



# Collocation Method With Shifted Legendre-Based Approach for Solution of Nonlinear Differential Equations

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**Abstract.** The technique of solving *nonlinear ordinary differential equation* (NODE) that we have used in the paper is the Shifted Legendre technique. This is achieved by rewriting the problem in a more stable way with Shifted Legendre polynomials which has been shown to be effective and precise in numerical solutions. Using this method allows solving complicated nonlinear equations and it is not difficult to obtain approximate solutions in a short time. The paper identifies the benefits of the given approach, that is, reduced computing requirements and enhanced precision and is supported by practical exercises. The up-to-date analysis of the solution of *nonlinear ordinary differential equations* (NODEs) by the Shifted Legendre technique provides a simple and easily applicable answer. Numerical findings of the applied method demonstrate the exact correspondence with the exact solution with the least errors that is better in comparison with the current methods. A number of examples are resolved to demonstrate the success of the suggested method with minimum errors that outperform the existing methods.

**Keywords.** Shifted Legendre Polynomial, Collocation method,  $L_2$ -Norm, Nonlinear Ordinary Differential Equations, Numerical Approximation

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## 1. Introduction

In mathematics, *differential equations* (DEs) are crucial in deriving as well as interpreting algorithms modeling the dynamics of systems in nearly all fields of science. Every differential equation describes relationships between variables, it describes how the functions relate to one another and how fast they change making it possible to trace transformations and determine how they will behave into the future (Braun [6]). In physics, biology, engineering and economics, among other fields, differential equations are essential to the movement of a simple pendulum, study of physical system, and the changes over a period of time.

During the last two centuries, the theory of DEs has experienced impressive evolution and resulted in very strong systems of analysis and numerical procedures (Butcher [7]). Classical methods were mainly aimed at finding precise solutions to a given equation whereas the current developments are solving problems that cannot be analytically solved. This combination of theory and practice shows the mathematical roots and the practical use of the DEs in the modeling of physical phenomena. With these improvements, the utility of the differential equations is still limited by the inability to find a general approach of solving the equations, as well as to interpret the equations reliably, especially when the equations are nonlinear, large parameters, or chaotic systems (Nagle *et al.* [23]). This difficulty offers continual possibilities to the advancement of effective and correct methodologies of solutions.

The problem with nonlinear differential equations is especially troublesome as they are complex and sensitive to initial conditions (Verhulst [30]). Nonlinear systems have phenomena like bifurcations, chaos and multistability unlike linear systems where the principle of superposition holds. Such properties lead to the development of new numerical methods that are accurate, stable, and efficient in the computations. *Ordinary differential equations* (ODEs) are one of the subclasses among other types of differential equations whose functions are of one independent variable and higher derivatives.

ODEs are applied to model many physical, engineering, and biological processes mathematically. Nonetheless, it is uncommon to get precise solutions, and thus numerical techniques are necessary. A strong alternative is spectral methods, which can be used when the problem is smooth and spectral methods are accurate and converge rapidly. As a result, many numerical methods, including Runge-Kutta methods (Wambecq [31]), predictor-corrector methods (Douglas and Jones [13]), finite difference methods, and finite element methods (Reddy [28]), have been designed. These methods usually divide the computational region into a finite number of points or elements and solve the solution locally.

During the last ten years, methods of Legendre polynomials have been highly advanced to effectively resolve initial and boundary value equations, fractional equations, and control problems. The techniques have been found to be strong, computationally efficient, and much flexibility. The strengths that are considered to be one of the main strengths of the Legendre polynomials are that they can convert the differential operators into non-algebraic forms and thus they become powerful in theoretical studies and also in practical studies. Legendre polynomials are a set of orthogonal polynomials that are defined on the interval  $[-1, 1)$ . They are very commonly used in numerical analysis due to their orthogonality property. Legendre polynomials (also known as zonal harmonics or, in some applications, as spherical harmonics)

are commonly applied in physics and other applied sciences. Significant uses are the calculation of electrostatic and gravitational potentials within spherical shells, steady-state heat conduction within homogeneous solid spheres (Hesthaven *et al.* [15]).

ODEs that can be solved using hypergeometric functions can frequently be represented using so-called Legendre polynomials that might be symmetric polynomials or nonsymmetric polynomials. These polynomials have a number of outstanding properties which make them even more significant in the fields of theoretical and applied mathematics. The existence and applicability of many classes of such polynomials, lots of which have been determined, are established by numerous studies; see, as examples, Boyd [5], Ismail and Koelink [20], and Jackson [21]. Legendre polynomials are particularly important in a number of applications, such as control theory, signal processing, and cryptography, due to their special properties. They are especially of use in approximation theory and numerical analysis since they are the exact solutions of the Legendre differentiation equation and can therefore be used as the natural base functions to approximate smooth functions on finite intervals. SLPs are the result of a mapping of the domain  $([-1, 1])$  to  $([0, 1])$  which makes them more apt in problems with finite domain. Due to their orthogonality, SLPs can now be used to solve a differential equation by representing a general solution as a finite linear sum of basis polynomials to obtain a small least-squares error and high approximation accuracy (Chang and Wang [10], and Mohammadi and Hosseini [22]).

The collocation method is one of the most widespread methods of using shifted Legendre polynomials to solve ordinary differential equations (Doha *et al.* [12]). In the method, an approximation to the unknown solution is obtained by a truncated SLP series and the approximation is imposed to satisfy the DE at chosen points within the domain which are called collocation points. These nodes are normally the shifted Legendre poly nodes or equally spaced nodes to provide numerical stability and accuracy. Application of the governing equation at these collocation points converts the initial ODE to a system of algebraic equations of the unknown series coefficients (Nemati [24]). This approach is a hybrid of the precision of spectral approximations and the numerical simplicity of the solution of finite-dimensional linear or nonlinear equations, and it is especially useful when a solution to a problem with initial and boundary conditions is needed.

The rest of this paper is structured in the following way. Section 2 gives the mathematical backgrounds, such as the key definitions, findings, and properties of Legendre polynomials needed to carry out the analysis proposed. Section 3 is a presentation of a collocation-based spectral scheme of solving nonlinear ordinary differential equations. In Section 4, the convergence analysis of the shifted Legendre polynomial expansion is discussed. Section 5 will be a series of numbers which shows the efficiency of the proposed method and the comparison of the results of the numerical calculations with the exact solutions with the graphical presentation of the results.

Ibrahim [17, 18] developed a new method of analysis of *nonlinear partial differential equations* (NPDEs) and a *nonlinear Schrodinger equation* (NLSE). The suggested approach is very versatile and applicable to the vast majority of NPDEs and NLSEs. Practically, the approach shows a very high potential in the use of control theory applications (Ibrahim and Köksal [19]).

Cesarano [9] generalized the classical Chebyshev polynomials of the first and second kind by applying operational techniques related to the Kampé de Fériet-type Hermite polynomials, which gave new integral presentations of the generalized Chebyshev polynomials. Also, Dattoli and Cesarano [11] used the methods of integral representation to derive the relations of the classical Hermite polynomials with a recently created family of functions dependent on the parabolic cylinder functions. In order to solve the nonlinear ODE and PDE problems, various works have been proposed and formulated by different researchers via different methods to provide efficient solutions to the problem, among them are: Agarwal *et al.* [3], Rahmoune *et al.* [26], and Shams *et al.* [29].

