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Research Article

On Automorphisms and Wreath Products in Crossed Modules

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Abstract. In this paper, we show that if $W_1 = A_1WrB_1$, $W_2 = A_2WrB_2$ are wreath products groups with $A_i = B_i$, $1 \leq i \leq 2$ nontrivial, then $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$ if and only if $A_i = B_i = C_2$, $1 \leq i \leq 2$. Moreover, we obtain some results for central automorphisms of crossed module (W_1,W_2,∂) when W_1, W_2 are wreath products groups.

Keywords. Crossed module; Central automorphism; Wreath product

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1. Introduction

The term crossed module was introduced by J.H.C. Whitehead in his work on combinatorial homotopy theory [8]. Actor crossed module of algebroid was defined by M. Alp in [2]. Actions and automorphisms of crossed modules was studied by Norrie, Alp and Wensley [1, 6]. Wreath products of various kinds have been used over the fifty years ago or so for solving a remarkable variety of problems of group theory. In [5] by Neumann provided a certain amount of information on the structure of wreath products. The structure of the automorphism group of a standard wreath product has been determined by Houghton [4]. Panagopoulos gave in [7] the structure of the group of central automorphisms of the standard wreath products.

The rest of the paper is organized as follows. In the second section we present basic concepts and notations of wreath products and crossed modules. The third section is dedicated to central automorphisms of crossed modules and wreath products groups; and we obtain some results for central automorphisms of crossed module (W_1,W_2,∂) when W_1, W_2 are wreath products groups.

2. Definitions and Notations

Let A be a group and S a non-empty set. The cartesian power, A^S of A is the set of all functions from S to A with multiplication defined componentwise. So, $A^S = \{f \mid f : S \to A\}$, and if $f,g \in A^S$, then $(fg)(s) = f(s)g(s)$, for all $s \in S$. If we have a group B of permutations of S, we can identify B in a natural way with a group of automorphisms of A^S. Namely, if $b \in B$, $f \in A^S$, and f^b denotes the image of f under the automorphism corresponding to b, then $f^b(s) = f(s^{b^{-1}})$, for all $s \in S$. Thus, the automorphisms corresponding to elements of B are transitions of the functions in A^S, or permutations of the factors, if A^S is considered as a cartesian product. We consider a group $\bar{W}$ which is the splitting extension of A^S by this group B of automorphisms. That is, $\bar{W} = \{(b,f) \mid b \in B \text{ and } f \in A^S\}$ and multiplication in $\bar{W}$ is defined by $(b,f)(c,g) = (bc, f^c g)$, $b,c \in B$; $f,g \in A^S$. If we denote by 1 the function in A^S for which $1(s) = 1 \in A$ for every $s \in S$, then the set of all elements $(b,1)$, $b \in B$, forms a subgroup of $\bar{W}$ isomorphic to B, which again we identify with B. Similarly, the set of all elements $(1,f)$ with $f \in A^S$ forms a subgroups of $\bar{W}$ which we identify with A^S. With these conventions, A^S is a normal subgroup of $\bar{W}$, and is complemented by B. Elements of $\bar{W}$ can be factorized in a unique way as products bf, with $b \in B$ and $f \in A^S$; and the automorphism of A^S induced by $b \in B$ is just the restriction to A^S of the inner automorphism of $\bar{W}$ induced by transformation by b. In a particular case that S is the set B itself, and the action of B is given by multiplication on the right, then if $f \in A^B$ and $b \in B$, f^b is given in terms of f by $f^b(\beta) = f(\beta b^{-1})$ for all $\beta \in B$. In this case the group we have the structure of unrestricted wreath product $AwrB$, of A by B makes as follows [5]: By defining the support $\sigma(f)$ by $\sigma(f) = \{b \in B \mid f(b) \neq 1\}$, then the functions whose supports are finite sets form the subgroup $A^{(B)}$ of A^B. This subgroup admits the automorphisms induced by B, and so in $AwrB$ we have the subgroup $B \cdot A^{(B)}$ which is the restricted wreath product $AwrB$. If B is finite, the restricted and unrestricted wreath products coincide. The direct power $A^{(B)}$ which goes into the making of $AwrB$ will be denoted throughout by $\mathcal{F}$, and the cartesian power A^B in $AwrB$ will be denoted by $\bar{\mathcal{F}}$. We call these groups the base groups in W and $\bar{W}$ respectively, and we refer to B as the top group, A as the bottom group.

