Communications in Mathematics and Applications
Vol. 17, No. 1, pp. 143–162, 2026
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com
DOI: 10.26713/cma.v17i1.3541

Research Article

Picture Fuzzy Operators with Application in Pattern Recognition

Mohammad Kamrul Hasan

Department of Mathematics and Statistics, Bangladesh University of Business and Technology, Dhaka, Bangladesh
krul.habi@yahoo.com

Received: January 9, 2026 **Revised:** February 26, 2026 **Accepted:** March 3, 2026

Abstract. In our daily lives, we face many uncertain situations where vague or imprecise data are the robust issue for consideration. Handling such imprecise data is more difficult than precise data. Picture fuzzy sets are powerful concepts to deal with these kinds of imprecise data effectively because of its diverse operations techniques in several dimensions. In this article, different sorts of operators for picture fuzzy sets are defined and various related properties of these operators are explored. Some mean operators in particular cases are discussed with some of their associated properties. Finally, a real life application is described in decision-making.

Keywords. Picture fuzzy set, Difference operator, Possibility operator, Necessary operator, Closure operator, Interior operator, Mean operator

Mathematics Subject Classification (2020). 03E72

1. Introduction

Modern innovations are going through research with data and these tasks become more difficult when dealing with vague data. In 1965, L. A. Zadeh [33] has achieved a great success for introducing the concept of fuzzy set to handle the vague data. Since then, many theories have been developed for treating imprecision and uncertainty. In fuzzy set, every element associated with a membership degree and the complement of the membership degree reflects the non-membership degree of that element. However, much research in application fields has found that the non-membership degree of an element is not always the direct complement of the membership degree because there arises some hesitation. In 1986, Atanassov [1] developed the concept of intuitionistic fuzzy set which is a generalization of the fuzzy set and the non-membership degree is less than or equal to the complement of the membership degree. Cuong [3], and Cuong and Kreinovich [4] introduced the concept of picture fuzzy set which is a direct extension of fuzzy set and intuitionistic fuzzy set by incorporating the idea of positive, negative and neutral membership degree of an element. Dutta and Ganju [7] introduced (α,δ,β)-cut and strong (α,δ,β)-cut, level set for picture fuzzy set and discussed some properties of them. Some algebraic operations such as algebraic sum, algebraic product, scalar multiplication, exponential operation for picture fuzzy sets are discussed by Silambarasan [22]. He also described arithmetic mean operation for picture fuzzy sets. The averaging operators on picture fuzzy sets along their applications are also becoming a great attention by numerous researchers. Khan *et al*. [14] investigated the information aggregation operators' method under the picture fuzzy environment with the help of Einstein norms operations and applied a group decision making problem. Wei [32] discussed the multiple attribute decision making problem based on the arithmetic, geometric aggregation operators and Hamacher operations of picture fuzzy. Luo and Long [15] studied picture fuzzy geometric aggregation operators based on a trapezoidal fuzzy number and applied it to multi-attribute decision making and pattern recognition. Hasan *et al*. [10] discussed some picture fuzzy mean operators of picture fuzzy sets and applied in decision-making problem. Picture fuzzy aggregation operators are also discussed some researchers, e.g., Garg [9], Jana *et al*. [11], Qiyas *et al*. [20], Tian *et al*. [28], Wang *et al*. [30], and Wei [31]. Besides these some other operations of picture fuzzy sets are also depicted by Chau *et al*. [2], Dinh and Thao [5], Dogra and Pal [6], Ganie *et al*. [8], Kadian and Kumar [12], Khan *et al*. [13], Luo and Zhang [16], Meksavang *et al*. [17], Nguyen *et al*. [18], Peng and Dai [19], Si *et al*. [21], Singh *et al*. [23], Singh and Ganie [24], Son [25], Son *et al*. [26], Thao [27], and Wang [29].

In this article, different types of operators such as difference operator, necessity operator, possibility operator, closure operator, interior operator for picture fuzzy set are defined and various related properties of these operators are explored. Finally, some mean operators in particular cases are discussed with a real life application.

The article is organized as follows: In Section 2, some basic definitions and operations are given essential to the rest of the paper. In Section 3, different kinds of operators for picture fuzzy set are discussed. In Section 4, the mean operators for picture fuzzy sets in some particular cases are discussed. In Section 5, a multi-attribute decision making problem is discussed. In Section 6, the conclusion remark is given.

2. Preliminaries

Definition 2.1 ([33]). A FS A in U is given by $A = \{(\theta,\xi_A(\theta)) : \theta \in U\}$, with $\xi_A : U \to [0,1]$ and $0 \le \xi_A(\theta) \le 1$, $\forall\ \alpha \in U$.

Definition 2.2 ([1]). An IFS A in U is defined by $A = \{(\theta,\xi_A(\theta),\zeta_A(\theta)) : \theta \in U\}$, with $\xi_A : U \to [0,1]$, $\zeta_A : U \to [0,1]$ and $0 \le \xi_A(\theta) + \zeta(\theta) \le 1$, $\forall\ \theta \in U$ and the hesitant value, $\pi_A(\theta) = 1-(\xi_A(\theta)+\zeta_A(\theta))$.

Definition 2.3 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *necessity operator of* A is given by $\Box A = \{(\theta, \xi_A(\theta), 1-\xi_A(\theta)) : \theta \in U\}$.