Three major formulations (Galerkin method, tau method and collocation method (pseudospectral method)) are popular amongst spectral methods. The spectral collocation method is the most appealing method as it only approximates the solution by imposing the governing differential equation at the collocation points, which are usually Gauss or GaussLobatto nodes related to orthogonal polynomials. Recently, a number of authors have suggested hybrid and advanced spectral models using Legendre, Chebyshev and other orthogonal polynomials as an attempt to enhance stability, computational efficiency and overall accuracy in solving nonlinear and fractional models.

## 2. Prelude

To facilitate the task, we begin with the basics of Legendre polynomials, then proceed to provide an introduction to the properties of shifted Legendre polynomials and finally get the necessary mathematical instruments to further the progress of the present paper. While spectral and polynomial methods are well established, the paper emphasizes the formulation of the collocation scheme using shifted Legendre polynomials together with its implementation for nonlinear problems and comparison with shifted Chebyshev methods.

### 2.1 Properties for Legendre and Shifted Legendre Polynomials

The following recurrence relation characterizes the most prevalent Legendre polynomials,  $L_n(x)$ , orthogonal polynomials of degree  $n$  in  $x$  defined on the interval  $[-1, 1]$ :

$$L_{n+1}(x) = \frac{(2n+1)}{(n+1)}xL_n(x) - \frac{n}{(n+1)}L_{n-1}(x), \quad n = 1, 2, \dots$$

with its initials as

$$L_0(x) = 1, \quad L_1(x) = x.$$

Form of analytical Legendre polynomials is as follows

$$L_i(x) = \sum_{k=0}^i \frac{(-1)^k (i+k)! (1-x)^k}{(i-k)! 2^k (k!)^2}, \quad i = 0, 1, 2, \dots \quad (1)$$

As  $w(x) = 1$  is the weight function, we can take the weighted space  $L_w^2(-1, 1) \equiv L^2(-1, 1)$  as normal with the following inner product and the norm:

$$\langle f(x), g(x) \rangle = \int_{-1}^1 f(x) g(x) w(x) dx, \quad \|f\|^2 = \langle f(x), f(x) \rangle.$$

Therefore, a full  $L^2(-1, 1)$ -orthogonal system is formed by the set of Legendre polynomials,

$$\langle L_n(x), L_m(x) \rangle = \frac{2}{2n+1} \delta_{nm}$$

and where the Kronecker function is denoted by  $\delta_{nm}$ . For  $t \in [0, l]$  the ICs are given by

$$L_0^*(t) = 1, \quad L_1^*(t) = 2t - 1.$$

The orthogonality of the polynomials on the interval  $[0, l]$  is decided with reference to the weighting function  $w_s(t) = 1$  under the following inner product:

$$\langle L_n^*(t), L_m^*(t) \rangle = \int_0^l L_n^*(t) L_m^*(t) w_s(t) dt = \frac{l}{2n+1} \delta_{nm}.$$

The shifted Legendre polynomials  $L_n^*(x)$  of degree  $n$  has explicit analytical form given by

$$L_n^*(t) = \sum_{k=0}^n (-1)^{(n+k)} \frac{\Gamma(n+k+1)}{\Gamma(n-k+1)(\Gamma(k+1))^2 l^k} t^k, \quad n \in \mathbb{Z}^+. \quad (2)$$

It follows that  $L_n^*(0) = (-1)^n$  and  $L_n^*(l) = 1$ .

### 3. Collocation Approach via Shifted Legendre Polynomial

#### 3.1 Overview of Collocation Method for Shifted Legendre Polynomial

The fundamental ones in the description of the various physical phenomena, including chemical reactions and spring-mass systems as well as the bending of beams, are the nonlinear differential equations. They also occur in ecological and economic investigations. The outstanding representatives of the family are Riccati, Emden-Fowler, and Duffing, Van der Pol, Rayleigh, and Yermakov equations. The larger portion of nonlinear differential equations cannot be solved in exact form and the alternative means of approximate and numerical solving must be applied. Among the existing numerical methods that study the nonlinear differential system are the Adomian decomposition method in [32], [2] the linearization method is studied in [27] and presented the sixth degree  $B$ -spline method in [8], decomposition method is also studied in [22] and recently the HPM method was investigated by Abbasbandy [1]. Abbasbandy [1] also considered the application of the quasilinearization procedures to address the nonlinear problems. Akyüz- Daşcıoğlu and Çerdik-Yaslan [4] incorporates the Chebyshev collocation matrix process in the process of coming up with solutions to nonlinear differential equations. The issue narrows down to a matrix equation that is the most powerful system of nonlinear algebraic equations with unknown Chebyshev coefficients of a special form related to the Chebyshev collocation points. The least square procedures were presented by Ibrahim [16]. Recently, the authors explored shifted Chebyshev technique in high order ODEs (Hassan and Ibrahim [14], and Öztürk and Gülsu [25]).

In this paper, the authors describe some modifications and improvements to the application of the shifted Legendre collocation method to solve generalizations of nonlinear differential equations.

$$\sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) y^s(x) y^{(k)}(x) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) y^{(s)}(x) y^{(k)}(x) = f(x), \quad (3)$$

with conditions

$$\sum_{k=0}^{m-1} (a_{ik}y^{(k)}(a) + b_{ik}y^{(k)}(b) + c_{ik}y^{(k)}(c)) = \alpha_i, \quad i = 0, 1, \dots, m - 1, \tag{4}$$

with  $y^{(0)}(x) = y(x)$  and  $y(x) \in C^m[0, L]$  as unknown function.  $Q_{k,s}(x)$ ,  $P_{k,s}(x)$  and  $f(x)$  are given functions on  $[0, M]$ .

We will consider the numerical solution of eq. (4) from eq. (5). Truncated shifted series of Legendre, in form:

$$y_N(x) = \sum_{r=0}^N a_r L_{M,r}^*(x), \quad x \in [0, M], \tag{5}$$

where  $N$  may assume any positive integer such that  $N > m$ . We denote it by notation such as  $L_{M,r}^*(x)$  whilst  $r = 0, 1, \dots, N$  to shifted Legendre polynomials, which can be obtained, also, by the following recurrence relation:

$$L_{M,r+1}^*(x) = 2 \left( \frac{2x}{L} - 1 \right) T_{L,r}^*(x) - T_{L,r-1}^*(x), \quad r = 1, 2, \dots,$$

with  $L_{M,0}^*(x) = 1$ ,  $T_{L,1}^*(x) = \frac{2x}{L} - 1$ . The analytical form of the shifted Legendre polynomials  $L_{M,r}^*(x)$  of degree  $r$  is as follow

$$L_{M,r}^*(x) = r \sum_{p=0}^r (-1)^{r-p} \frac{(r+p-1)! 2^{2p}}{(r-p)!(2p)! M^p} x^p, \tag{6}$$

with  $L_{M,r}^*(0) = (-1)^r$  and  $L_{M,r}^*(M) = 1$ . The orthogonality condition is given as

$$\int_0^L T_{M,j}^*(x) T_{M,i}^*(x) w_M(x) dx = h_k \delta_{ji},$$

where  $w_M(x) = (Mx - x^2)^{-1/2}$  and  $h_i = b_i \pi / 2$ ,  $b_0 = 2$ ,  $b_i = 1$ ,  $k \geq 1$ . By eq. (4), the  $k$ -th derivatives for  $T_{M,r}^*(x)$  is realized

$$(L_{M,r}^*)^{(k)}(x) = L_{M,r}^{*,k}(x) = r \sum_{p=m}^r (-1)^{r-p} p(p-1) \dots (p-k+1) \frac{(r+p-1)! 2^{2p}}{(r-p)!(2p)! M^k} x^{p-k}, \tag{7}$$

where  $p \geq m - 1$ .

