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The Best Uniform Cubic Approximation of Circular Arcs with High Accuracy

Research Article

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Abstract. In this article, the issue of the best uniform cubic approximation of circular arcs with parametrically defined polynomial curves is considered. By a proper choice of the Bézier points, the best uniform approximation of degree 3 to a circular arc is given in explicit form. The approximation is constructed so that the error function is the Chebyshev polynomial of degree 6; the error function equioscillates 7 times; the approximation order is 6. The numerical examples demonstrate the efficiency and simplicity of the approximation method as well as satisfying the properties of the best uniform approximation and yielding the best approximation of least deviation and the highest possible accuracy.

Keywords. Bézier curves; Best uniform approximation; Circular arc; High accuracy; Approximation order; Equioscillation.

MSC. 41A10; 65D07; 65D17

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1. Introduction

Throughout this paper, we consider the circular arc c of angular width 2θ as follows, see Figure 1:

$$c : t \mapsto (\cos(t), \sin(t)), \; -\theta \le t \le \theta, \;\; \text{where } \theta \in [-\pi, \pi].$$

It is not possible to exactly represent a circle with a polynomial curve. It is representable only using rational Bézier curves. Therefore, approximating a circular arc with highest possible accuracy is a very important issue. Thus, it is of high demand to find a parametrically defined polynomial curve $p : t \mapsto (x(t), y(t))$, $0 \le t \le 1$, where $x(t)$ and $y(t)$ are polynomials of degree 3, that approximates c with "minimum" error. The proper distance function to measure the error between p and c is the Euclidean error function:

$$E(t) := \sqrt{x^2(t) + y^2(t)} - 1. \tag{1}$$

$E(t)$ will be replaced by the following error function

$$e(t) := x^2(t) + y^2(t) - 1. \tag{2}$$

Since $e(t) = 0$ is the implicit equation of the unit circle; this implies that the $e(t)$ error function is a suitable measure to test if $x(t)$ and $y(t)$ satisfy this equation and to measure the error.

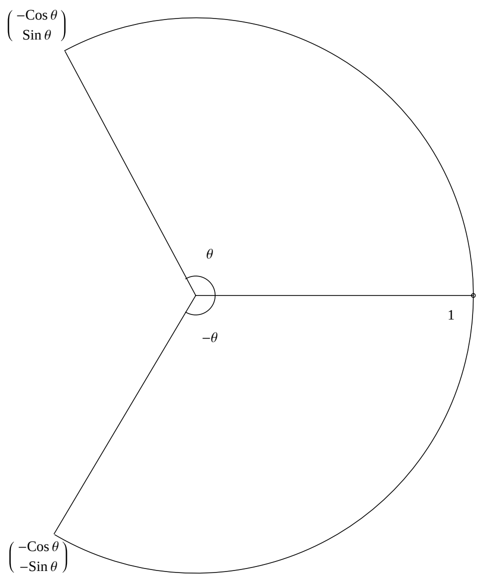

Figure 1. A circular arc.

Both error functions share the same roots and critical points, see Propositions I and II.

The approximation problem in this paper is to find $p : t \mapsto (x(t), y(t))$, $0 \le t \le 1$, where $x(t)$ and $y(t)$ are polynomials of degree 3, that approximates c and satisfies the following three conditions:

(1) p minimizes $\max_{t \in [0,1]} |e(t)|$,

(2) $e(t)$ equioscillates 7 times over $[0,1]$, and

(3) p approximates c with order 6.

The solution to this problem is graphed in Figure 3, and the corresponding error is graphed in Figure 4.