We label the coordinate subgroups in $\mathcal{F}$, $\bar{\mathcal{F}}$ by elements of B in the obvious way, if $b \in B$, then

$$\begin{aligned} A_b &= \{f \in \mathcal{F} \mid f(\beta) = 1 \text{ if } \beta \neq b\} \\ &= \{f \in \bar{\mathcal{F}} \mid f(\beta) = 1 \text{ if } \beta \neq b\} \\ &= \{f \in \mathcal{F} \mid \sigma(f) \subseteq \{b\}\}. \end{aligned}$$

However, to avoid confusion, we write A_e instead of A_1 for the coordinate subgroup corresponding to $1 \in B$. When we wish to identify A_b with A we use the natural isomorphism

$\nu_b : A_b \to A$ given by $\nu_b(f) = f(b)$ for all $f \in A_b$. The diagonal $\bar{D}$ of $\bar{\mathcal{F}}$ is defined to be the set of all constant function: $\bar{D} = \{f \in \bar{\mathcal{F}} \mid f(\beta) = f(1)$, for all $\beta \in B\}$. $\bar{D}$ lies in $\mathcal{F}$ if and only if B is finite, but we make the convention that again the diagonal D of $\mathcal{F}$ is the set of all constant functions in $\mathcal{F}$, i.e., $D = \{f \in \mathcal{F} \mid f(\beta) = f(1)$, for all $\beta \in B\}$.

Proposition 2.1 ([4])**.** *Expect when B has order two and A is a dihedral group of order $4m+2$ or is of order two, the base group is characteristic in W.*

In view of Proposition 2.1, we assume from now on that when B has order 2, A is not of the type specified there, so that the base group is characteristic in the wreath product. We have extension of automorphisms of A and B to automorphisms of their wreath product W as follows:

Proposition 2.2 ([4])**.** *If $\alpha \in Aut(A)$, we define $\alpha^* \in Aut(W)$ by $(bf)^{\alpha^*} = bf^{\alpha^*}$ for all $b \in B$, $f \in \mathcal{F}$, where $f^{\alpha^*}(x) = (f(x))^{\alpha}$, for all $x \in B$, then the group A^* of all such automorphisms is isomorphic to $Aut(A)$.*

Proposition 2.3. *If $\beta \in Aut(B)$, we define $\beta^* \in Aut(W)$ by $(bf)^{\beta^*} = b^{\beta} f^{\beta^*}$ for all $b \in B$, $f \in \mathcal{F}$, where $f^{\beta^*}(x) = f(x^{\beta^{-1}})$ for all $x \in B$, then the group B^* of all such automorphisms is isomorphic to $Aut(B)$.*

Theorem 2.4. (1) *The automorphism group of the wreath product W of two groups A and B can be expressed as a product $Aut(W) = KI_1B^*$, where*

- *K is the subgroup of $Aut(W)$ consisting of those automorphisms which leave B elementwise fixed.*
- *I_1 is the subgroup of $Aut(W)$ consisting of those inner automorphisms corresponding to transformation by elements of the base group $\mathcal{F}$.*
- *B^* is defined as in Proposition 2.3.*

(2) *The group K can be written as A^*H, where*

- *A^* is defined as in Proposition 2.2.*
- *H is the subgroup of $Aut(W)$ consisting of those automorphisms which leave both B and diagonal elementwise fixed.*