### 3.2 Solution Method

Applying eq. (5) leads to the  $ks$ -th derivatives of numerical solution  $y_N(x)$

$$y_N^{(k)}(x) = \sum_{r=0}^N a_r (T_{L,r}^*)^{(k)}(x) = \sum_{r=k}^N a_r T_{L,r}^{*,k}(x). \tag{8}$$

Using eq. (3), eq. (5) and eq. (8), we obtain

$$\begin{aligned} & \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) \left( \sum_{r=0}^N a_r L_{M,r}^*(x) \right)^s \left( \sum_{r=k}^N a_r L_{M,r}^{*,k} \right) \\ & + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) \left( \sum_{r=s}^N a_r L_{M,r}^{*,s} \right) \left( \sum_{r=k}^N a_r L_{M,r}^{*,k} \right) = f(x). \end{aligned} \tag{9}$$

juxtaposing eq. (9) at  $N - m + 1$  points  $x_p$ ,  $p = 0, 1, \dots, N - m$  gives

$$\sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x_q) \left( \sum_{r=0}^N a_r L_{M,r}^*(x_q) \right)^s \left( \sum_{r=k}^N a_r L_{M,r}^{*,k}(x_q) \right) \tag{10}$$

$$+ \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x_q) \left( \sum_{r=s}^N a_r L_{M,r}^{*,s}(x_q) \right) \left( \sum_{r=k}^N a_r L_{M,r}^{*,k}(x_q) \right) = f(x_q), \tag{11}$$

where  $x_p$  are the roots of  $L_{M,m}^*(x)$ . Further, now from eq. (8) and inserting the conditions in eq. (4) we obtain the  $k$  equations as follows:

$$\sum_{k=0}^{m-1} \left( a_{ik} \sum_{r=k}^N a_r L_{M,r}^{*,k}(a) + b_{ik} \sum_{r=0}^N a_r L_{M,r}^{*,k}(b) + c_{ik} \sum_{r=0}^N a_r L_{M,r}^{*,k}(c) \right) = \alpha_i, \quad i = 0, 1, \dots, m-1. \tag{12}$$

Eq. (11) along with the  $ks$ -equations in the conditions of the conditions eq. (12) provide the  $(N + 1)$  non-linear algebraic equations (NAEs). Using the NAEs, we can get the unknown shifted Legendre coefficients  $a_r, r = 0, 1, \dots, N$ . With the obtained value of the  $y_N(x)$  we can then have an approximation of  $y(x)$  as in the eq. (4).

### 3.3 Error Analysis

In section section, we consider how the mentioned method is put together: we suppose that  $y(x)$  is smooth on  $[0, 1]$  and  $I_N(x)$  is the polynomial that interpolates  $y$  at the nodes  $x_i, i = 0, 1, \dots, n$  with  $x_i$ , being the Legendre-Gauss node points. We then have:

$$y(x) - I_N(x) = \frac{y^{(N+1)}(\lambda)}{(N + 1)!} \prod_{i=0}^N (x - x_i), \quad \lambda \in [0, 1]. \tag{13}$$

Also, we obtain

$$|y(x) - I_N(x)| \leq \frac{1}{2^{2N+1}} \|y^{(N+1)}(x)\|_\infty. \tag{14}$$

The least-square norm is given as

$$\|y\|_2 = \left( \int_a^b w(x) |y(x)|^2 dx \right)^{1/2},$$

where  $w(x)$  is a weight function which is non-negative.

**Theorem 3.1.** Assume that the functions in eq. (3) are real  $(N + 1)$ -times continuously differentiable functions on the interval  $[0, 1]$  and

$$y_N(x) = \sum_{r=0}^N a_r L_{M,r}^*(x),$$

are the SCPs expansion of the exact solution.

Let

$$\bar{y}_N(x) = \sum_{r=0}^N \bar{a}_r L_{M,r}^*(x),$$

be the approximate result formed by the LeCM method,  $\exists \Re \alpha$  such that

$$\|y(x) - y_N(x)\|_2 \leq \alpha \frac{1}{2^{2N+1}} \|y^{(N+1)}(x)\|_\infty + \sqrt{\frac{3\pi}{8}} \|A - \bar{A}\|_2, \tag{15}$$

where

$$A = [a_0 \ a_1 \ \dots \ a_N] \quad \text{and} \quad \bar{A} = [\bar{a}_0 \ \bar{a}_1 \ \dots \ \bar{a}_N].$$

*Proof.* Let  $y_N(x)$  be a real-valued polynomials of degree  $\leq N$  and  $y_N(x)$  is the best approximation of  $y(x)$ . We define

$$\|y(x) - y_N(x)\|_2 \leq \|y(x) - \bar{y}_N(x)\|_2 + \|\bar{y}_N(x) - y_N(x)\|_2.$$

From eq. (11), we get

$$\begin{aligned} \|y(x) - y_N(x)\|_2 &= \left( \int_0^1 |y(x) - y_N(x)|^2 dx \right)^{1/2} \\ &\leq \left( \int_0^1 \left[ \frac{1}{2^{N+1}(N+1)!} \|y^{(N+1)}(x)\|_\infty \right]^2 dx \right)^{1/2} \\ &= \sqrt{L} \frac{1}{2^{2N+1}(N+1)!} (\|y^{(N+1)}(x)\|_\infty)^{N+1}, \end{aligned}$$

and we have

$$\begin{aligned} \|y(x) - \bar{y}_N(x)\|_2 &= \left( \int_0^1 \left[ \sum_{r=0}^N (a_r - \bar{a}_r) L_r^*(x) \right]^2 dx \right)^{1/2} \\ &\leq \left( \int_0^1 \left[ \sum_{r=0}^N (a_r - \bar{a}_r)^2 \right] \left[ \sum_{r=0}^N |L_r^*(x)|^2 \right] dx \right)^{1/2} \\ &= \left[ \sum_{r=0}^N (a_r - \bar{a}_r)^2 \right]^{1/2} \left( \sum_{r=0}^N \int_0^1 |L_r^*(x)|^2 dx \right)^{1/2} \\ &= \sqrt{\frac{3\pi}{8}} \|A - \bar{A}\|. \end{aligned}$$

Moreover, the method can easily verified. This is because the truncated Legendre series (2) is an approximation of the solutions of the equation, eq. (1) and in that case when we replace the term,  $y_j^N(x), j = 0, 1, \dots, m$  and replace in the equation, the first derivatives of this term in eq. (1) must be roughly satisfied; that is, for  $x_i \in [0, 1], i = 0, 1, 2, \dots,$

$$\left| \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x_i) y^s(x_i) y^{(k)}(x_i) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x_i) y^{(s)}(x_i) y^{(k)}(x_i) - f(x_i) \right| \cong 0.$$

Thus, the estimated error may be provided by the following function

$$E_N(x) = \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) y^s(x) y^{(k)}(x) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) y^{(s)}(x) y^{(k)}(x) - f(x).$$

#### 4. Description and Interpretation of the Findings

Here, we consider homogeneous and non-homogeneous Painlevé equations in order to prove the efficiency and applicability of the given method.