2. Previous Works

Bézier introduced in [1] a cubic parametric curve by interpolating the end points and a point in the middle of the circular arc. A quarter of a circle is approximated by a cubic curve by

Blinn using the values and tangents at the end points in [2]. A general cubic scheme of order 6 is presented by de Boor, Höllig and Sabin in [3]. They used values of positions, tangents, and curvatures at the endpoints. Dokken, Dæhlen, Lyche and Mørken constructed cubic approximation for the circle of order 6 in [4]. Different types of cubic approximations of circular arcs of order 6 are considered by Goldapp in [6]. The author partially proved the conjecture of high order approximation in [10]; the circular arc is given as an example, see also [5, 9, 11, 14] and the papers therein.

The method in this paper represents a circular arc in a very easy way while satisfying the approximation conditions of best uniform approximation. The numerical results of this method are superior over the above mentioned methods, see the numerical comparisons in Section 6.

3. Preliminaries

A monic polynomial $Q_6(u)$ of degree 6 satisfies, see [15],

$$\max_{u\in[-1,1]} |Q_6(u)| \geq \frac{1}{32}. \tag{3}$$

The monic Chebyshev polynomial, $\widetilde{T}_6(u)$, $u \in [-1,1]$, is the unique polynomial of degree 6 that satisfies the equality in (3) and equioscillates 7 times between $\pm\frac{1}{32}$.

The Bézier curve $p(t)$ of degree 3 (see Figure 2), is given by, see [7],

$$p(t) = \sum_{i=0}^{3} p_i B_i^3(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad 0 \leq t \leq 1, \tag{4}$$

where p_0, p_1, p_2 and p_3 are the Bézier points, and $B_0^3(t) = (1-t)^3$, $B_1^3(t) = 3t(1-t)^2$, $B_2^3(t) = 3t^2(1-t)$, and $B_3^3(t) = t^3$ are the Bernstein polynomials of degree 3.

Since it is intended to represent the circular arc with polynomial curve with minimum error, it is not important where the errors occur, at the end points with G^k-continuity, like in [12, 13], or elsewhere; it is important to keep this error as small as possible than where the error occurs. To represent a circular arc of the unit circle, the Bézier points are chosen to explore symmetry properties of the circle. So, let $p_0 = \left(-\alpha_0\cos(\theta), -\beta_0\sin(\theta)\right)$, then by the symmetry $p_3 = \left(-\alpha_0\cos(\theta),\ \beta_0\sin(\theta)\right)$. Let $p_1 = \left(\gamma, -\zeta\right)$ then by symmetry $p_2 = \left(\gamma, \zeta\right)$. By making the substitution $\alpha = \alpha_0\cos(\theta)$, $\beta = \beta_0\sin(\theta)$, then the proper choice for the Bézier points should be, see Figure 2,

$$p_0 = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}, \quad p_1 = \begin{pmatrix} \gamma \\ -\zeta \end{pmatrix}, \quad p_2 = \begin{pmatrix} \gamma \\ \zeta \end{pmatrix}, \quad p_3 = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}. \tag{5}$$

The circular arc c begins in the third quadrant, goes counter clockwise through fourth and first quadrants and ends in the second quadrant. In order to have the Bézier curve p follow the same path as c, the following conditions should be satisfied

$$\alpha, \beta, \zeta > 0, \quad \gamma > 1. \tag{6}$$

The Bézier curve $p(t)$ in (4) with Bézier points in (5) is given by

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\alpha\left(B_0^3(t) + B_3^3(t)\right) + \gamma(B_1^3(t) + B_2^3(t)) \\ \beta\left(B_3^3(t) - B_0^3(t)\right) + \zeta\left(B_2^3(t) - B_1^3(t)\right) \end{pmatrix}, \quad 0 \leq t \leq 1. \tag{7}$$

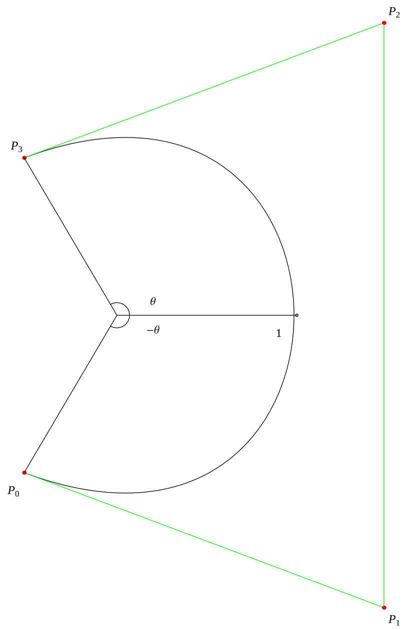

Figure 2. Possible Bézier points of circular arc.

