauchy sequence is g.m.s. convergent in X.

Recently, Samet et al. [12] introduced the following notions:

Definition 1.4. Let (X,d) be a metric space and $T:X\to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y))$$

for all $x,y\in X$.

Definition 1.5. Let $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$. T is said to be α-admissible if $x,y\in X$, $\alpha(x,y)\ge1\Rightarrow\alpha(Tx,Ty)\ge1$.

Very recently, Karapinar [6] gave the analog of the notion of an α-ψ-contractive mapping, in the context of generalized metric spaces as follows:

Definition 1.6. Let (X,d) be a generalized metric space and $f:X\to X$ be a given mapping. f is said to be an α-ψ-contractive mapping if there exists two functions $\alpha:X\times X\to[0,\infty)$ and a certain ψ such that

$$\alpha(x,y)d(fx,fy)\le\psi(d(x,y)),$$

for all $x,y\in X$.

The following result introduced by Kirk and Shahzad [9] will be used to prove our results:

Proposition 1.7. *Suppose that $\{x_n\}$ is a Cauchy sequence in a g.m.s. (X,d) with $\lim_{n\to\infty}d(x_n,z)=0$, where $z\in X$. Then $\lim_{n\to\infty}d(x_n,t)=d(z,t)$, for all $t\in X$. In particular, the sequence $\{x_n\}$ does not converge to t if $t\ne z$.*

Recently, Aydi *et al.* [1] considered the following family of functions and introduced the notion of generalized (α,ψ)-contractive mapping in the context of a generalized metric space.

Let Ψ be the family of functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following:

(i) ψ is upper semicontinuous;

(ii) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$, for all $t>0$;

(iii) $\psi(t)<t$, for any $t>0$.

Definition 1.8. Let (X,d) be a generalized metric space and $T:X\to X$ be a given mapping. T is said to be a generalized (α,ψ)-contractive mapping of type I if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi(M(x,y)),\quad\text{for all } x,y\in X,$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}.$$

Definition 1.9. Let (X,d) be a generalized metric space and $T:X\to X$ be a given mapping. T is said to be a generalized (α,ψ)-contractive mapping of type II if there exists two functions

$\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(N(x,y)), \quad \text{for all } x,y \in X,$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}\right\}.$$

In what follows, we recall the main results of Aydi *et al.* [1].

Theorem 1.10. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized (α,ψ)-contractive mapping of type I. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 1.11. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized (α,ψ)-contractive mapping of type II. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *T is continuous.*

Then there exists an $u \in X$ such that $Tu = u$.

Theorem 1.12. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized (α,ψ)-contractive mapping of type I. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all k.*

Then there exists an $u \in X$ such that $Tu = u$.

Theorem 1.13. *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized (α,ψ)-contractive mapping of type II. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all k.*

Then there exists an $u \in X$ such that $Tu = u$.

2. Main Results

Throughout this section, we make use of the standard notations and terminologies of nonlienar analysis. We introduce a new notion of generalized (α,ψ)-expansive mappings in the context of generalized metric spaces as follows:

Definition 2.1. Let (X,d) be a generalized metric space and $T:X\to X$ be a given mapping. T is said to be a generalized (α,ψ)-expansive mapping of type I if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\psi(d(Tx,Ty))\geq\alpha(x,y)M(x,y),\quad \text{for all } x,y\in X, \tag{2.1}$$

where

$$M(x,y)=\min\{d(x,y),d(x,Tx),d(y,Ty)\}. \tag{2.2}$$

Definition 2.2. Let (X,d) be a generalized metric space and $T:X\to X$ be a given mapping. T is said to be a generalized (α,ψ)-contractive mapping of type II if there exists two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\psi(d(Tx,Ty))\geq\alpha(x,y)N(x,y),\quad \text{for all } x,y\in X, \tag{2.3}$$

where

$$N(x,y)=\min\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2}\right\}. \tag{2.4}$$

Now, we state our first fixed point result.