(3) *The subgroups A^*HI_1, HI_1B^*, HI_1, and I_1 are normal in $Aut(W)$ and $Aut(W)$ is splitting extension of A^*HI_1 by B^*. Furthermore, A^* intersects HB^* trivially.*

We recall some definitions and properties of the crossed module category. A crossed module (T,G,∂) consist of a group homomorphism $\partial : T \to G$ called the boundary map, together with an action $(g,t) \to {}^g t$ of G on T satisfying

(1) $\partial({}^g t) = g\partial(t)g^{-1}$,

(2) ${}^{\partial(s)}t = sts^{-1}$,

for all $g \in G$ and $s,t \in T$. The automorphism group $Aut N$ of a group N comes equipped with the canonical homomorphism $\tau : N \to Aut(N)$ which has image $I_{nn}N$, the group of inner automorphism of N. The inner automorphism τ is one of the standard examples of crossed module. Other standard examples of crossed modules are:

The inclusion of a normal subgroup $N \to G$; a G-module M with the zero homomorphism $M \to G$; any epimorphism $E \to G$ with central kernel. We note at once certain consequences of the definition of a crossed module:

(1) $\ker\partial$ lies in $Z(T)$; the center of T;

(2) $\partial(T)$ is a normal subgroup of G;

(3) The action of G on T induces a natural $(G/\partial(T))$-module structure on $Z(T)$, and $\ker\partial$ is a submodule of $Z(T)$.

We say that (S,H,∂') is a subcrossed module of the crossed module (T,G,∂) if

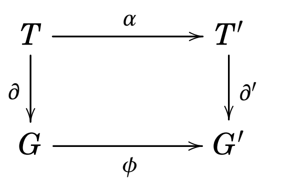
- S is a subgroup of T, and H is a subgroup of G;
- ∂' is the restriction of ∂ to S;
- the action of H on S is included by the action of G on T.

A subcrossed module (S,H,∂) of (T,G,∂) is normal if

- H is a normal subgroup of G;
- ${}^{g}s \in S$ for all $g \in G$, $s \in S$;
- ${}^{h}tt^{-1} \in S$ for all $h \in H$; $t \in T$.

In this case we consider the triple $(T/S,G/H,\bar{\partial})$, where $\bar{\partial} : T/S \to G/H$ is induced by ∂, and the new action is given by ${}^{gH}(tS) = ({}^{g}t)S$. This is the quotient crossed module of (T,G,∂) by (S,H,∂).

A crossed module morphism $\langle\alpha,\phi\rangle : (T,G,\partial) \to (T',G',\partial')$ is a commutative diagram of homomorphisms of groups



such that for all $x \in G$ and $t \in T$; we have $\alpha({}^{x}t) = {}^{\phi(x)}\alpha(t)$. We say that $\langle\alpha,\phi\rangle$ is an isomorphism if α and ϕ are both isomorphisms. We denote the group of automorphisms of (T,G,∂) by $Aut(T,G,\partial)$. The kernel of the crossed module morphism $\langle\alpha,\phi\rangle$ is the normal subcrossed module $(\ker\alpha,\ker\phi,\partial)$ of (T,G,∂), denoted by $\ker\langle\alpha,\phi\rangle$. The image $im\langle\alpha,\phi\rangle$ of $\langle\alpha,\phi\rangle$ is the subcrossed module $(im\,\alpha, im\,\phi,\partial')$ of (T',G',∂'). For a crossed module (T,G,∂); denote by $Der(G,T)$. The set of all derivations from G to T; i.e., all maps $\chi : G \to T$ such that for all $x,y \in G$, $\chi(xy) = \chi(x)^{x}\chi(y)$. Each such derivation χ defines endomorphisms $\sigma = (\sigma_x)$ and $\theta(=\theta_x)$ of G,T respectively; given by $\sigma(x)) = \partial\chi(x)x$; $\theta(t) = \chi\partial(t)t$ and $\sigma\partial(t) = \partial\theta(t)$; $\theta\chi(x) = \chi\partial(x)$; $\theta({}^{x}t) = {}^{\sigma(x)}\theta(t)$. We define a multiplication in $Der(G,T)$ by the formula $\chi_1 \circ \chi_2 = \chi$, where