4. The Best Uniform Approximation

Theorem I. *Consider the Bézier curve in* (7) *with the Bézier points in* (5)*, where*

$$\beta = \beta^* := \frac{1}{4}\left(-5 + 2\left(32 - \sqrt{1023}\right)^{\frac{1}{3}} + 2\left(32 + \sqrt{1023}\right)^{\frac{1}{3}}\right) = 0.8748473632, \tag{8}$$

$$\alpha = \alpha^* := \frac{1}{4}\sqrt{\frac{33}{2} - 16\beta^2} = 0.5156472545, \tag{9}$$

$$\gamma = \gamma^* := \frac{16\sqrt{1+4\beta} - 3\sqrt{33-32\beta^2}}{12\sqrt{2}} = 1.484217064, \tag{10}$$

$$\zeta = \zeta^* := \frac{4+\beta}{3} = 1.624949121. \tag{11}$$

Then p satisfies the following 3 *conditions: p minimizes the uniform error* $\max_{t\in[0,1]} |e(t)|$ *and approximates c with order* 6*, and the error function $e(t)$ equioscillates* 7 *times in* $[0,1]$*. More precisely, the error functions satisfy:*

$$-\frac{1}{2^5} \le e(t) \le \frac{1}{2^5}, \quad -\frac{1}{2^6} \lessapprox E(t) \le \frac{1}{2^6}, \quad \text{for all } t \in [0,1], \tag{12}$$

where $\lessapprox$ means approximately less than or equal with error no more than $\frac{1}{2^{12}}$; the exact values are given in Proposition III.

Proof. Substituting the components of $p(t)$ from equation (7) into equation (2) for the error function $e(t)$ gives

$$\begin{aligned} e(t) = {} & 4(\beta - 3\zeta)^2 t^6 - 12\left(\beta - 3\zeta\right)^2 t^5 + 3\left(3\alpha^2 + 7\beta^2 + 6\alpha\gamma + 3\gamma^2 - 34\beta\zeta + 39\zeta^2\right) t^4 \\ & - 2\left(9\alpha^2 + 11\beta^2 + 18\alpha\gamma - 42\beta\zeta + 9\left(\gamma^2 + 3\zeta^2\right)\right) t^3 \end{aligned}$$

$$+3\left(5\alpha^2+5\beta^2+8\alpha\gamma-12\beta\zeta+3\left(\gamma^2+\zeta^2\right)\right)t^2-6\left((\alpha^2+\alpha\gamma+\beta(\beta-\zeta)\right)t+\left(\alpha^2+\beta^2-1\right).$$

The last one is a polynomial of degree 6. Substituting the values of $\alpha=\alpha^*$, $\beta=\beta^*$, $\gamma=\gamma^*$, $\zeta=\zeta^*$ from equations (8)-(11) and doing some simplifications gives

$$e(t)=\frac{1}{32}-\frac{9}{4}t+\frac{105}{4}t^2-112t^3+216t^4-192t^5+64t^6, \quad t\in[0,1].$$

Making the substitution $t=\frac{u+1}{2}$ yields

$$e(u)=-\frac{1}{32}+\frac{9}{16}u^2-\frac{3}{2}u^4+u^6, \quad u\in[-1,1].$$

The last polynomial is the monic Chebyshev polynomial $\widetilde{T}_6(u)$, $u\in[-1,1]$, which is the unique polynomial of degree 6 that satisfies the equality in (3) and equioscillates 7 times between $\pm\frac{1}{2^5}$ for all $u\in[-1,1]$. This shows that $e(t)$ equioscillates 7 times in $[0,1]$, p approximates c with order 6 and minimizes the uniform error $\max\limits_{t\in[0,1]}|e(t)|$.