Definition 2.4 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *possibility operator of* A is given by $\Diamond A = \{(\theta, 1-\zeta_A(\theta), \zeta_A(\theta)) : \theta \in U\}$.

Definition 2.5 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *closure operator of* A is given by

$$C(A) = \{(\theta, P, Q) : \theta \in U\},$$

where $P = \max_{u \in U} \xi_A(u)$ and $Q = \min_{u \in U} \zeta_A(u)$.

Definition 2.6 ([1]). Let $A = \{(\theta, \xi_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be an IFS on U, then the *interior operator of* A is given by

$$I(A) = \{(\theta, p, q) : \theta \in U\},$$

where $p = \min_{u \in U} \xi_A(u)$ and $q = \max_{u \in U} \zeta_A(u)$.

Definition 2.7 ([4, 11]). A PFS A on U is given by $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ with $\xi_A(\theta), \tau_A(\theta), \zeta_A(\theta) \in [0,1]$ and $0 \le \xi_A(\theta) + \tau_A(\theta) + \zeta_A(\theta) \le 1$, $\forall\ \theta \in U$ and the refusal value, $\pi_A(\theta) = 1 - (\xi_A(\theta) + \tau_A(\theta) + \zeta_A(\theta))$, $\forall\ \theta \in U$.

In the whole paper, PFS(U) represents the collection of all PFSs in U.

Definition 2.8 ([3, 4]). Let $X, Y \in$ PFS(U), then

(i) $X \subseteq Y$ iff $\forall\ \theta \in U$, $\xi_X(\theta) \le \xi_Y(\theta)$, $\tau_X(\theta) \le \tau_Y(\theta)$ and $\zeta_X(\theta) \ge \zeta_Y(\theta)$;

(ii) $X = Y$ iff $\forall\ \theta \in U$, $\xi_X(\theta) = \xi_Y(\theta)$, $\tau_X(\theta) = \tau_Y(\theta)$ and $\zeta_X(\theta) = \zeta_Y(\theta)$;

(iii) $X \cup Y = \{(\theta, \max(\xi_X(\theta), \xi_Y(\theta)), \min(\tau_X(\theta), \tau_Y(\theta)), \min(\zeta_X(\theta), \zeta_Y(\theta))) : \theta \in U\}$;

(iv) $X \cap Y = \{(\theta, \min(\xi_X(\theta), \xi_Y(\theta)), \min(\tau_X(\theta), \tau_Y(\theta)), \max(\zeta_X(\theta), \zeta_Y(\theta))) : \theta \in U\}$;

(v) $X^c = \{(\theta, \zeta_X(\theta), \tau_X(\theta), \xi_X(\theta)) : \theta \in U\}$.

Definition 2.9 ([22]). Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U, then the *arithmetic mean operator of* A *and* B is denoted by $A@B$ and defined as

$$A@B = \left\{\left(\theta, \frac{\xi_A(\theta)+\xi_B(\theta)}{2}, \frac{\tau_A(\theta)+\tau_B(\theta)}{2}, \frac{\zeta_A(\theta)+\zeta_B(\theta)}{2}\right) : \theta \in U\right\}.$$

Definition 2.10 ([10]). Let $X = (\xi_A, \tau_A, \zeta_A)$ be a picture fuzzy value. Then, the score and accuracy functions are respectively given by $S(X) = \xi_A + \tau_A - \zeta_A$ and $H(X) = \xi_A + \tau_A + \zeta_A$, where $S(X) \in [-1,1]$ and $H(X) \in [0,1]$.

Definition 2.11 ([10]). Let $X = (\xi_A, \tau_A, \zeta_A)$ and $Y = (\xi_B, \tau_B, \zeta_B)$.

(i) If $S(X) > S(Y)$, then X is greater than Y, denoted by $X \succ Y$.

(ii) If $S(X) = S(Y)$, then

(a) $H(X) = H(Y)$, indicates that $X = Y$.

(b) $H(X) > H(Y)$, indicates that X is greater than Y, denoted by $X \succ Y$.

3. Operators for Picture Fuzzy Sets

Definition 3.1. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U, then the *difference between A and B* is denoted by $A \ominus B$, and defined as

$$A \ominus B = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

Obviously, for $A, B \in \text{PFS}(U)$, $A \ominus B \in \text{PFS}(U)$.

Theorem 3.2. *Let $A, B \in \text{PFS}(U)$, then*

(i) $A \ominus B = A \cap B^c$,

(ii) $A \ominus B = B \ominus A$ *if* $A = B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \ominus B = \{(a, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

But

$$B^c = \{(\theta, \zeta_B(\theta), \tau_B(\theta), \xi_B(\theta)) : \theta \in U\}$$

$$\implies \quad A \cap B^c = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}$$
$$= A \ominus B.$$

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \ominus B = \{(\theta, \min\{\xi_A(\theta), \zeta_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\zeta_A(\theta), \xi_B(\theta)\}) : \theta \in U\}.$$

If $A = B$, then $\xi_A(\theta) = \xi_B(\theta)$, $\tau_A(\theta) = \tau_B(\theta)$ and $\zeta_A(\theta) = \zeta_B(\theta)$, $\forall\ \theta \in U$. □

From this, it is certain that $B \ominus A = A \ominus B$ and the result follows.