Theorem 2.3. *Let (X,d) be a complete g.m.s, and $T:X\to X$ be a bijective, generalized (α,ψ)-expansive mapping of type I. Suppose that*

(i) *T^{-1} is α-admissible;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,T^{-1}x_0)\geq1$ and $\alpha(x_0,T^{-2}x_0)\geq1$;*

(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $z\in X$ such that $Tz=z$.

Proof. Let $x_0\in X$ be such that $\alpha(x_0,T^{-1}x_0)\geq1$ and $\alpha(x_0,T^{-2}x_0)\geq1$. We define the sequence $\{x_n\}$ in X by

$$x_n=Tx_{n+1},\quad \text{for all } n\in\mathbb{N}.$$

Now, if $x_n=x_{n+1}$ for any $n\in\mathbb{N}$, one sees that x_n is a fixed point of T from the definition. Without loss of generality, we can suppose

$$x_n\neq x_{n+1},\quad \text{for all } n\in\mathbb{N}. \tag{2.5}$$

Since T^{-1} is an α-admissible mapping and $\alpha(x_0,T^{-1}x_0)\geq1$, we deduce that

$$\alpha(x_0,x_1)=\alpha(x_0,T^{-1}x_0)\geq1\ \Rightarrow\ \alpha(x_1,x_2)=\alpha(T^{-1}x_0,T^{-1}x_1)\geq1.$$

By repeating the above, we get

$$\alpha(x_n,x_{n+1})\geq1,\quad \text{for all } n=0,1,2,\ldots. \tag{2.6}$$

By using the same techniques above, we get

$$\alpha(x_0,x_2)=\alpha(x_0,T^{-2}x_0)\geq 1 \quad \Rightarrow \quad \alpha(x_1,x3)=\alpha(T^{-1}x_0,T^{-1}x_2)\geq 1.$$

Continuing this process, we get

$$\alpha(x_n,x_{n+2})\geq 1, \quad \text{for all } n=0,1,2,\dots. \tag{2.7}$$

Step 1: We shall prove

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.8}$$

Applying inequality (2.1) with $x=x_n,\ y=x_{n+1}$, we obtain

$$d(x_{n-1},x_n)>\psi(d(Tx_n,Tx_{n+1}))\geq \alpha(x_n,x_{n+1})M(x_n,x_{n+1}).$$

Owing to the fact that $\alpha(x_n,x_{n+1})\geq 1$, for all $n\in\mathbb{N}$, we have

$$d(x_{n-1},x_n)>\psi(d(Tx_n,Tx_{n+1}))\geq M(x_n,x_{n+1}), \tag{2.9}$$

where

$$\begin{aligned} M(x_n,x_{n+1}) &= \min\{d(x_n,x_{n+1}),d(x_n,Tx_n),d(x_{n+1},Tx_{n+1})\} \\ &= \min\{d(x_n,x_{n+1}),d(x_n,x_{n-1}),d(x_{n+1},x_n)\} \\ &= \min\{d(x_n,x_{n+1}),d(x_n,x_{n-1})\}. \end{aligned}$$

If for some n, $M(x_n,x_{n+1})=d(x_n,x_{n-1})$, then the inequality (2.9) becomes

$$d(x_{n-1},x_n)>\psi(d(Tx_n,Tx_{n+1}))\geq d(x_{n-1},x_n), \tag{2.10}$$

which is a contradiction. Hence, $M(x_n,x_{n+1})=d(x_n,x_{n+1})$, for all $n\in\mathbb{N}$, and (2.9) becomes

$$\psi(d(Tx_n,Tx_{n+1}))\geq d(x_n,x_{n+1}), \quad \text{for all } n\in\mathbb{N}. \tag{2.11}$$

This yields

$$d(x_n,x_{n+1})<d(x_{n-1},x_n), \quad \text{for all } n\in\mathbb{N}. \tag{2.12}$$

By induction, (2.11) yields

$$d(x_n,x_{n+1})\leq \psi^n(d(x_0,x_1)), \quad \text{for all } n\in\mathbb{N}. \tag{2.13}$$

By the property of ψ, it is evident that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

Step 2: Now, we shall prove

$$\lim_{n\to\infty} d(x_n,x_{n+2})=0. \tag{2.14}$$

Applying inequality (2.1) with $x=x_n,\ y=x_{n+2}$, we obtain

$$d(x_{n-1},x_{n+1})>\psi(d(Tx_n,Tx_{n+2}))\geq \alpha(x_n,x_{n+2})M(x_n,x_{n+2}).$$