Since $\frac{-1}{2^5}\le e(t)\le\frac{1}{2^5}$, and $E(t)$ is a bounded function, then we have

$$-1+\sqrt{1-\frac{1}{2^5}}\le E(t)\le -1+\sqrt{1+\frac{1}{2^5}}.$$

Since $\sqrt{1+\frac{1}{2^5}}\le 1+\frac{1}{2^6}$, then we have $E(t)\le\frac{1}{2^6}$, for all $t\in[0,1]$. For the other inequality, since $\sqrt{1-\frac{1}{2^5}}\approx 1-\frac{1}{2^6}$ with error no more than $\frac{1}{2^{13}\left(1-\frac{1}{2^5}\right)^{\frac{3}{2}}}\le\frac{1}{2^{12}}$, then $-1+\sqrt{1-\frac{1}{2^5}}\approx-\frac{1}{2^6}$; in this case, we write $E(t)\gtrapprox-\frac{1}{2^6}$ with error no more than $\frac{1}{2^{12}}$. This completes the proof of Theorem I. □

Figure 3 shows the circular arc and the approximating Bézier curve with the Bézier polygon, and Figure 4 shows the corresponding error.

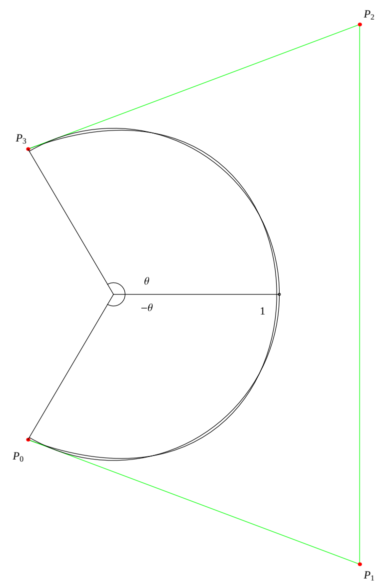

Figure 3. The circular arc and the best uniform approximation in Theorem I.

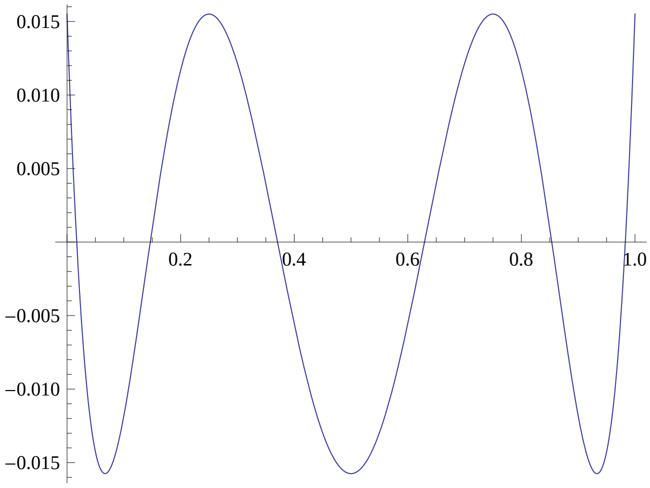

Figure 4. Euclidean error of the solution in Theorem I.

One would not, in general, expect a cubic polynomial to approximate more than 2/3rd, the angle $2\theta = 241°$, of the circle more precisely than this.