Theorem 3.3. *Let $A, B \in \text{PFS}(U)$, then*

(i) $A \ominus A^c = A$,

(ii) $A^c \ominus B^c = B \ominus A$,

(iii) $B^c \ominus A^c = A \ominus B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$A^c = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$A \ominus A^c = \{(\theta, \min\{\xi_A(\theta), \xi_A(\theta)\}, \min\{\tau_A(\theta), \tau_A(\theta)\}, \max\{\zeta_A(\theta), \zeta_A(\theta)\}) : \theta \in U\}$$
$$= \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = A.$$

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A^c = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : a \in U\}$$

and

$$B^c = \{(\theta, \zeta_B(\theta), \tau_B(\theta), \xi_B(\theta)) : \theta \in U\}.$$

Now,

$$\begin{aligned} A^c \ominus B^c &= \{(\theta, \min\{\zeta_A(\theta), \xi_B(\theta)\}, \min\{\tau_A(\theta), \tau_B(\theta)\}, \max\{\xi_A(\theta), \zeta_B(\theta)\}) : \theta \in U\} \\ &= \{(\theta, \min\{\xi_B(\theta), \zeta_A(\theta)\}, \min\{\tau_B(\theta), \tau_A(\theta)\}, \max\{\zeta_B(\theta), \xi_A(\theta)\}) : \theta \in U\} \\ &= B \ominus A. \end{aligned}$$

(iii): Straightforward. □

Definition 3.4. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U, then the *necessity operator of* A is denoted by $\Box A$ and defined as

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Definition 3.5. Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U, then the *possibility operator of* A is denoted by $\Diamond A$ and defined as

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Theorem 3.6. *For every PFS A, we have*

(i) $\overline{\Box \overline{A}} = \Diamond A$,

(ii) $\overline{\Diamond \overline{A}} = \Box A$,

(iii) $\Box A \subset A \subset \Diamond A$,

(iv) $\Box\Box A = \Box A$,

(v) $\Box\Diamond A = \Diamond A$,

(vi) $\Diamond\Box A = \Box A$,

(vii) $\Diamond\Diamond A = \Diamond A$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$\Box\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \zeta_A(\theta)) : \theta \in U\}.$$

Therefore,

$$\overline{\Box\overline{A}} = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Again,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}.$$

Thus, $\overline{\Box\overline{A}} = \Diamond A$.

(ii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}.$$

Now,

$$\Diamond\overline{A} = \{(\theta, 1 - \tau_A(\theta) - \xi_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}$$

and so

$$\overline{\Box\overline{A}} = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Again,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}.$$

Thus, $\overline{\Diamond \overline{A}} = \Box A$.

(iii) Trivial.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\Box\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} = \Box A.$$

(v) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$$

and

$$\begin{aligned}\Box\Diamond A &= \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - 1 + \tau_A(\theta) + \zeta_A(\theta)) : \theta \in U\} \\ &= \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = \Diamond A.\end{aligned}$$

(vi) Let $A = \{(\theta, \xi\varphi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\begin{aligned}\Diamond\Box A &= \{(\theta, 1 - \tau_A(\theta) - 1 + \tau_A(\theta) + \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} \\ &= \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\} = \Box A.\end{aligned}$$

(vii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$$

and

$$\Diamond\Diamond A = \{(\theta, 1 - \tau_A(\theta) - \zeta_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\} = \Diamond A. \qquad \Box$$

Theorem 3.7. *For every two PFSs A and B,*

(i) $\Box(A \cap B) = \Box A \cap \Box B$,

(ii) $\Box(A \cup B) = \Box A \cup \Box B$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then,

$$A \cap B = \{\theta, \min(\xi_A(\theta), \xi_B(\theta)), \min(\tau_A(\theta), \tau_B(\theta)), \max(\zeta_A(\theta), \zeta_B(\theta)) : \theta \in U\}.$$

Now,

$$\Box A = \{(\theta, \xi_A(\theta), \tau_A(\theta), 1 - \tau_A(\theta) - \xi_A(\theta)) : \theta \in U\}$$

and

$$\Box B = \{(\theta, \xi_B(\theta), \tau_B(\theta), 1 - \tau_B(\theta) - \xi_B(\theta)) : \theta \in Ut\}.$$

Therefore,

$$\Box A \cap \Box B = \{\theta, \min(\xi_A(\theta), \xi_B(\theta)), \min(\tau_A(\theta), \tau_B(\theta)), \max((1 - \tau_A(\theta) - \xi_A(\theta)), (1 - \tau_B(\theta) - \xi_B(\theta))) : \theta \in U\}.$$

Again,

$$\Box(A\cap B)=\{\theta,\min(\xi_A(\theta),\xi_B(\theta)),\min(\tau_A(\theta),\tau_B(\theta)),1-\min(\tau_A(\theta),\tau_B(\theta))-\min(\xi_A(\theta),\xi_B(\theta)):\theta\in U\}$$
$$=\{\theta,\min(\xi_A(\theta),\xi_B(\theta)),\min(\tau_A(\theta),\tau_B(\theta)),\max((1-\tau_A(\theta)-\xi_A(\theta)),(1-\tau_B(\theta)-\xi_B(\theta))):\theta\in U\}$$
$$=\Box A\cap\Box B.$$

(ii) Trivial. □

Theorem 3.8. *For every two PFSs A and B,*

(i) $\Box(A@B)=\Box A@\Box B$,

(ii) $\Diamond(A@B)=\Diamond A@\Diamond B$.