$$\chi(x) = \chi_1\sigma_2(x)\chi_2(x) \ \ (= \theta_1\,\chi_2(x)\chi_1(x)).$$

This turns $Der(G,T)$ into a semigroup; with identity element the derivation which maps each element of G into identity element of T. Moreover, if $\chi = \chi_1 \circ \chi_2$ then $\sigma = \sigma_1\sigma_2$. The whitehead group $D(G,T)$ is defined to be the group of units of $Der(G,T)$, and the elements of $D(G,T)$ are called regular derivations.

Proposition 2.5. *The following statements are equivalent:*

(1) $\chi \in D(G,T)$;

(2) $\sigma \in Aut(G)$;

(3) $\theta \in Aut(T)$.

The map $\Delta : D(G,T) \to Aut(T,G,\partial)$ defined by $\Delta(X) = \langle \sigma,\theta\rangle$ is a homomorphisms of groups and there is an action of $Aut(T,G,\partial)$ on $D(G,T)$ given by ${}^{\langle \alpha,\phi\rangle}\chi = \alpha\chi\phi^{-1}$; which makes $(D(G,T), Aut(T,G,\partial),\Delta)$ a crossed module. This crossed module is called the actor crossed module $\mathcal{A}(T,G,\partial)$ of the crossed module (T,G,∂). There is a morphism of crossed modules $\langle \eta,\gamma\rangle : (T,G,\partial) \to \mathcal{A}(T,G,\partial)$ defined as follows. If $t \in T$, then $\eta_t : G \to T$ defined by $\eta_t(x) = t\,{}^x t^{-1}$ is a derivation, and the map $t \to \eta_t$ defines a homomorphism $\eta : T \to D(G,T)$ of groups. Let $\gamma : G \to \mathcal{A}(T,G,\partial)$ be the homomorphism $y \to \langle \alpha_y, \phi_y\rangle$, where $\alpha_y(t) = {}^y t$ and $\phi_y(x) = yxy^{-1}$ for $t \in T$ and $y, x \in G$.

3. Central Automorphisms of Crossed Modules and Wreath Products Groups

Let (T,G,∂) be a crossed module. Center of (T,G,∂) is the crossed module kernel $Z(T,G,\partial)$ of $\langle \eta,\gamma\rangle$. Thus, $Z(T,G,\partial)$ is the crossed module $(T^G, St_G(T) \cap Z(G),\partial)$, where T^G denotes the fixed point subgroup of T; that is, $T^G = \{t \in T \mid {}^x t = t \text{ for all } x \in G\}$. $St_G(T)$ is the stabilizer in G of T, that is: $St_G(T) = \{x \in G \mid {}^x t = t \text{ for all } t \in T\}$ and $Z(T)$ is the center of G. Note that T^G is central in T. Let (T,G,∂) be a crossed module and (T',G',∂) a normal subcrossed module its and $\langle \alpha,\phi\rangle \in Aut(T,G,\partial)$. Then, $\langle \alpha,\phi\rangle$ induces a $\langle \bar\alpha,\bar\phi\rangle$ in $Aut(T/T', G/G', \bar\partial)$ such that

$$\bar\partial : \frac{T}{T'} \to \frac{G}{G'}, \quad \bar\partial(tT') = \partial(t)T'.$$

Definition 3.1. *Let (T,G,∂) be a crossed module and $Z(T,G,\partial)$; center of it and $<\alpha,\phi> \in Aut(T,G,\partial)$. If $\langle \bar\alpha,\bar\phi\rangle$ induced of $\langle \alpha,\phi\rangle$ in $Aut\left(\frac{T}{T^G}, \frac{G}{St_G(T)\cap Z(G)}, \bar\partial\right)$; is identity, then $\langle \alpha,\phi\rangle$ is called central automorphism of crossed module (T,G,∂).*