Owing to the fact that $\alpha(x_n,x_{n+2})\geq 1$, for all n, we have

$$d(x_{n-1},x_{n+1})>\psi(d(Tx_n,Tx_{n+2}))\geq M(x_n,x_{n+2}),$$

where

$$\begin{aligned} M(x_n,x_{n+2}) &= \min\{d(x_n,x_{n+2}),d(x_n,Tx_n),d(x_{n+2},Tx_{n+2})\} \\ &= \min\{d(x_n,x_{n+2}),d(x_n,x_{n-1}),d(x_{n+2},x_{n+1})\}. \end{aligned} \tag{2.15}$$

By (2.12), we have

$$M(x_n,x_{n+2}) = \min\{d(x_n,x_{n+1}),d(x_n,x_{n-1})\}.$$

Take $a_n = d(x_{n+1},x_{n+3})$ and $b_n = d(x_{n+2},x_{n+3})$. Thus, from (2.15)

$$\begin{aligned} a_{n-2} = d(x_{n-1},x_{n+1}) &> \psi(d(x_{n-1},x_{n+1})) = \psi(d(Tx_n,Tx_{n+2})) \\ &\geq M(x_n,x_{n+2}) = \min\{a_{n-1},b_{n-1}\}. \end{aligned} \tag{2.16}$$

Again, by (2.12)

$$b_{n-2} \geq b_{n-1} \geq \min\{a_{n-1},b_{n-1}\}.$$

Therefore

$$\min\{a_{n-1},b_{n-1}\} \leq \min\{a_{n-2},b_{n-2}\}, \quad \text{for all } n \in \mathbb{N}.$$

Then the sequence $\{\min\{a_n,b_n\}\}$ is monotone non-increasing, so it converges to some $t \geq 0$. By (2.8) assume that $t > 0$, we have

$$\lim_{n\to\infty}\sup a_n = \lim_{n\to\infty}\sup \min\{a_n,b_n\} = \lim_{n\to\infty}\min\{a_n,b_n\} = t.$$

Making $n \to \infty$ in (2.16), we get

$$t = \lim_{n\to\infty}\sup a_{n-2} > \lim_{n\to\infty}\sup \psi(d(x_{n-1},x_{n+1})) \geq \lim_{n\to\infty}\sup \min\{a_{n-1},b_{n-1}\} = t,$$

which is a contradiction, that is, (2.14) is proved.

Step 3: We shall prove that

$$x_n \neq x_m, \quad \text{for all } n \neq m. \tag{2.17}$$

On the contrary, assume that $x_n = x_m$ for some $m,n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p,x_{p+1}) > 0$ for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n+1$. Now we consider

$$\begin{aligned} \psi(d(x_m,x_{m-1})) &= \psi(d(x_m,Tx_m)) \\ &= \psi(d(x_n,Tx_n)) \\ &= \psi(d(Tx_{n+1},Tx_n)) \\ &\geq \alpha(x_{n+1},x_n)M(x_{n+1},x_n) \\ &\geq M(x_{n+1},x_n), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} M(x_{n+1},x_n) &= \min\{d(x_{n+1},x_n),d(x_{n+1},Tx_{n+1}),d(x_n,Tx_n)\} \\ &= \min\{d(x_{n+1},x_n),d(x_{n+1},x_n),d(x_n,x_{n-1})\} \\ &= \min\{d(x_{n+1},x_n),d(x_n,x_{n-1})\}. \end{aligned} \tag{2.19}$$

If $M(x_{n+1},x_n) = d(x_{n+1},x_n)$, then from (2.18), we get

$$\psi(d(x_m,x_{m-1})) \geq d(x_{n+1},x_n),$$

that is,

$$\begin{aligned} d(x_{n+1},x_n) &\leq \psi(d(x_m,x_{m-1})) \\ &\leq \psi^{m-n}(d(x_{n+1},x_n)). \end{aligned} \tag{2.20}$$