Proof. (i) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ and $B=\{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)):\theta\in U\}$ be two PFSs on U, then

$$A@B=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{\zeta_A(\theta)+\zeta_B(\theta)}{2}\right):\theta\in U\right\}.$$

Now,

$$\Box A=\{(\theta,\xi_A(\theta),\tau_A(\theta),1-\tau_A(\theta)-\xi_A(\theta)):\theta\in U\}$$

and

$$\Box B=\{(\theta,\xi_B(\theta),\tau_B(\theta),1-\tau_B(\theta)-\xi_B(\theta)):\theta\in U\}.$$

Therefore,

$$\Box A@\Box B=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{(1-\tau_A(\theta)-\xi_A(\theta))+(1-\tau_B(\theta)-\xi_B(\theta))}{2}\right):\theta\in U\right\}$$
$$=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{2+(-\tau_A(\theta)-\xi_A(\theta))+(-\tau_B(\theta)-\xi_B(\theta))}{2}\right):\theta\in U\right\}$$
$$=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},\frac{2-(\xi_A(\theta)+\xi_B(\theta))-(\tau_A(\theta)+\tau_B(\theta))}{2}\right):\theta\in U\right\}$$
$$=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},1-\frac{\xi_A(\theta)+\xi_B(\theta)}{2}-\frac{\tau_A(\theta)+\tau_B(\theta)}{2}\right):\theta\in U\right\}.$$

Again,

$$(A@B)=\left\{\left(\theta,\frac{\xi_A(\theta)+\xi_B(\theta)}{2},\frac{\tau_A(\theta)+\tau_B(\theta)}{2},1-\frac{\xi_A(\theta)+\xi_B(\theta)}{2}-\frac{\tau_A(\theta)+\tau_B(\theta)}{2}\right):\theta\in U\right\}=A@B.$$

(ii) Straightforward. □

Definition 3.9. Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U, then the *closure operator of A* is denoted by $C(A)$ and defined as

$$C(A)=\{(\theta,P,Q,R):\theta\in U\},$$

where

$$P=\max_{u\in U}\xi_A(u),\ Q=\min_{u\in U}\tau_A(u)\ \text{ and }\ R=\min_{u\in U}\zeta_A(u).$$

Definition 3.10. Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U, then the *interior operator of A* is denoted by $I(A)$ and defined as

$$I(A)=\{(\theta,p,q,r):\theta\in U\},$$

where

$$p=\min_{u\in U}\xi_A(u),\ q=\min_{u\in U}\tau_A(u)\ \text{ and }\ r=\max_{u\in U}\zeta_A(u).$$

Clearly, $I(A)\subset A$ and $I(A)\subset C(A)$.

Theorem 3.11. *For every PFSs A and B,*

(i) $C(A)$ *is a PFS,*

(ii) $I(A)$ *is a PFS,*

(iii) $I(A)\subset A$,

(iv) $I(A)\subset C(A)$,

(v) $C(C(A))=C(A)$,

(vi) $I(I(A))=I(A)$,

(vii) $\overline{I(\overline{A})}=C(A)$.

Proof. Prats (i) and (ii) are trivial.

(iii) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U. Then,

$$I(A)=\{(\theta,\min_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\max_{u\in U}\zeta_A(u)):\theta\in U\}.$$

Clearly,

$$\min_{u\in U}\xi_A(u)\le\xi_A(\theta),\ \min_{u\in U}\tau_A(u)\le\tau_A(\theta)\ \text{ and }\ \max_{u\in U}\zeta_A(u)\ge\zeta_A(\theta).$$

Therefore, $I(A)\subset A$.

(iv) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U. Then,

$$I(A)=\left\{\left(\theta,\min_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\max_{u\in U}\zeta_A(u)\right):\theta\in U\right\}$$

and

$$C(A)=\left\{\left(\theta,\max_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\min_{u\in U}\zeta_A(u)\right):\theta\in U\right\}.$$

Clearly,

$$\min_{u\in U}\xi_A(u)\le\max_{u\in U}\xi_A(u),\ \min_{u\in U}\tau_A(u)\le\min_{u\in U}\tau_A(u)\ \text{ and }\ \max_{u\in U}\zeta_A(u)\ge\min_{u\in U}\zeta_A(u).$$

Therefore, $I(A)\subset C(A)$.

(v) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U. Then,

$$C(A)=\left\{\left(\theta,\max_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\min_{u\in U}\zeta_A(u)\right):\theta\in U\right\},$$

$$C(C(A))=\left\{\left(\theta,\max_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\min_{u\in U}\zeta_A(u)\right):\theta\in U\right\}=C(A).$$

(vi) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ be a PFS on U. Then,

$$I(A)=\left\{\left(\theta,\min_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\max_{u\in U}\zeta_A(u)\right):\theta\in U\right\}$$

and

$$I(I(A))=\left\{\left(\theta,\min_{u\in U}\xi_A(u),\min_{u\in U}\tau_A(u),\max_{u\in U}\zeta_A(u)\right):\theta\in U\right\}=I(A).$$