**Example 4.1.** Take the first order IVP as

$$(1 + \sqrt{2})y'(x) + xy(x) = 0, \quad y(x) = 0. \tag{16}$$

The analytical solution of eq. (16) is provided as

$$y(x) = e^{\frac{x^2}{2} - \frac{x^2}{\sqrt{2}}}. \tag{17}$$

Considering the shifted Legendre polynomial in eqs. (5) and (6), for  $N = 6$ , we get

$$\begin{aligned}
 y_N x = & a_0 + (-1 + 2x)a_1 + (1 - 6x + x^2)a_2 + (-1 + 12x - 30x^2 + 20x^3)a_3 \\
 & + (1 - 20x + 90x^2 - 140x^3 + 70x^4)a_4 + (-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5)a_5 \\
 & + (1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)a_6.
 \end{aligned} \tag{18}$$

On applying the initial conditions in eq. (16) to eq. (18),

$$y(0) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 1. \tag{19}$$

Simplifying eq. (18) gives

$$\begin{aligned}
 R = & xa_0 + 2a_1 + 2\sqrt{2}a_1 - xa_1 + 2x^2a_1 - 6a_2 - 6\sqrt{2}a_2 + 3xa_2 + 2\sqrt{2}xa_2 - 6x^2a_2 \\
 & + x^3a_2 + 12a_3 + 12\sqrt{2}a_3 - 61xa_3 - 60\sqrt{2}xa_3 + 72x^2a_3 + 60\sqrt{2}x^2a_3 - 30x^3a_3 \\
 & + 20x^4a_3 - 20a_4 - 20\sqrt{2}a_4 + 181xa_4 + 180\sqrt{2}xa_4 - 440x^2a_4 - 420\sqrt{2}x^2a_4 \\
 & + 370x^3a_4 + 280\sqrt{2}x^3a_4 - 140x^4a_4 + 70x^5a_4 + 30a_5 + 30\sqrt{2}a_5 - 421xa_5 \\
 & - 420\sqrt{2}xa_5 + 1710x^2a_5 + 1680\sqrt{2}x^2a_5 - 2730x^3a_5 - 2520\sqrt{2}x^3a_5 + 1820x^4a_5 \\
 & + 1260\sqrt{2}x^4a_5 - 630x^5a_5 + 252x^6a_5 - 42a_6 - 42\sqrt{2}a_6 + 841xa_6 + 840\sqrt{2}xa_6 \\
 & - 5082x^2a_6 - 5040\sqrt{2}x^2a_6 + 13020x^3a_6 + 12600\sqrt{2}x^3a_6 - 15540x^4a_6 \\
 & - 13860\sqrt{2}x^4a_6 + 8694x^5a_6 + 5544\sqrt{2}x^5a_6 - 2772x^6a_6 + 924x^7a_6 = 0.
 \end{aligned} \tag{20}$$

The roots of the shifted Legendre polynomial for  $N = 6$  are provided as

$$x = 0.38069; \quad x = 0.61931; \quad x = 0.169395; \quad x = 0.830605; \quad x = 0.0337652; \quad x = 0.966235. \tag{21}$$

Replacing eq. (21) in the eq. (20) will give seven algebraic equations, which are non-linear and consist of seven unknowns

$$\left. \begin{aligned}
 0.38069a_0 + 4.73759a_1 - 13.0808a_2 - 5.05736a_3 + 7.55999a_4 + 2.32548a_5 - 10.2794a_6 &= 0, \\
 0.61931a_0 + 4.97621a_1 - 12.9394a_2 - 5.38133a_3 - 7.38433a_4 + 2.6601a_5 + 10.2794a_6 &= 0, \\
 0.169395a_0 + 4.71642a_1 - 13.6653a_2 + 8.63528a_3 - 0.554506a_4 - 9.97022a_5 + 15.1542a_6 &= 0, \\
 0.830605a_0 + 5.37763a_1 - 13.2105a_2 + 8.36617a_3 + 0.12626a_4 - 10.2646a_5 - 15.1542a_6 &= 0, \\
 0.0337652a_0 + 4.79694a_1 - 14.2953a_2 + 24.2235a_3 - 34.727a_4 + 42.5764a_5 - 45.6672a_6 &= 0, \\
 0.966235a_0 + 5.72941a_1 - 13.5532a_2 + 24.8518a_3 + 35.1491a_4 + 42.7821a_5 + 45.6672a_6 &= 0, \\
 a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 &= 1.
 \end{aligned} \right\} \tag{22}$$

The solution of the nonlinear system of eq. (22) results to

$$\left. \begin{aligned}
 a_0 = 0.64695, \quad a_1 = -0.52758, \quad a_2 = -0.17286, \quad a_3 = 0.0019124, \quad a_4 = 0.000227, \\
 a_5 = -0.000015, \quad a_6 = -1.04488 \times 10^{-6}.
 \end{aligned} \right\} \tag{23}$$

And the numerical solution is as

$$\begin{aligned}
 y_N = & 1 - 1.70654 \times 10^{-6}x - 0.207071x^2 - 0.000232621x^3 + 0.0221142x^4 \\
 & - 0.000910427x^5 - 0.000965469x^6.
 \end{aligned} \tag{24}$$

The following figures show the graph of exact with the numerical solutions and the error for  $N = 6$ .