If $M(x_{n+1},x_n)=d(x_n,x_{n-1})$, the inequality (2.18) becomes

$$\psi(d(x_m,x_{m-1}))\geq d(x_n,x_{n-1}),$$

that is,

$$\begin{aligned} d(x_n,x_{n-1}) &\leq \psi(d(x_m,x_{m-1})) \\ &\leq \psi^{m-n+1}(d(x_n,x_{n-1})). \end{aligned} \tag{2.21}$$

Due to a property of ψ, the inequalities (2.20) and (2.21) yield

$$d(x_{n+1},x_n)\leq \psi^{m-n}(d(x_{n+1},x_n))$$

and

$$d(x_n,x_{n-1})\leq \psi^{m-n+1}(d(x_n,x_{n-1})),$$

respectively. In each case, there is a contradiction.

Step 4: We shall prove that $\{x_n\}$ is a Cauchy sequence in (X,d), that is,

$$\lim_{n\to\infty} d(x_n,x_{n+k})=0, \quad \text{for all } k\in\mathbb{N}. \tag{2.22}$$

The cases $k=1$ and $k=2$ are proved, respectively, by (2.8) and (2.14). Now, we take $k\geq 3$ arbitrary. It is sufficient to examine two cases.

Case (I): Suppose that $k=2m+1$, where $m\geq 1$. Then by using step 3 and the quadrilateral inequality together with (2.13), we find

$$\begin{aligned} d(x_n,x_{n+k}) &= d(x_n,x_{n+2m+1}) \\ &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{n+2m},x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p(d(x_0,x_1)) \\ &\leq \sum_{p=n}^{\infty} \psi^p(d(x_0,x_1)) \to 0 \text{ as } n\to\infty. \end{aligned} \tag{2.23}$$

Case (II): Suppose that $k=2m$, where $m\geq 2$. Again, by using step 3 and the quadrilateral inequality together with (2.13), we find

$$\begin{aligned} d(x_n,x_{n+k}) &= d(x_n,x_{n+2m}) \\ &\leq d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+\ldots+d(x_{n+2m-1},x_{n+2m}) \\ &\leq d(x_n,x_{n+2})+\sum_{p=n+2}^{n+2m-1} \psi^p(d(x_0,x_1)) \\ &\leq d(x_n,x_{n+2})+\sum_{p=n}^{\infty} \psi^p(d(x_0,x_1)) \to 0 \text{ as } n\to\infty. \end{aligned} \tag{2.24}$$

By combining the expressions (2.23) and (2.24), we have

$$\lim_{n\to\infty} d(x_n,x_{n+k})=0, \quad \text{for all } k\geq 3.$$

It follows that $\{x_n\}$ is a Cauchy sequence in the complete generalized metric space (X,d). So, there exists $z\in X$ such that

$$\lim_{n\to\infty} d(x_n,z)=0.$$

From the continuity of T, it follows that

$$\lim_{n\to\infty} d(x_{n+1},Tz) = \lim_{n\to\infty} d(Tx_n,Tz) = 0,$$

that is, $\lim_{n\to\infty} x_{n+1} = Tz$. Taking Proposition 1.7 into account, we conclude that $Tz = z$, that is, z is a fixed point of T. □

The following result can be deduced using the same arguments:

Theorem 2.4. *Let (X,d) be a complete g.m.s, and $T: X \to X$ be a bijective, generalized (α,ψ)-expansive mapping of type II. Suppose that*

(i) *T^{-1} is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,T^{-1}x_0) \geq 1$ and $\alpha(x_0,T^{-2}x_0) \geq 1$;*

(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $z \in X$ such that $Tz = z$.

Assuming the following condition, we prove that Theorem 2.3 and Theorem 2.4 still holds for T not necessarily continuous:

(M) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$, for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)},x) \geq 1, \quad \text{for all } k \in N. \tag{2.25}$$

Theorem 2.5. *If we replace the continuity of T by the condition* (M) *in Theorem* 2.3*, then the result holds true.*

Proof. From Theorem 2.3, we know that the sequence $\{x_n\}$ defined by $x_n = Tx_{n+1}$, for all $n \geq 0$, is Cauchy and converges to some $z \in X$. In view of Proposition 1.7, we get

$$\lim_{k\to\infty} d(x_{n(k)+1},Tz) = d(z,Tz). \tag{2.26}$$

We shall show that $Tz = z$. Suppose, on the contrary, that $Tz \neq z$.