(vii) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\overline{A} = \{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\},$$

$$\begin{aligned} \overline{I(\overline{A})} &= \overline{I\{(\theta, \zeta_A(\theta), \tau_A(\theta), \xi_A(\theta)) : \theta \in U\}} \\ &= \overline{\left\{\left(\theta, \min_{u\in U} \zeta_A(u), \min_{u\in U} \tau_A(u), \max_{u\in U} \xi_A(u)\right) : \theta \in U\right\}} \\ &= \left\{\left(\theta, \max_{u\in U} \xi_A(u), \min_{u\in U} \tau_A(u), \min_{u\in U} \zeta_A(u)\right) : \theta \in U\right\} \\ &= C(A). \end{aligned}$$

□

4. Mean Operators for Picture Fuzzy Sets

Definition 4.1 (Picture Fuzzy Geometric Mean (PFGM)). Let $A, B \in \text{PFS}(U)$, then the *geometric mean between A and B* is denoted by $A\$B$, and defined as

$$A\$B = \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}, \sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}, \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}) : \theta \in U\}.$$

Proposition 4.2. *For any two PFSs A and B, A$B is also PFS.*

Proof. For $A\$B$,

$$\begin{aligned} 0 &\le \sqrt{\xi_A(\theta)\cdot\xi_B(\theta)} + \sqrt{\tau_A(\theta)\cdot\tau_B(\theta)} + \sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)} \\ &\le \frac{\xi_A(\theta)+\xi_B(\theta)}{2} + \frac{\tau_A(\theta)+\tau_B(\theta)}{2} + \frac{\zeta_A(\theta)+\zeta_B(\theta)}{2} \\ &\le \frac{\xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta)}{2} + \frac{\xi_B(\theta)+\tau_B(\theta)+\zeta_B(\theta)}{2} \\ &\le \frac{1}{2} + \frac{1}{2} \\ &\le 1 \quad (\text{as } 0 \le \xi_A(\theta)+\tau_A(\theta)+\zeta_A(\theta) \le 1). \end{aligned}$$

□

Theorem 4.3. *For every $A, B, C \in \text{PFS}(U)$, the following hold:*

(i) $A\$A = A$,

(ii) $A\$B = B\$A$,

(iii) $A\$(B \cup C) = (A\$B) \cup (A\$C)$,

(iv) $A\$(B \cap C) = (A\$B) \cap (A\$C)$,

(v) $(A\$B)\$C = (A\$C)\$(B\$C)$,

(vi) $(A^c\$B^c)^c = A\$B$,

(vii) $(A\$B)^c = A^c\$B^c$.

Proof. (i) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ be a PFS on U. Then,

$$\begin{aligned} A\$A &= \{(\theta, \sqrt{\xi_A(\theta)\cdot\xi_A(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_A(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_A(\theta)}) : \theta \in U\} \\ &= \{(\theta, \sqrt{(\xi_A(\theta))^2}\sqrt{(\tau_A(\theta))^2}\sqrt{(\zeta_A(\theta))^2}) : \theta \in U\} \\ &= \{(\theta, \xi_A(\theta)\tau_A(\theta)\zeta_A(\theta)) : \theta \in U\} = A. \end{aligned}$$

(ii) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ and $B=\{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)):\theta\in U\}$ be two PFSs on U. Then,

$$A\$B=\{(\theta,\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}):\theta\in U\},$$
$$=\{(\theta,\sqrt{\xi_B(\theta)\cdot\xi_A(\theta)}\sqrt{\tau_B(\theta)\cdot\tau_A(\theta)}\sqrt{\zeta_B(\theta)\cdot\zeta_A(\theta)}):\theta\in U\}=B\$A.$$

(iii) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$, $B=\{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)):\theta\in U\}$ and $C=\{(\theta,\xi_C(\theta),\xi_C(\theta),\zeta_C(\theta)):\theta\in U\}$ be three PFSs on U. Then,

$$B\cup C=\{(\theta,\max\{\xi_B(\theta),\xi_C(\theta)\},\min\{\tau_B(\theta),\tau_C(\theta)\},\min\{\zeta_B(\theta),\zeta_C(\theta)\}):\theta\in U\}.$$

Now,

$$\begin{aligned}A\$(B\cup C)&=\{(\theta,\sqrt{\xi_A(\theta)\cdot\max\{\xi_B(\theta),\xi_C(\theta)\}}\sqrt{\tau_A(\theta)\cdot\min\{\tau_B(\theta),\tau_C(\theta)\}}\\&\quad\cdot\sqrt{\zeta_A(\theta)\cdot\min\{\zeta_B(\theta),\zeta_C(\theta)\}}):\theta\in U\}\\&=\{(\theta,\sqrt{\max\{\xi_A(\theta)\cdot\xi_B(\theta)\xi_A(\theta)\cdot\xi_C(\theta)\}}\sqrt{\min\{\tau_A(\theta)\cdot\tau_B(\theta),\tau_A(\theta)\cdot\tau_C(\theta)\}}\\&\quad\cdot\sqrt{\min\{\zeta_A(\theta)\cdot\zeta_B(\theta)\zeta_A(\theta)\cdot\zeta_C(\theta)\}}):\theta\in U\}\\&=\{(\theta,\max\{\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)},\sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\}\\&\quad\cdot\min\{\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)},\sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\},\\&\quad\min\{\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)},\sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}\}):\theta\in U\}.\end{aligned}$$

Now,

$$A\$B=\{(\theta,\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}):\theta\in U\},$$
$$A\$C=\{(\theta,\sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}):\theta\in U\}$$

and

$$\begin{aligned}(A\$B)\cup(A\$C)&=\{(\theta,\max\{\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)},\sqrt{\xi_A(\theta)\cdot\xi_C(\theta)}\}\\&\quad\min\{\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)},\sqrt{\tau_A(\theta)\cdot\tau_C(\theta)}\},\\&\quad\min\{\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)},\sqrt{\zeta_A(\theta)\cdot\zeta_C(\theta)}\}):\theta\in U\}.\end{aligned}$$

Thus, $A\$(B\cup C)=(A\$B)\cup(A\$C)$.