5. Properties of the Approximating Bézier Curve

Proposition I. *The roots of the error functions $e(t)$ and $E(t)$ are:*

$$t_1 = \frac{1}{2}\left(1+\cos\left(\frac{\pi}{12}\right)\right) = 0.982963, t_2 = \frac{1}{2}\left(1+\cos\left(\frac{3\pi}{12}\right)\right) = 0.853553,$$
$$t_3 = \frac{1}{2}\left(1+\cos\left(\frac{5\pi}{12}\right)\right) = 0.62941, t_4 = \frac{1}{2}\left(1-\cos\left(\frac{5\pi}{12}\right)\right) = 0.37059,$$
$$t_5 = \frac{1}{2}\left(1-\cos\left(\frac{3\pi}{12}\right)\right) = 0.146447, t_6 = \frac{1}{2}\left(1-\cos\left(\frac{\pi}{12}\right)\right) = 0.0170371.$$

Proof. Substituting t_i in $e(t)$ gives $e(t_i) = 0$, $i = 1,\dots,6$. Since $e(t)$ is a polynomial of degree 6 and thus has 6 roots; these are all the roots. The error function $E(t)$ has the same roots as $e(t)$ because $E(t) = 0$ iff $\sqrt{x^2(t)+y^2(t)} = 1$ iff $x^2(t)+y^2(t) = 1$ iff $e(t) = 0$. □

Proposition II. *The extreme values of $e(t)$ and $E(t)$ occur at the following parameter values:*

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}\left(1+\frac{\sqrt{3}}{\sqrt{2}}\right) = 0.933013, \quad \tilde{t}_2 = \frac{3}{4}, \quad \tilde{t}_3 = \frac{1}{2},$$
$$\tilde{t}_4 = \frac{1}{4}, \quad \tilde{t}_5 = \frac{1}{2}\left(1-\frac{\sqrt{3}}{\sqrt{2}}\right) = 0.0669873, \quad \tilde{t}_6 = 0.$$

Proof. Differentiating $e(t)$ gives a polynomial of degree 5. Substituting $\tilde{t}_1,\dots,\tilde{t}_5$ gives $e'(\tilde{t}_i) = 0$, $i = 1,\dots,5$. Since $e'(t)$ is of degree 5 then these are all interior critical points. Checking at the end points adds $\tilde{t}_0 = 1$, $\tilde{t}_6 = 0$ to the critical points. Since $\sqrt{x^2(t)+y^2(t)} \neq 0$, for all $t \in [0,1]$, thus differentiating $E(t)$ and equating to 0 gives $\frac{e'(t)}{\sqrt{x^2(t)+y^2(t)}} = 0$ iff $e'(t) = 0$. Thus $e(t)$ and $E(t)$ attain the extrema at the same values. This completes the proof of the proposition. □

Proposition III. *The values of $e(t)$ and $E(t)$ at $\tilde{t}_i$ are given by*

$$e(\tilde{t}_0)=e(\tilde{t}_2)=e(\tilde{t}_4)=e(\tilde{t}_6)=\frac{1}{32}, \qquad e(\tilde{t}_1)=e(\tilde{t}_3)=e(\tilde{t}_5)=\frac{-1}{32}.$$

$$E(\tilde{t}_0)=E(\tilde{t}_2)=E(\tilde{t}_4)=E(\tilde{t}_6)=0.0155048, \quad E(\tilde{t}_1)=E(\tilde{t}_3)=E(\tilde{t}_5)=-0.015749.$$

Thus

$$\frac{-1}{32}\le e(t)\le\frac{1}{32}=2(0.015625), \quad -0.015749\le E(t)\le 0.0155048, \; t\in[0,1].$$

Proof. These equalities and inequalities can be proved by direct substitution into $e(t)$ and $E(t)$. □