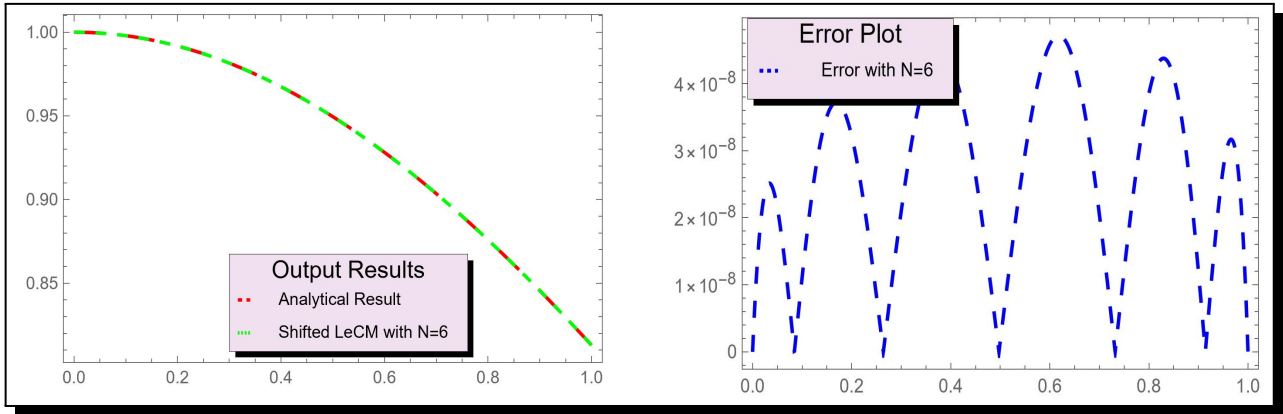


Figure 1. Plot of exact with approximate solutions, and error for  $N = 6$

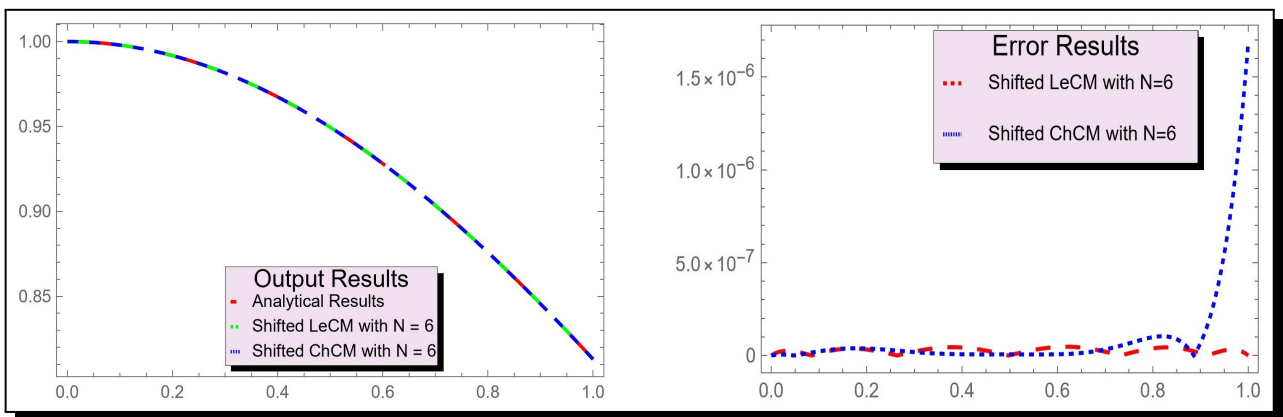


Figure 2. Comparison plot between shifted Chebyshev method and the proposed method for the approximate solutions and the error using  $N = 6$

Table 1. Error for Example 4.1 for  $N = 6$

	Exact solution	Shifted LeCM	Shifted ChCM	Error for Shifted LeCM	Error for Shifted ChCM
0	1	1	1	$1.11011 \times 10^{-16}$	$1.11022 \times 10^{-16}$
0.1	0.9979	0.9979	0.9979	$1.11061 \times 10^{-8}$	$2.238591 \times 10^{-8}$
0.2	0.99175	0.99175	0.99175	$3.2180 \times 10^{-8}$	$3.7459 \times 10^{-8}$
0.3	0.9815	0.9815	0.9815	$2.09994 \times 10^{-8}$	$1.8468 \times 10^{-8}$
0.4	0.96741	0.96741	0.96741	$4.28713 \times 10^{-8}$	$6.49594 \times 10^{-9}$
0.5	0.949541	0.949541	0.949541	$1.605793 \times 10^{-9}$	$5.954895 \times 10^{-9}$
0.5	0.949541	0.949541	0.949541	$1.605793 \times 10^{-9}$	$5.954895 \times 10^{-9}$
0.6	0.92815	0.92815	0.92815	$4.58282 \times 10^{-8}$	$6.55413 \times 10^{-9}$
0.7	0.9035	0.903497	0.903497	$2.04881 \times 10^{-8}$	$3.26184 \times 10^{-8}$
0.8	0.875861	0.875861	0.875861	$3.85121 \times 10^{-8}$	$9.86707 \times 10^{-8}$
0.9	0.84556	0.84556	0.84556	$1.189183 \times 10^{-8}$	$6.74423 \times 10^{-8}$
1	$e^{-\frac{1}{2(1+\sqrt{2})}}$	0.812933	0.812931	$4.26881 \times 10^{-13}$	$1.68531 \times 10^{-6}$

**Example 4.2.** Take the first order IVP as

$$y''(x) + \frac{1}{\sqrt{2}}y'(x) + e^x y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0.9. \quad (25)$$

The analytical solution of eq. (25) is provided as

$$y_N(x) = -3.72463e^{-\frac{x}{\sqrt{2}}} \left( 1. \text{BesselJ} \left[ -\frac{1}{\sqrt{2}}, 2\sqrt{e^x} \right] + 0.259053 \text{BesselJ} \left[ \frac{1}{\sqrt{2}}, 2\sqrt{e^x} \right] \right). \quad (26)$$

Considering the shifted Legendre polynomials in eqs. (5) and (6), for  $N = 3$ , we get

$$Y = a_0 + a_0 + (-1 + 2x)a_1 + (1 - 6x + x^2)a_2 + (-1 + 12x - 30x^2 + 20x^3)a_3. \quad (27)$$

On applying the initial conditions, we find

$$\left. \begin{aligned} y(0) &= a_0 - a_1 + a_2 - a_3 = 1, \\ y'(0) &= 2a_1 - 6a_2 + 12a_3 = 0.9. \end{aligned} \right\} \quad (28)$$

Simplifying eq. (28) gives

$$\begin{aligned} R &= e^x a_0 + \sqrt{2}a_1 - e^x a_1 + 2e^x x a_1 - 3\sqrt{2}a_2 + e^x a_2 + 6\sqrt{2}x a_2 - 6e^x x a_2 + 6e^x x^2 a_2 + 120a_3 \\ &\quad + 6\sqrt{2}a_3 - e^x a_3 - 30\sqrt{2}x a_3 + 12e^x x a_3 + 30\sqrt{2}x^2 a_3 - 30e^x x^2 a_3 + 20e^x x^3 a_3 \\ &= 0. \end{aligned} \quad (29)$$

The roots of the shifted Legendre polynomial for  $N = 3$  is provided as

$$x = \frac{1}{6}(3 - \sqrt{3}), \quad x = \frac{1}{6}(3 + \sqrt{3}). \quad (30)$$

Replacing eq. (30) in the eq. (29) will give four algebraic equations, which are non-linear and consist of four unknowns:

$$\left. \begin{aligned} 1.23531a_0 + 0.701005a_1 - 2.44949a_2 + 121.89a_3 &= 0, \\ 2.20048a_0 + 2.68466a_1 + 2.44949a_2 + 120.567a_3, \\ a_0 - a_1 + a_2 - a_3 &= 1, \\ 2a_1 - 8a_2 + 18a_3 &= 0.9. \end{aligned} \right\} \quad (31)$$

The solution of nonlinear system of eq. (31) results to

$$a_0 = 1.112408, \quad a_1 = -0.0614794, \quad a_2 = -0.200634, \quad a_3 = -0.0150706. \quad (32)$$

And the numerical solution is as

$$y_N = 1 + 0.9x - 0.85169x^2 - 0.08201x^3. \quad (33)$$

By considering Example 4.2 for  $N = 6$  and using the shifted Legendre polynomial in eqs. (3) and (4), for  $N = 6$ , we obtain

$$\begin{aligned} Y &= a_0 + (-1 + 2x)a_1 + (1 - 6x + 6x^2)a_2 + (-1 + 12x - 30x^2 + 20x^3)a_3 \\ &\quad + (1 - 20x + 90x^2 - 140x^3 + 70x^4)a_4 + (-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5)a_5 \\ &\quad + (1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)a_6. \end{aligned} \quad (34)$$

On applying the initial equation, we find

$$\left. \begin{aligned} y(0) &= a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 1, \\ y'(0) &= 2a_1 - 8a_2 + 18a_3 - 32a_4 + 50a_5 - 72a_6 = 0.9. \end{aligned} \right\} \quad (35)$$