(vii) Let $A=\{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)):\theta\in U\}$ and $B=\{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)):\theta\in U\}$ be two PFSs on U. Then, we have

$$A\$B=\{(\theta,\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}):\theta\in U\},$$

which implies

$$(A\$B)^c=\{(\theta,\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}):\theta\in U\}.$$

Again,

$$A^c=\{(\theta,\zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)):\theta\in U\}\quad\text{and}\quad B^c=\{(\theta,\zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)):\theta\in U\}.$$

Therefore,

$$A^c\$B^c=\{(\theta,\sqrt{\zeta_A(\theta)\cdot\zeta_B(\theta)}\sqrt{\tau_A(\theta)\cdot\tau_B(\theta)}\sqrt{\xi_A(\theta)\cdot\xi_B(\theta)}):\theta\in U\},$$

which implies that

$$(A\$B)^c=A^c\$B^c.$$

□

Theorem 4.4. *For any two PFSs A and B, we have*

(i) *If $A \subset B$, then $A\$B \subset B$,*

(ii) $A\$(A \cup B) \subset A \cup B$,

(iii) $A\$(A \cap B) \subset A$

Proof. Trivial. □

Definition 4.5. Let $A_i \in \text{PFS}(U)$, where $i = 1,2,\ldots,n$. Then, the *geometric mean of A_i* is denoted by $\$(A_i)$, and defined as

$$\begin{aligned}\$(A_i) &= \{(\theta, \sqrt[n]{\xi_{A_1}(\theta)\cdot\xi_{A_2}(\theta)\ldots\xi_{A_n}(\theta)}\sqrt[n]{\tau_{A_1}(\theta)\cdot\tau_{A_2}(\theta)\ldots\tau_{A_n}(\theta)}\\ &\qquad\cdot\sqrt[n]{\zeta_{A_1}(\theta)\cdot\zeta_{A_2}(\theta)\ldots\zeta_{A_n}(\theta)}):\theta\in U\}\\ &= \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}\xi_{A_i}(\theta)}\sqrt[n]{\prod_{i=1}^{n}\tau_{A_i}(\theta)}\sqrt[n]{\prod_{i=1}^{n}\zeta_{A_i}(\theta)}\right):\theta\in U\right\}.\end{aligned}$$

Proposition 4.6. *For any $A_i \in \text{PFS}(U)$, where $i = 1,2,\ldots,n$, $\$(A_i) \in \text{PFS}(U)$.*

Proof. For $\$(A_i)$,

$$\begin{aligned}0 &\le \sqrt[n]{\prod_{i=1}^{n}\xi_{A_i}(\theta)} + \sqrt[n]{\prod_{i=1}^{n}\tau_{A_i}(\theta)} + \sqrt[n]{\prod_{i=1}^{n}\zeta_{A_i}(\theta)}\\ &\le \frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n} + \frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n} + \frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n}\\ &\le \frac{1}{n}\sum_{i=1}^{n}[\xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta)]\\ &\le \frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}\\ &\le \frac{1+1+\ldots+1}{n}\\ &\le \frac{n}{n} = 1 \quad (\text{as, } 0 \le \xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta) \le 1).\end{aligned}$$

□

Theorem 4.7. *For every $A, B, C \in \text{PFS}(U)$, the followings hold:*

(i) $\$(A_i) = A_i$;

(ii) $(\$(A_i))^c = \$(A_i^c)$;

(iii) $(\$(A_i^c))^c = \$(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta, \xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \text{PFS}(U)$. Then

$$\$(A_i) = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}\right):\theta\in U\right\}.$$

Therefore,

$$(\$(A_i))^c = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\right):\theta\in U\right\}.$$

Again,

$$A_i^c = \{(\theta, \zeta_{A_i}(\theta), \tau_{A_i}(\theta), \xi_{A_i}(\theta)) : \theta \in U\},$$

then we have

$$\$(A_i^c) = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))}, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))}\right) : \theta \in U\right\},$$

which implies that,

$$(\$(A_i^c))^c = \left\{\left(\theta, \sqrt[n]{\prod_{i=1}^{n}(\xi_{A_i}(\theta))} \sqrt[n]{\prod_{i=1}^{n}(\tau_{A_i}(\theta))} \sqrt[n]{\prod_{i=1}^{n}(\zeta_{A_i}(\theta))}\right) : \theta \in U\right\} = \$(A_i).$$

Therefore, $(\$(A_i^c))^c = \$(A_i)$. □

Definition 4.8 (Picture Fuzzy Harmonic Mean (PFHM))**.** Let $A, B \in \text{PFS}(U)$, then the *harmonic mean between A and B* is denoted by $A©B$, and defined as

$$A©B = \left\{\left(\theta, \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)}\right) : \theta \in U\right\}.$$

Theorem 4.9. *For every $A, B, C \in \text{PFS}(U)$, the followings hold:*

(i) $A©A = A$;

(ii) $A©B = B©A$;

(iii) $(A^c©B^c)^c = A©B$;

(iv) $(A©B)^c = A^c©B^c$;

(v) $(A©B)©C \neq A©(B©C)$.