From (2.6) and the condition (M), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},z) \geq 1$, for all k.

By applying (2.1), we get

$$d(x_{n(k)-1},z) > \psi(d(Tx_{n(k)}),Tz) \geq \alpha(x_{n(k)},z)M(x_{n(k)},z), \tag{2.27}$$

where

$$\begin{aligned} M(x_{n(k)},z) &= \min\{d(x_{n(k)},z), d(x_{n(k)},Tx_{n(k)}), d(z,Tz)\} \\ &= \min\{d(x_{n(k)},z), d(x_{n(k)},x_{n(k)-1}), d(z,Tz)\}. \end{aligned}$$

By (2.8) and (2.26), we have

$$\lim_{k\to\infty} M(x_(n(k)),z) = d(z,Tz). \tag{2.28}$$

Making $k \to \infty$ in (2.27) and regarding that ψ is upper semi continuous

$$d(z,Tz) \leq \psi(d(z,Tz)) < d(z,Tz),$$

which is a contradiction. Hence, we find that z is a fixed point of T, that is, $Tz = z$. □

In the following, the hypothesis of upper semicontinuity of ψ is not required. Similar to Theorem 2.5, for the generalized (α,ψ)-expansive mapping of type II, we have the following.

Theorem 2.6. *If we replace the continuity of T by the condition (M) in Theorem 2.4, then the result holds true.*

Proof. From Theorem 2.4 (which is same as Theorem 2.3), we know that the sequence $\{x_n\}$ defined by $x_n = Tx_{n+1}$, for all $n \geq 0$, is Cauchy and converges to some $z \in X$. Similarly, in view of Proposition 1.7,

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tz) = d(z, Tz). \tag{2.29}$$

We shall show that $Tz = z$. Suppose, on the contrary, that $Tz \neq z$.

From (2.6) and the condition (M), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, z) \geq 1$, for all k.

By applying (2.3), we get

$$d(x_{n(k)-1}, z) > \psi(d(Tx_{n(k)}), Tz) \geq \alpha(x_{n(k)}, z) N(x_{n(k)}, z), \tag{2.30}$$

where

$$\begin{aligned} N(x_{n(k)}, z) &= \min\left\{ d(x_{n(k)}, z), \frac{d(x_(n(k),)Tx_{n(k)}) + d(z,Tz)}{2} \right\} \\ &= \min\left\{ d(x_{n(k)}, z), \frac{d(x_(n(k),)x_(n(k)-1)) + d(z,Tz)}{2} \right\}. \end{aligned} \tag{2.31}$$

Letting $k \to \infty$ in (2.31), we have

$$\lim_{k\to\infty} N(x_(n(k)), z) = \frac{d(z,Tz)}{2}. \tag{2.32}$$

From (2.32), for k large enough, we have $N(x_{n(k)}, z) > 0$, which implies that

$$\psi(N(x_{n(k)}, z)) < N(x_{n(k)}, z).$$

Making $k \to \infty$ in (2.30), and using (2.32), we get

$$d(z, Tz) \leq \frac{d(z,Tz)}{2},$$

which is a contradiction. Hence, we find that z is a fixed point of T, that is, $Tz = z$. □

If we take the special case $\alpha(x, y) = 1$ in Theorem 2.3, many existing results in the literature can be easily deduced from our main results (see also [7]).

3. Some Consequences

Aydi, Karapinar and Samet [1] proved the following fixed point result on g.ms. endowed with a partial order using the Corollary 3.8 of Karapinar and Samet [8].

Corollary 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and d be a generalized metric on X such that (X,d) is complete. Let $T : X \to X$ be a non-decreasing mapping with respect to $\preccurlyeq$. Suppose* *that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(M(x,y)),$$

for all $x,y \in X$ with $x \succcurlyeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$ and $x_0 \preccurlyeq T^2x_0$;*

(ii) *either T is continuous or X is regular.*

Then T has a fixed point u in X.