Simplifying eq. (34) gives

$$\begin{aligned}
 Y = & e^x a_0 + \sqrt{2} a_1 - e^x a_1 + 2e^x x a_1 - 3\sqrt{2} a_2 + e^x a_2 + 6\sqrt{2} x a_2 - 6e^x x a_2 + 6e^x x^2 a_2 + 120 a_3 \\
 & + 6\sqrt{2} a_3 - e^x a_3 - 30\sqrt{2} x a_3 + 12e^x x a_3 + 30\sqrt{2} x^2 a_3 - 30e^x x^2 a_3 + 20e^x x^3 a_3 - 840 a_4 \\
 & - 10\sqrt{2} a_4 + e^x a_4 + 1680 x a_4 + 90\sqrt{2} x a_4 - 20e^x x a_4 - 210\sqrt{2} x^2 a_4 + 90e^x x^2 a_4 \\
 & + 140\sqrt{2} x^3 a_4 - 140e^x x^3 a_4 + 70e^x x^4 a_4 + 3360 a_5 + 15\sqrt{2} a_5 - e^x a_5 - 15120 x a_5 \\
 & - 210\sqrt{2} x a_5 + 30e^x x a_5 + 15120 x^2 a_5 + 840\sqrt{2} x^2 a_5 - 210e^x x^2 a_5 - 1260\sqrt{2} x^3 a_5 \\
 & + 560e^x x^3 a_5 + 630\sqrt{2} x^4 a_5 - 630e^x x^4 a_5 + 252e^x x^5 a_5 - 10080 a_6 - 21\sqrt{2} a_6 + e^x a_6 \\
 & + 75600 x a_6 + 420\sqrt{2} x a_6 - 42e^x x a_6 - 166320 x^2 a_6 - 2520\sqrt{2} x^2 a_6 + 420e^x x^2 a_6 \\
 & + 110880 x^3 a_6 + 6300\sqrt{2} x^3 a_6 - 1680e^x x^3 a_6 - 6930\sqrt{2} x^4 a_6 + 3150e^x x^4 a_6 \\
 & + 2772\sqrt{2} x^5 a_6 - 2772e^x x^5 a_6 + 924e^x x^6 a_6.
 \end{aligned} \tag{36}$$

The roots of the shifted Legendre polynomial for  $N = 6$  is provided as

$$\begin{aligned}
 x = \frac{1}{2}; \quad x = \frac{1}{6} \left( 3 - \sqrt{\frac{1}{7}(35 - 2\sqrt{70})} \right); \quad x = \frac{1}{6} \left( 3 + \sqrt{\frac{1}{7}(35 - 2\sqrt{70})} \right); \\
 x = \frac{1}{6} \left( 3 - \sqrt{\frac{1}{7}(35 + 2\sqrt{70})} \right); \quad x = \frac{1}{6} \left( 3 + \sqrt{\frac{1}{7}(35 + 2\sqrt{70})} \right).
 \end{aligned} \tag{37}$$

Replacing eq. (37) in eq. (36) will give seven algebraic equations, which are non-linear and consist of seven unknowns:

$$\left. \begin{aligned}
 1.64872a_0 + 1.41421a_1 + 14.3513a_2 - 4.24264a_3 - 62.3513a_4 + 7.07107a_5 + 142.351a_6 &= 0, \\
 1.22889a_0 + 0.691893a_1 + 12.2952a_2 - 53.6381a_3 + 71.7845a_4 + 13.4608a_5 - 186.494a_6 &= 0, \\
 2.21199a_0 + 2.71439a_1 + 18.6415a_2 + 55.9442a_3 + 62.7693a_4 - 30.9415a_5 - 198.029a_6 &= 0, \\
 1.02477a_0 + 0.439596a_1 + 11.4491a_2 - 80.7964a_3 + 266.238a_4 - 621.716a_5 + 1175.22a_6 &= 0, \\
 2.65257a_0 + 3.93696a_1 + 23.526a_2 + 103.968a_3 + 301.561a_4 + 667.481a_5 + 1226.95a_6 &= 0, \\
 a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 &= 1, \\
 2a_1 - 6a_2 + 12a_3 - 20a_4 + 30a_5 - 42a_6 &= 0.9.
 \end{aligned} \right\} \tag{38}$$

Solving eq. (38) leads to

$$\left. \begin{aligned}
 a_0 = 1.10497, \quad a_1 = -0.01569, \quad a_2 = -0.12353, \quad a_3 = -0.00236, \\
 a_4 = 0.00062, \quad a_5 = 0.00012, \quad a_6 = 8.53599 \times 10^{-6}.
 \end{aligned} \right\} \tag{39}$$

And the approximate solution is given as

$$\begin{aligned}
 y_N(x) = & 1 + 0.9x - 0.818148x^2 - 0.124678x^3 - 0.0227134x^4 + 0.0121948x^5 \\
 & + 0.0174817x^6.
 \end{aligned} \tag{40}$$

The following figures show the graph of exact with the numerical solutions and error for  $N = 3$  and that of  $N = 6$ .

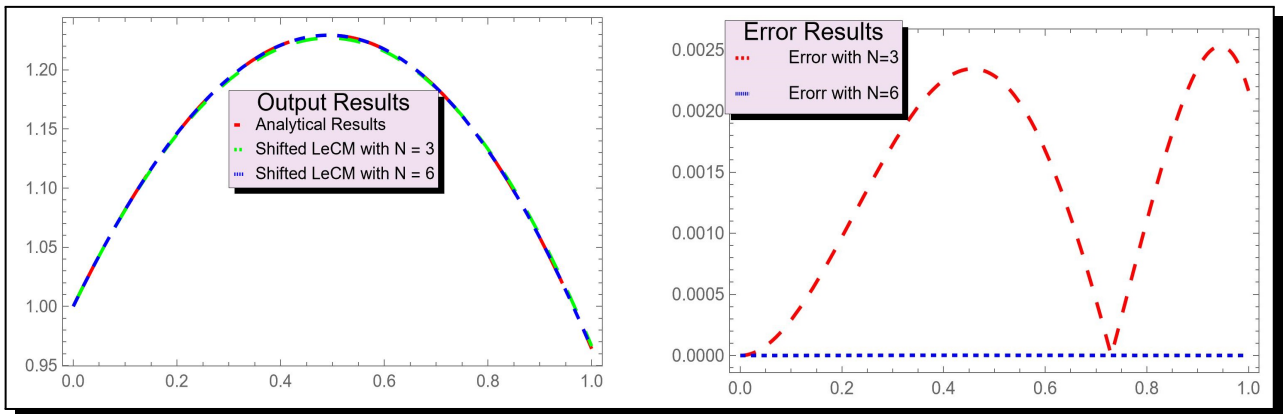


Figure 3. Plot of Example 4.2 of exact with approximate solutions, and error for  $N = 3$  and that of  $N = 6$