Proof. Parts (i), (ii) and (iii) trivial.

(iv) Let $A = \{(\theta, \xi_A(\theta), \tau_A(\theta), \zeta_A(\theta)) : \theta \in U\}$ and $B = \{(\theta, \xi_B(\theta), \tau_B(\theta), \zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U. Then, we have

$$A©B = \left\{\left(\theta, \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)}\right) : \theta \in U\right\},$$

which implies

$$(A©B)^c = \left\{\left(\theta, \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)}\right) : \theta \in U\right\}.$$

Again,

$$A^c = \{(\theta, \zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)) : \theta \in U\} \text{ and } B^c = \{(\theta, \zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)) : \theta \in U\}.$$

Therefore,

$$A^c©B^c = \left\{\left(\theta, \frac{2\zeta_A(\theta)\cdot\zeta_B(\theta)}{\zeta_A(\theta)+\zeta_B(\theta)} \frac{2\tau_A(\theta)\cdot\tau_B(\theta)}{\tau_A(\theta)+\tau_B(\theta)} \frac{2\xi_A(\theta)\cdot\xi_B(\theta)}{\xi_A(\theta)+\xi_B(\theta)}\right) : \theta \in U\right\}.$$

Therefore, $(A©B)^c = A^c©B^c$. □

Definition 4.10. Let $A_i \in \text{PFS}(U)$, where $i = 1, 2, \ldots, n$. Then, the *harmonic mean of A_i* is denoted by $©(A_i)$, and defined as

$$©(A_i) = \left\{\left(\theta, \frac{n}{\frac{1}{\xi_{A_1}(\theta)} + \frac{1}{\xi_{A_2}(\theta)} + \ldots + \frac{1}{\xi_{A_n}(\theta)}} \frac{n}{\frac{1}{\tau_{A_1}(\theta)} + \frac{1}{\tau_{A_2}(\theta)} + \ldots + \frac{1}{\tau_{A_n}(\theta)}}\right.\right.$$

$$\left.\frac{n}{\frac{1}{\zeta_{A_1}(\theta)}+\frac{1}{\zeta_{A_2}(\theta)}+\ldots+\frac{1}{\zeta_{A_n}(\theta)}}\right):\theta\in U\Bigg\}$$

$$=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}.$$

Theorem 4.11. *For every $A_i \in \mathrm{PFS}(U)$, where $i=1,2,\ldots,n$. Then the following hold:*

(i) $©(A_i)=A_i$;

(ii) $(©(A_i))^c = ©(A_i^c)$;

(iii) $(©(A_i^c))^c = ©(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i=\{(\theta,\xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)):\theta\in U\}\in \mathrm{PFS}(U)$. Then,

$$©(A_i)=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}.$$

Again,

$$A_i^c=\{(\theta,\zeta_{A_i}(\theta),\tau_{A_i}(\theta),\xi_{A_i}(\theta)):\theta\in U\},$$

then

$$©(A_i^c)=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\right):\theta\in U\right\},$$

which implies

$$(©(A_i^c))^c=\left\{\left(\theta,\frac{n}{\sum_{i=1}^{n}\frac{1}{\xi_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\tau_{A_i}(\theta)}}\frac{n}{\sum_{i=1}^{n}\frac{1}{\zeta_{A_i}(\theta)}}\right):\theta\in U\right\}=©(A_i).$$

Therefore, $(©(A_i^c))^c = ©(A_i)$. □

Definition 4.12 (Picture Fuzzy Arithmetic Mean (PFAM))**.** Let $A_i\in \mathrm{PFS}(U)$, where $i=1,2,\ldots,n$. Then, the *arithmetic mean of* A_i is denoted by $@(A_i)$, and defined as

$$@(A_i)=\left\{\left(\theta,\frac{\xi_{A_1}(\theta)+\xi_{A_2}(\theta)+\ldots+\xi_{A_n}(\theta)}{n},\frac{\tau_{A_1}(\theta)+\tau_{A_2}(\theta)+\ldots+\tau_{A_n}(\theta)}{n},\right.\right.$$

$$\left.\left.\frac{\zeta_{A_1}(\theta)+\zeta_{A_2}(\theta)+\ldots+\zeta_{A_n}(\theta)}{n}\right):\theta\in U\right\}$$

$$=\left\{\left(\theta,\frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n},\frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n}\frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n}\right):\theta\in U\right\}.$$

Proposition 4.13. *For any $A_i\in \mathrm{PFS}(U)$, where $i=1,2,\ldots,n$, $@(A_i)\in \mathrm{PFS}(U)$.*