Theorem 3.2. *If $(\alpha,\theta) \in Aut_C(W_1,W_2,\partial)$, then*

(a) *$\alpha = k_1 i_1$, $k_1 \in K_1$, $i_1 \in I_1$ and the inner automorphism i_1 is induced by an element $g_1 \in A_1^{B_1}$ with $g_1 \in Z(A_1^{B_1})$ and $[b_1, g_1] \in Z(W_1)$ for all $b_1 \in B_1$.*

(b) *$\theta = k_2 i_2$, $k_2 \in K_2$, $i_2 \in I_2$ and the inner automorphism i_2 is induced by an element $g_2 \in A_2^{B_2}$ with $g_2 \in Z(A_2^{B_2})$ and $[b_2, g_2] \in Z(W_2)$ for all $b_2 \in B_2$.*

Proof. (a) Suppose that $b_1 \in B_1$ and $b_1^{\alpha} \equiv b'(\mathrm{mod} A_1^{B_1})$ for some $b' \in B_1$. We define, the map β_1 by $\beta_1 : B_1 \to B_1$ such that $b_1^{\beta} = b'$ for all $b \in B_1$. β_1 is an automorphisms of B_1. If β_1^* is the extension of β_1, then by (2)-(4) $\alpha = \beta_1^* k_1 i_1$, $k_1 \in K_1$, $i_1 \in I_1$. But α is central, so we have$b_1^{\alpha} \equiv b_1(\mathrm{mod} Z(W_1))$ for all $b_1 \in B_1$. In addition, by [5], $Z(W_1) \leqq A_1^{B_1}$, hence $b_1^{\beta_1} = b_1$ for all $b_1 \in B_1$, that is the automorphism β_1 is trivial and so is trivial the automorphism β_1^*. Thus, the automorphism α is $\alpha = k_1 i_1$, $k_1 \in K_1$, $i_1 \in I_1$. Suppose that the automorphism i_1 is induced by the element $g_1 \in {A_1}^{B_1}$. So, $g(x) = f_x(x)$ for all $x \in B_1$, $f_x \in {A_1}^{B_1}$ and $f_x = x^{-1} x^{(\beta_1^*)^{-1}\alpha}$ for all $x \in B_1$, by Theorem 2.4. But $\beta_1^* = 1$, so we have $f_x = x^{-1}x^{\alpha}$ and if $x^{\alpha} = x h_x$, $h_x \in Z(W_1)$, then $f_x = h_x \in Z(W_1)$ for all $x \in B_1$. Thus, $g \in Z(A_1^{B_1})$. Moreover, $b_1^{\alpha_1} = b_1^{k_1 i_1} = b_1^{i_1} = b_1[b_1, g_1]$ for all $b_1 \in B_1$. Since $\alpha \in Aut_C(W_1)$, it follows that $[b_1, g_1] \in Z(W_1)$ for all $b_1 \in B_1$.

(b) The proof is similar to (a). □

We consider for simpleness $K_C = Aut_C(W) \cap K$ and $I_C = Aut_C(W) \cap I_1$.

Theorem 3.3. *Let W_1 and W_2 be wreath products and (W_1, W_2, ∂) crossed module. Then, $Aut_C(W_1, W_2, \partial) = \langle K_{1C} \times I_{1C}, K_{2C} \times I_{2C} \rangle$.*