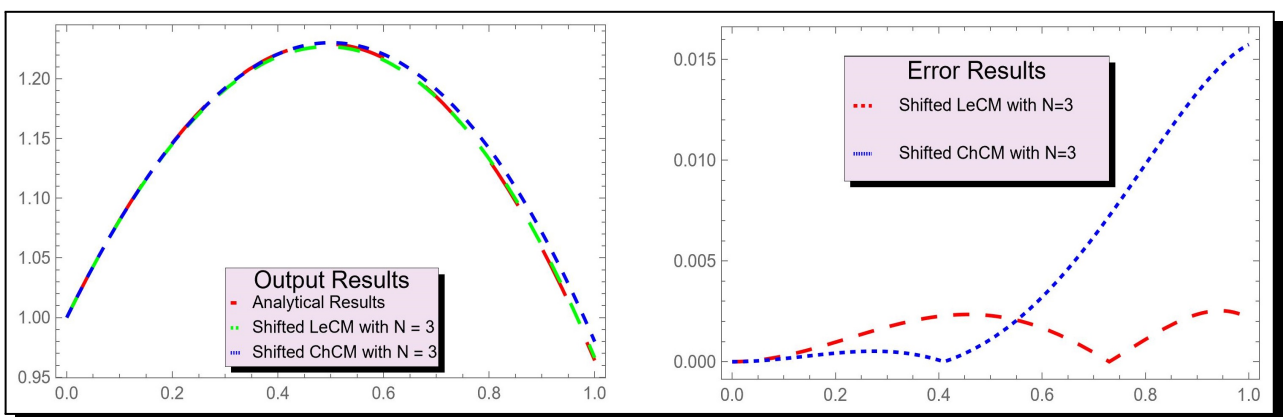


Figure 4. Comparison plot between shifted Chebyshev method and the proposed method for the approximate solutions and the error using  $N = 3$

Table 2. Error of Example 4.2 for  $N = 3$

	Exact solution	Shifted LeCM	Shifted ChCM	Error for Shifted LeCM	Error for Shifted ChCM
0	1.	1.	1.	0.	$1.11022 \times 10^{-16}$
0.1	1.08169	1.0814	1.08154	0.000290642	0.000154872
0.2	1.14625	1.14528	1.14582	0.000969663	0.000426583
0.3	1.19286	1.19113	1.19236	0.00172638	0.000504451
0.4	1.22073	1.21848	1.22065	0.00225222	0.0000798985
0.5	1.22911	1.22683	1.23022	0.00228743	0.00110682
0.5	1.22911	1.22683	1.23022	0.00228743	0.00110682
0.6	1.21736	1.21568	1.22057	0.00167927	0.00320846
0.7	1.185	1.18454	1.1912	0.000452743	0.00619999
0.8	1.13183	1.13293	1.14162	0.00110534	0.00979462
0.9	1.058	1.06035	1.07135	0.0023493	0.0133467
1	0.964138	0.966304	0.979881	0.00216576	0.0157428

The exact solution is

$$y = -3.72463e^{-\frac{x}{2\sqrt{2}}} \left( 1. \text{BesselJ} \left[ -\frac{1}{\sqrt{2}}, 2\sqrt{e^x} \right] + 0.259053 \text{BesselJ} \left[ \frac{1}{\sqrt{2}}, 2\sqrt{e^x} \right] \right).$$

The approximate solution when  $N = 3$  is

$$y_N(x) = 1 + 0.9x - 0.85169x^2 - 0.08201x^3.$$

The approximate solution when  $N = 6$  is

$$y_N(x) = 1 + 0.9x - 0.818148x^2 - 0.124678x^3 - 0.0227134x^4 + 0.0121948x^5 + 0.0174817x^6.$$

**Example 4.3** (Painleve Equation). Take the first order IVP as

$$y'(x) - y(x)^2 = x - 1, \quad y(0) = 0.5. \tag{41}$$

The analytical solution of eq. (41) is provided as

$$y = -\left( (0.5 + 0.866025i)(1. + 0.i) \text{AiryAiPrime}[-(-1)^{\frac{1}{3}}(1-x)] + (0.52228 + 0.0809953i) \text{AiryBiPrime}[-(-1)^{\frac{1}{3}}(1-x)] \right) / \left( (1. + 0.i) \text{AiryAi}[-(-1)^{\frac{1}{3}}(1-x)] + (0.52228 + 0.0809953i) \text{AiryBi}[-(-1)^{\frac{1}{3}}(1-x)] \right). \tag{42}$$

Considering the shifted Legendre polynomials in eqs. (6) and (7), for  $N = 3$ , we get

$$Y = 12a_3(5x^2 - 5x + 1) - (-a_2(6x^2 - 6x + 1) - a_3(20x^3 - 30x^2 + 12x - 1) + a_1(2x - 1) + a_0)^2 + 6a_2(2x - 1) + 2a_1 - x + 1 \tag{43}$$

On applying the initial equation, we find

$$y(0) = a_0 - a_1 + a_2 - a_3 = 0.5. \tag{44}$$

Simplifying eq. (43) gives

$$Y = 1 + 2a_1 - 6a_2 + 12xa_2 + 12a_3 - 60xa_3 + 60x^2a_3x + a_0^2 - 2a_0a_1 + 4xa_0a_1 + a_1^2 - 4xa_1^2 + 4x^2a_1^2 + 2a_0a_2 - 12xa_0a_2 + 12x^2a_0a_2 - 2a_1a_2 + 16xa_1a_2 - 36x^2a_1a_2 + 24x^3a_1a_2 + a_2^2 - 12xa_2^2 + 48x^2a_2^2 - 72x^3a_2^2 + 36x^4a_2^2 - 2a_0a_3 + 24xa_0a_3 - 60x^2a_0a_3 + 40x^3a_0a_3 + 2a_1a_3 - 28xa_1a_3 + 108x^2a_1a_3 - 160x^3a_1a_3 + 80x^4a_1a_3 - 2a_2a_3 + 36xa_2a_3 - 216x^2a_2a_3 + 544x^3a_2a_3 - 600x^4a_2a_3 + 240x^5a_2a_3 + a_3^2 - 24xa_3^2 + 204x^2a_3^2 - 760x^3a_3^2 + 1380x^4a_3^2 - 1200x^5a_3^2 + 400x^6a_3^2. \tag{45}$$

The roots of the shifted Legendre polynomial for  $N = 3$  are given as

$$x = \frac{1}{2}; \quad x = \frac{1}{10}(5 - \sqrt{15}); \quad x = \frac{1}{10}(5 + \sqrt{15}). \tag{46}$$

Replacing eq. (46) in eq. (45) will give four algebraic equations, which are non-linear and consist of four unknowns:

$$\begin{aligned} 1 + 2a_1 - 3a_3 &= 0.5 + a_0^2 - 1. a_0a_2 + 0.25a_2^2 \\ 1 + 2a_1 - 4.64758a_2 + 6a_3 &= 0.112702 + a_0^2 - 1.54919a_0a_1 + 0.6a_1^2 + 0.8a_0a_2 - \\ &0.619677a_1a_2 + 0.16a_2^2 - 4.441 \times 10^{-16}a_0a_3 + 2.220 \times 10^{-16}a_1a_3 + 4.441 \times 10^{-16}a_3^2 \\ 1 + 2a_1 + 4.64758a_2 + 6a_3 &= 0.887298 + a_0^2 + 1.54919a_0a_1 + \\ &0.6a_1^2 + 0.8a_0a_2 + 0.6197a_1a_2 + 0.16a_2^2 + 3.553 \times 10^{-15}a_0a_3 - 1.137 \times 10^{-13}a_3^2 \\ a_0 - a_1 + a_2 - a_3 &= 0.5. \end{aligned} \tag{47}$$