Inspired by this result, we give some consequences of the main result presented above. Basically, we apply our results to g.m.s. endowed with a partial order.

Definition 3.2. Let $(X,\preccurlyeq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be non-decreasing with respect to $\preccurlyeq$ if for every

$$x,y \in X, \quad x \preccurlyeq y \;\Rightarrow\; Tx \preccurlyeq Ty. \tag{3.1}$$

Definition 3.3. Let $(X,\preccurlyeq)$ be a partially ordered set. If $x_n \preccurlyeq x_{n+1}$, for all n, then the sequence $\{x_n\}$ in X is said to be non-decreasing with respect to $\preccurlyeq$.

Definition 3.4. Let $(X,d,\preccurlyeq)$ be a partially ordered generalized metric space. X is called regular g.m.s. if, whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$, for all $k \in \mathbb{N}$.

Corollary 3.5. *Let $(X,d,\preccurlyeq)$ be a partially ordered complete generalized metric space. Let T be a bijective self map on X be such that T^{-1} is a non-decreasing map with respect to $\preccurlyeq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$\psi(d(Tx,Ty)) \ge M(x,y), \tag{3.2}$$

for all $x,y \in X$ with $x \succcurlyeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq T^{-1}x_0$ and $x_0 \preccurlyeq T^{-2}x_0$;*

(ii) *either T is continuous or X is regular.*

Then T has a fixed point, say, $z \in X$ such that $Tz = z$.

Proof. Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preccurlyeq y \text{ or } x \succcurlyeq y, \\ 0 & \text{otherwise.} \end{cases}$$

Here, T is a generalized (α,ψ)-expansive mapping of type I, that is,

$$\psi(d(Tx,Ty)) \ge \alpha(x,y)M(x,y), \tag{3.3}$$

for all $x,y \in X$. Using condition (i), we have $\alpha(x_0,T^{-1}x_0) \ge 1$ and $\alpha(x_0,T^{-2}x_0) \ge 1$. Owing to the monotonicity of T^{-1}, we get

$$\alpha(x,y) \ge 1 \Rightarrow x \succcurlyeq y \text{ or } x \preccurlyeq y \Rightarrow T^{-1}x \succcurlyeq T^{-1}y \text{ or}$$
$$T^{-1}x \preccurlyeq T^{-1}y \Rightarrow \alpha(T^{-1}x,T^{-1}y) \ge 1.$$

This shows that T^{-1} is α-admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 2.3. Now, suppose that $(X,\preccurlyeq,d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$ and $x_n\to x\in X$ as $n\to\infty$. Due to the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\preccurlyeq x$ for all k. Now, in view of the definition of α, we obtain $\alpha(x_{n(k)},x)\geq 1$ for all k. Thus, we get the existence of a fixed point in this case from Theorem 2.5. □

Regarding remark above, one can deduce the following result using above corollary:

Corollary 3.6. *Let $(X,d,\preccurlyeq)$ be a partially ordered complete generalized metric space. Let T be a bijective self map on X be such that T^{-1} is a non-decreasing map with respect to $\preccurlyeq$. Suppose that there exists a function $\psi\in\Psi$ such that*

$$\psi(d(Tx,Ty))\geq ad(x,y)+bd(x,Tx)+cd(y,Ty), \tag{3.4}$$

for all $x,y\in X$ with $x\succcurlyeq y$ and $a+b+c>1$. Suppose also that the following conditions hold:

(i) *there exists $x_0\in X$ such that $x_0\preccurlyeq T^{-1}x_0$ and $x_0\preccurlyeq T^{-2}x_0$;*

(ii) *either T is continuous or X is regular.*

Then T has a fixed point, that is, there exists $z\in X$ such that $Tz=z$.

Corollary 3.7. *Let $(X,d,\preccurlyeq)$ be a partially ordered complete generalized metric space. Let T be a bijective self map on X be such that T^{-1} is a non-decreasing map with respect to $\preccurlyeq$. Suppose that there exists a function $\psi\in\Psi$ such that*

$$\psi(d(Tx,Ty))\geq d(x,y), \tag{3.5}$$

for all $x,y\in X$ with $x\succcurlyeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0\in X$ such that $x_0\preccurlyeq T^{-1}x_0$ and $x_0\preccurlyeq T^{-2}x_0$;*

(ii) *either T is continuous or X is regular.*

Then T has a fixed point, say, $z\in X$ such that $Tz=z$.