Proof. If $(\alpha, \theta) \in Aut_C(W_1, W_2, \partial)$, then $\alpha \in Aut(W_1)$, $\bar{\alpha} : \frac{W_1}{Z(W_1)} \to \frac{W_1}{Z(W_1)}$ is identity, and by Theorem 3.2, $\alpha = k_1 i_1$, where i_1 is induced by an element $g_1 \in Z(A_1^{B_1})$ with $[b_1, g_1] \in Z(W_1)$ for all $b_1 \in B_1$. Therefore, we obtain

$$(b_1 f)^{i_1} = (b_1 f)^{g_1} = b_1 f[b_1 f, g_1] = b_1 f [b_1, g_1]^f [f, g_1] = b_1 f [b_1, g_1]$$

for all $b_1 \in B$ and $f \in A_1^{B_1}$. So $i_1 \in Aut_C(W_1)$ which together with $\alpha \in Aut_C(W_1)$ we have $k_1 \in Aut_C(W_1)$. Hence, $Aut_{C_1}(W_1) = K_{1C} I_{1C}$. Now, we prove that $K_{1C} \cap I_{1C} = 1$. Let $i_1 \in K_{1C} \cap I_{1C}$ and i_1 is induced by the element $h \in A_1^{B_1}$. But i_1 is in K_{1C}. So, $b_1^{i1} = b_1$. Therefore, $h^{-1} b_1 h = b_1$ or $b_1 h = h b_1$ for all $b_1 \in B_1$. This means that $h \in C_{W_1}(B_1) = Z(B_1) \times D_1$, where D_1 is the diagonal of $A_1^{B_1}$ [4]. But $h \in A_1^{B_1}$ and this means that $h \in D_1$. Now, we show that $h \in Z(D_1) = Z(W_1)$. Suppose that $f \in A_1^{B_1}$. Then, $f^{i_1} = f[f, h]$, where $[f, h] \in Z(W_1)$. Let $a \in A$ and the function $f \in A_1^{B_1}$ with $f(1) = a$ and $f(x) = 1$, for all $x \in B_1$, $x \neq 1$. However, $[f, h](1) = [f(1), h(1)]$ and $[f, h](x) = [f(x), h(x)] = 1$ for all $x \in B_1$, $x \neq 1$. Since $[f, h] \in Z(W_1) = Z(D_1)$, it follows that $[f(1), h(1)] = 1$ or $[a, h(1)] = 1$, for all $a \in A_1$. Thus, $h(1) \in Z(A_1)$ and $h \in Z(W_1)$. So, $i_1 = 1$ and $K_{1C} \cap I_{1C} = 1$. But $k_1 i_1 = i_1 k_1$, where $k_1 \in K_{1C}$ and $i_1 \in I_{1C}$, because the group $Aut_C(W_1)$ is the centralizer in $Aut(W_1)$ of the group $I(W_1)$ of inner automorphisms of W_1, and the proof is completed. The proof of part $K_{2C} \times I_{2C}$ is straightforward. □

In the following we shall need some proposition and lemma of Baumslag and Panagopoulos [3, 7]. If G is a group and G' its derived subgroup, then every inner automorphism of G induces the identity on the group $\frac{G}{G'}$. Let K_G be the subgroup of $Aut(G)$ which consists of those automorphisms which induce the identity on $\frac{G}{G'}$. Clearly, $K_G \geq I(G)$ where $I(G)$ is the group of inner automorphisms. The group $I(G)$ is in general different from K_G.

Definition 3.4. *A group G is called semicomplete if $K_G = I(G)$.*

Proposition 3.5. *Let $\bar{W} = AWrB$ (resp, $W = AWrB$), semicompleteand B, be abelian. Then, A is directly indecomposable.*

Proposition 3.6. *The restricted wreath product of two groups A and B is nilpotent if and only if A and B are nilpotent p-groups for the same prime p, with A of finite exponent and B finite.*

Lemma 3.7. *Let $W = AWrB$, where A is not of exponent 2 when $|B| = 2$. Then, $Z_2(W) = \{f \mid f \in Z(A^B)$ and $[f,x] \in Z(W)$ for all $x \in B\}$.*