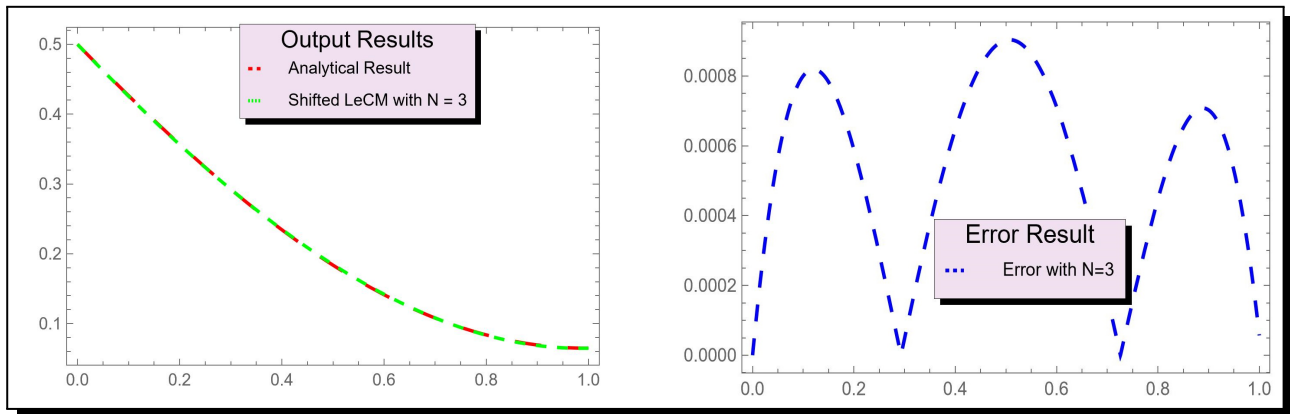
Solving eq. (47) leads to

$$a_0 = 217161, \quad a_1 = -0.22378, \quad a_2 = 0.046519, \quad a_3 = 0.00613. \quad (48)$$

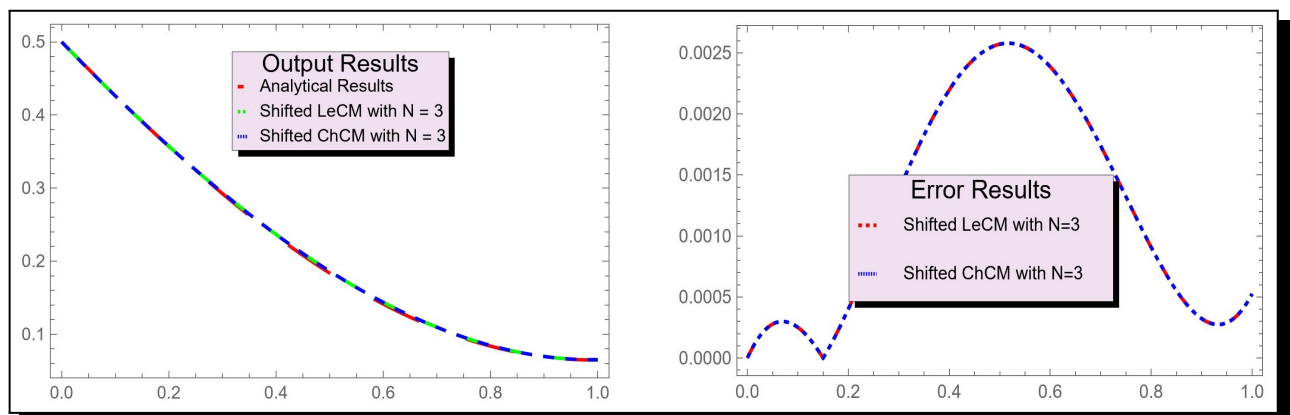
The numerical solution is as

$$Y_N(x) = 0.5 - 0.765166x + 0.207321x^2 + 0.122545x^3. \quad (49)$$

The figures below depict the graph of exact with approximate solutions and the error for  $N = 3$ .



**Figure 5.** Plot of the approximate solutions, exact solution, and error for  $N = 3$



**Figure 6.** Comparison plot between shifted Chebyshev method and the proposed method for the approximate solutions and error using  $N = 3$

The effectiveness of the presented shifted Legendre collocation method is proven by the resulting data being compared with the data obtained using the proposed technique called the shifted Chebyshev collocation technique, see Hassan and Ibrahim [14]. The numerical data show that the errors of the proposed method are by far smaller than those of the shifted Chebyshev collocation method, which are shown in both Figure 2 and Table 1. This comparison indicates that the proposed method offers more reliable approximations of the addressed problems of ordinary differential equations, which validates the efficiency and effectiveness of the developed method. The same comparison is made with regard to Example 4.2, as shown in Figure 4 and Table 2. The solutions that are obtained through the proposed method are compared to the solutions obtained through the shifted Chebyshev collocation method of the same problem. It

**Table 3.** Error of Example 4.3 for  $N = 3$ 

	Exact solution	Shifted LeCM	Shifted ChCM	Error for Shifted LeCM	Error for Shifted ChCM
0	$0.5 + 2.78 \times 10^{-17}$	0.5	0.5	$2.78 \times 10^{-17}$	$2.78 \times 10^{-17}$
0.1	$0.43 + 5.55 \times 10^{-17}$	0.4257	0.42623	0.00080	0.00025
0.2	$0.36 + 8.33 \times 10^{-17}$	0.3562	0.35723	0.00059	0.00040
0.3	$0.29 + 1.39 \times 10^{-16}$	0.2924	0.293744	0.00005	0.00137
0.4	$0.23 + 1.67 \times 10^{-16}$	0.2350	0.2365	0.00065	0.00220
0.5	$0.18 + 1.80 \times 10^{-16}$	0.1846	0.18623	0.00090	0.00257
0.5	$0.18 + 1.80 \times 10^{-16}$	0.1846	0.18623	0.00090	0.00257
0.6	$0.14 + 1.67 \times 10^{-16}$	0.1420	0.14368	0.00071	0.00239
0.7	$0.11 + 2.50 \times 10^{-16}$	0.1080	0.10958	0.00017	0.00174
0.8	$0.08 + 1.94 \times 10^{-16}$	0.08330	0.08465	0.00045	0.00091
0.9	$0.07 + 2.43 \times 10^{-16}$	0.0686	0.06964	0.00070	0.00032
1	$0.07 + 2.60 \times 10^{-16}$	0.06470	0.06528	0.00006	0.00053

is evident that the suggested method has lower numerical errors as the results demonstrate, which contributes to the accuracy, validity, and effectiveness of the suggested method.

## 5. Concluding remarks

The Legendre technique is a proper technique for solving nonlinear ordinary differential equations. It is efficient and accurate because of its capability to solve complex problems in the simplified manner through Legendre polynomials. It uses less computations than the other approaches and therefore is applicable in finding solutions of complex problems in most of the various fields of science and engineering. As demonstrated in this paper, the Shifted Legendre method may be used to provide useful and accurate solutions. This makes it the ideal approach to learners and researchers who want an effective and clear approach of solving nonlinear ordinary differential equations. The proposed approach has a restricted scope of linear and nonlinear ODEs, it will be open to future researchers to apply this concept to the PDEs, system of ODEs, fractional differential equations.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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