Proof. By taking $M(x,y)=d(x,y)$ in Corollary 3.5, we get the result. □

Corollary 3.8. *Let $(X,d,\preccurlyeq)$ be a partially ordered complete generalized metric space. Let T be a bijective self map on X be such that T^{-1} is a non-decreasing map with respect to $\preccurlyeq$ satisfying the following condition for all $x,y\in X$ with $x\succcurlyeq y$;*

$$d(Tx,Ty)\geq kd(x,y), \tag{3.6}$$

where $k>1$. Suppose also that the following conditions hold:

(i) *there exists $x_0\in X$ such that $x_0\preccurlyeq T^{-1}x_0$ and $x_0\preccurlyeq T^{-2}x_0$;*

(ii) *either T is continuous or X is regular.*

Then T has a fixed point, say, $z\in X$ such that $Tz=z$.

Proof. By taking $\psi(t)=at$, where $a<1$ and $k=\frac{1}{a}$ in Corollary 3.6, we get the result. □

4. Conclusion

We introduced a new notion of generalized (α,ψ)-expansive mappings and established various fixed point theorems for such mappings in complete generalized metric spaces. The presented theorems extended, generalized and improved many existing results of Aydi et al. [1] and Karapinar and Samet [8].

Acknowledgements

This project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. Aydi, E. Karapinar and B. Samet, Fixed points for generalized (α,ψ)-contractions on generalized metric spaces, *Journal of Inequalities and Applications* **2014** (2014), Article ID 229.

[2] A. Azam and M. Arshad, Kannan fixed point theorem on generalized metric spaces, *Journal of Nonlinear Sciences and Applications* **1** (2008), 45–48.

[3] S. Banach, Sur les operations dans les ensembles abstraitsetleur applications itegrales, *Fundam. Math.* **3** (1922), 133–181.

[4] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. (Debr.)* **57** (2000), 31–37.

[5] P. Das, A fixed point theorem on a class of generalized metric spaces, *Korean Mathematical Society* **9** (2002), 29–33.

[6] E. Karapinar and B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.* **2012** (2012), Article ID 793486.

[7] E. Karapinar, P. Shahi, J. Kaur and S.S. Bhatia, Generalized (ξ,α)-expansive mappings and related fixed-point theorems, *Journal of Inequalities and Applications* **2014** (2014), Article ID 22.

[8] E. Karapinar, Discussion on (α,ψ)-contractions on generalized metric spaces, *Abstr. Appl. Anal.* **2014** (2014), Article ID 962784.

[9] W.A. Kirk and N. Shahzad, Generalized metrics and Caristi's theorem, *Fixed Point Theory Appl.* **2013** (2013), Article ID 129.

[10] H. Lakzian and B. Samet, Fixed points for (ψ,φ)-weakly contractive mappings in generalized metric spaces, *Applied Mathematics Letters* **25** (2012), 902–906.

[11] D. Mihet, On Kannan fixed point principle in generalized metric spaces, *Journal of Nonlinear Science and Applications* **2** (2009), 92–96.

[12] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.* **75**, 2154–2165 (2012).

[13] B. Samet, A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type, *International Journal of Mathematical Analysis* **3** (2009), 25–28.

[14] B. Samet, Discussion on a fixed point theorem of Banach-Caccioli type on a class of generalized metric spaces, *Publicationes Mathematicae Debrecen* **76** (2010), 493–494.

[15] P. Shahi, J. Kaur and S.S. Bhatia, Fixed point theorems for (ξ,α)-expansive mappings in complete metric spaces, *Fixed Point Theory and Applications* **2012** (2012), Article ID 157.

[16] S.Z. Wang, B.Y. Li, Z.M. Gao and K. Iseki, Some fixed point theorems on expansion mappings, *Math. Jpn.* **29** (1984), 631–636.