Lemma 3.8. *Let $W = AWrB$, with A of exponent 2 and $|B| = 2$. Then, $Z_2(W) = W$, i.e., W is nilpotent of class 2.*

Lemma 3.9. *If A and B are nontrivial and $W = AWrB$ is nilpotent of class 2, then both A and B are abelian.*

Theorem 3.10. *Let $W_1 = A_1WrB_1$ and $W_2 = A_2WrB_2$.*

(1) *If A_i, $1 \le i \le 2$ is not of exponent 2 when $|B| = 2$, then $Aut_C(W_1,W_2,\partial) = \langle K_{1C} \times I_{1C}, K_{2C} \times I_{2C}\rangle$ with $I_{1C} \cong \frac{Z_2(W_1)}{Z(W_1)}$ and $I_{2C} \cong \frac{Z_2(W_2)}{Z(W_2)}$.*

(2) *If A_i, $1 \le i \le 2$ is of exponent 2, $A_i \ne C_2$, $1 \le i \le 2$ and $|B_i| = 2$, $1 \le i \le 2$, then $Aut_C(W_1,W_2,\partial) = \langle K_{1C} \times I_1, K_{2C} \times I_2\rangle$.*

Proof. By Theorem 3.3 and Lemmas 3.8, 3.9, the proof is straightforward. □

Theorem 3.11. *Let $W_1 = A_1WrB_1$, $W_2 = A_2WrB_2$ with $A_i = B_i$, $1 \le i \le 2$ nontrivial. Then, $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$ if and only if $A_i = B_i = C_2$, $1 \le i \le 2$.*

Proof. Suppose that the group B_1 and B_2 are infinite and $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$. Then, W_1 and W_2 are nilpotent of class 2 and so the restricted wreath products A_1 by B_1 and A_2 by B_2 are nilpotent as subgroups of W_1, W_2 respectively. Then, B_1 and B_2 are finite according by Proposition 3.6. But this is a contradiction. Let B_1 and B_2 be finite and $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$. Then, W_1 and W_2 are nilpotent of class 2 and from Lemma 3.7 and Proposition 3.6 we have that A_1 and A_2 are abelian p-groups of finite exponent and B_1, B_2 are abelian p-groups. Furthermore, $Z_2(W_1) = W_1$ and $Z_2(W_2) = W_2$, so $W_1' \le Z(W_1)$ and $W_2' \le Z(W_2)$ and therefore $K_{W_1} \le Aut_C(W_1)$ and $K_{W_2} \le Aut_C(W_2)$. Now, $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$, $I_{nn}(W_1) \le K_{W_1}$ and $I_{nn}(W_2) \le K_{W_2}$. Therefore, we obtain $I_{nn}(W_1,W_2,\partial) = Aut_C(W_1,W_2,\partial)$. So, W_1 and W_2 are semicomplete and we conclude that A_1 and A_2 are directly indecomposable by Proposition 3.5. We distinguish two cases for the group A_1 and A_2:

(1) Let A_1 and A_2 be infinite. Since A_1 and A_2 are abelian p-groups of finite exponent, it follows that A_1 and A_2 are direct product of cyclic groups. This is a contradiction.

(2) Let A_1 and A_2 be finite. Since B_1 and B_2 are finite, it follows that W_1 and W_2 are semicomplete if and only if $A_1 = A_2 = B_1 = B_2 = C_2$ by [7]. But for the group $W = C_2wrC_2$ we have that $Aut_C(W_1,W_2,\partial) = I_{nn}(W_1,W_2,\partial)$. □

4. Conclusion

We studied the connections between wreath products, automorphisms and crossed modules. Also, we investigated some results related to central automorphisms of crossed module (W_1, W_2, ∂), where W_1, W_2 are wreath products of groups. Hope that this work will develop a deep impact on the upcoming research works in this particular field and at the same time, it will be very helpful in the scholastic study of other concerned fields to open up new horizons of interest, erudition and innovations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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