Proposition IV. *For every $t\in[0,1]$, the errors of approximating the circular arc using the Bézier curve in Theorem* I *are given by:*

$$e(t)=\frac{1}{32}-\frac{9}{4}t+\frac{105}{4}t^2-112t^3+216t^4-192t^5+64t^6, \quad \textit{for all } t\in[0,1].$$

$$E(t)\cong\frac{1}{64}-\frac{9}{8}t+\frac{105}{8}t^2-56t^3+108t^4-96t^5+32t^6, \quad \textit{for all } t\in[0,1].$$

Proof. The approximate equality for $E(t)$ is proved from the equality for $e(t)$ and applying the equality $E(t)=\frac{e(t)}{2+E(t)}$. □

6. Examples and Comparisons

Theorem I gives the best uniform approximation for $\theta=120.5^\circ$. What about other angles? Using the subdivision algorithm guarantees this, but the error is not altered accordingly. To take advantage of the small error of Theorem I, we divided the error function $\widetilde{T}_6(t)$ by a constant and by trial and test we got the constant 322 that when the equations are solved then the Bézier points correspond to the quarter of the circle. Figure 5 shows the whole circle approximated by rotating one quarter. The resulting error is graphed in Figure 6. It is shown in this paper that our solution is the best uniform approximation, so it is anticipated that this fact is tested by comparing the numerical results with the numerical results of other existing methods. In this section, we compare between the different methods that are developed by many researchers to approximate a quarter of a unit circle. Table 1 summarizes these values.

Table 1. Comparison between errors of approximating a quarter of circle using different methods.

Researcher(s)	[Article]	Error
Bézier	[1]	3×10^{-4}
Blinn	[2]	4×10^{-4}
de Boor, Höllig and Sabin	[3]	2×10^{-3}
Dokken, Dæhlen, Lyche and Mørken	[4]	2×10^{-4}
Goldapp	[6]	2×10^{-4}
Rababah	[9, 10]	2×10^{-3}
Rababah	This article	4×10^{-5}

It is clear that the method developed in this article in Theorem I is superior over other existing methods. This is a natural consequence of the construction of the method so that the error function is the monic Chebyshev polynomial which has the least uniform deviation from 0.

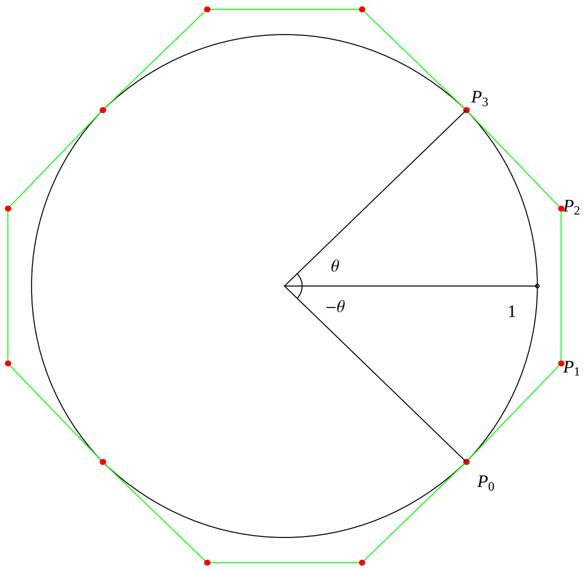

Figure 5. The figure of the full circle using 4 Bézier curves.

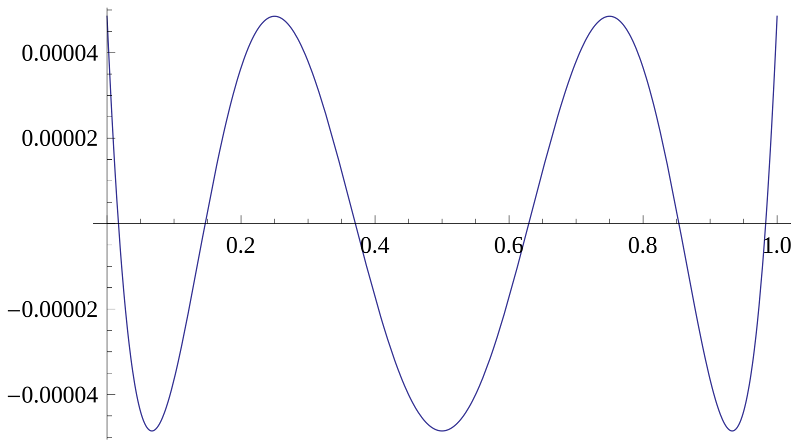

Figure 6. The error of one out of four quarters of the full circle.

7. Conclusions

In this article, the best uniform approximation of circular arcs with parametrically defined polynomial curves of degree 3 is given in explicit and closed form. The error function equioscillates 7 times; the approximation intersects the circular arc 6 times with approximation order 6. Numerical examples and comparisons are given in section 6 to demonstrate the efficiency of the approximation method. The comparison shows that this approximation method is superior over other existing methods. The approximation method is efficient, simple, satisfies the properties of the best uniform approximation, and yields the best approximation of least deviation and highest possible accuracy. As further investigations, the following issues are suggested.

(1) Defining splines with minimal defect with high approximation and compare their performance with the splines with minimal defect that are studied in [8].

(2) Is it possible to use the results in this paper to find sharp embedding for holomorphic based on Lorentz spaces and compare it with the results in [16].

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Competing Interests

The author declares that he has no competing interest.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] P. Bézier, *The Mathematical Basis of the UNISURF CAD System*, Butterworth-Heinemann Newton, MA, USA (1986).

[2] J. Blinn, How many ways can you draw a circle?, *Computer Graphics and Applications, IEEE* **7** (8) (1987), 39–44.

[3] C.de Boor, K. Höllig and M. Sabin, High accuracy geometric Hermite interpolation, *Comput. Aided Geom. Design* **4** (1987), 269–278.

[4] T. Dokken, M. Dæhlen, T. Lyche and K. Mørken, Good approximation of circles by curvature-continuous Bézier curves, *Comput. Aided Geom. Design* **7** (1990), 33–41.

[5] M. Floater, High order approximation of conic sections by quadratic splines, *Comput. Aided Geom. Design* **12** (1995), 617–637.

[6] M. Goldapp, Approximation of circular arcs by cubic polynomials, *Comput. Aided Geom. Design* **8** (1991), 227–238.

[7] K. Höllig and J. Hörner, Approximation and Modeling with B-Splines, *SIAM, Titles in Applied Mathematics* **132** (2013).

[8] A. Makarov, Splines with minimal defect and decomposition matrices, *Communications in Mathematics and Applications* **3** (3) (2012), 355–367.

[9] A. Rababah, Taylor theorem for planar curves, *Proc. Amer. Math. Soc.* **119** (3) (1993), 803–810.

[10] A. Rababah, High order approximation method for curves, *Comput. Aided Geom. Design* **12** (1995), 89–102.

[11] A. Rababah, High accuracy Hermite approximation for space curves in $\Re^d$, *Journal of Mathematical Analysis and Applications* **325** (2) (2007), 920–931.

[12] A. Rababah and S. Mann, Iterative process for G^2-multi degree reduction of Bézier curves, *Applied Mathematics and Computation* **217** (20) (2011), 8126–8133.

[13] A. Rababah and S. Mann, Linear methods for G^1, G^2, and G^3-multi-degree reduction of Bézier curves, *Computer-Aided Design* **45** (2) (2013), 405–414.

[14] A. Rababah, The best uniform quadratic approximation of circular arcs with high accuracy, *Open Mathematics* **14** (1) (2016), 118–127.

[15] J. Rice, The approximation of functions, Vol. 1: linear theory. Addison-Wesley, 1964.

[16] R. Shamoyan and W. Xu, A note on sharp embedding theorems for holomorphic classes based on Lorentz spaces on a unit circle, *Communications in Mathematics and Applications* **1** (1) (2010), 47–51.