Proof. For $@(A_i)$,

$$0\le \frac{\sum_{i=1}^{n}\xi_{A_i}(\theta)}{n}+\frac{\sum_{i=1}^{n}\tau_{A_i}(\theta)}{n}+\frac{\sum_{i=1}^{n}\zeta_{A_i}(\theta)}{n}$$

$$\le \frac{1}{n}\sum_{i=1}^{n}[\xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta)]$$

$$\le \frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}$$

$$\leq \frac{1+1+\ldots+1}{n}$$
$$\leq \frac{n}{n} = 1 \quad (\text{as, } 0 \leq \xi_{A_i}(\theta)+\tau_{A_i}(\theta)+\zeta_{A_i}(\theta) \leq 1). \qquad \square$$

Theorem 4.14. *For every* $A,B,C \in \text{PFS}(U)$, *then the following hold:*

(i) $@(A_i) = A_i$;

(ii) $(@(A_i))^c = @(A_i^c)$;

(iii) $(@(A_i^c))^c = @(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta,\xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \text{PFS}(U)$. Then

$$@(A_i) = \left\{\left(\theta, \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}\right) : \theta \in U\right\}.$$

Therefore,

$$(@A_i)^c = \left\{\left(\theta, \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}\right) : \theta \in U\right\}.$$

Again,

$$A_i^c = \{(\theta,\zeta_{A_i}(\theta)\tau_{A_i}(\theta)\xi_{A_i}(\theta)) : \theta \in U\}$$

and

$$@(A_i^c) = \left\{\left(\theta, \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}\right) : \theta \in U\right\},$$

which implies that,

$$(@(A_i^c))^c = \left\{\left(\theta, \frac{\sum_{i=1}^n \xi_{A_i}(\theta)}{n}, \frac{\sum_{i=1}^n \tau_{A_i}(\theta)}{n} \frac{\sum_{i=1}^n \zeta_{A_i}(\theta)}{n}\right) : \theta \in U\right\} = @(A_i).$$

Therefore, $(@(A_i^c))^c = @(A_i)$. $\square$

Definition 4.15. Let $A,B \in \text{PFS}(U)$, then the *weighted mean between* A *and* B is denoted by $A \circledcirc B$, and defined as

$$A \circledcirc B = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2} \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2} \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\},$$

where $w_1+w_2 \in [0,1]$ and $0 < w_1+w_2 \leq 1$.

Theorem 4.16. *For every* $A,B,C \in \text{PFS}(U)$, *the following hold:*

(i) $A \circledcirc A = A$;

(ii) $A \circledcirc B = B \circledcirc A$ *if* $A = B$;

(iii) $(A \circledcirc B)^c = A^c \circledcirc B^c$;

(iv) $(A^c \circledcirc B^c)^c = A \circledcirc B$.

Proof. Parts (i), (ii) and (iii) are trivial.

(iv) Let $A = \{(\theta,\xi_A(\theta),\tau_A(\theta),\zeta_A(\theta)) : a \in U\}$ and $B = \{(\theta,\xi_B(\theta),\tau_B(\theta),\zeta_B(\theta)) : \theta \in U\}$ be two PFSs on U.

Then,

$$A \circledcirc B = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2} \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2} \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\}.$$

Again,

$$A^c = \{(\theta, \zeta_A(\theta)\tau_A(\theta)\xi_A(\theta)) : \theta \in U\}$$

and

$$B^c = \{(\theta, \zeta_B(\theta)\tau_B(\theta)\xi_B(\theta)) : \theta \in U\},$$

then we have

$$A^c \circledcirc B^c = \left\{\left(\theta, \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}, \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2}, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\},$$

which implies

$$(A^c \circledcirc B^c)^c = \left\{\left(\theta, \frac{w_1\xi_A(\theta)+w_2\xi_B(\theta)}{w_1+w_2}, \frac{w_1\tau_A(\theta)+w_2\tau_B(\theta)}{w_1+w_2}, \frac{w_1\zeta_A(\theta)+w_2\zeta_B(\theta)}{w_1+w_2}\right) : \theta \in U\right\} = A \circledcirc B.$$

Therefore, $(A^c \circledcirc B^c)^c = A \circledcirc B$. □

Definition 4.17. Let $A_i \in \text{PFS}(U)$, where $i = 1,2,\dots,n$. Then, the *weighted mean of* A_i is denoted by $\circledcirc(A_i)$, and defined as

$$\circledcirc(A_i) = \left\{\left(\theta, \frac{w_1\xi_{A_1}(\theta)+w_2\xi_{A_2}(\theta)+\dots\dots+w_n\xi_{A_n}(\theta)}{w_1+w_2+\dots\dots+w_n} \frac{w_1\tau_{A_1}(\theta)+w_2\tau_{A_2}(\theta)+\dots+w_n\tau_{A_n}(\theta)}{w_1+w_2+\dots+w_n}\right.\right.$$
$$\left.\left.\frac{w_1\zeta_{A_1}(\theta)+w_2\zeta_{A_2}(\theta)+\dots+w_n\zeta_{A_n}(\theta)}{w_1+w_2+\dots\dots+w_n}\right) : \theta \in U\right\}$$
$$= \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\},$$

where $w_i \in [0,1]$ and $0 < \sum_{i=1}^n w_i \le 1$.

Theorem 4.18. *For every* $A_i \in \text{PFS}(U)$*, where* $i = 1,2,\dots,n$*, the following hold:*

(i) $\circledcirc(A_i) = A_i$;

(ii) $(\circledcirc(A_i))^c = \circledcirc(A_i^c)$;

(iii) $(\circledcirc(A_i^c))^c = \circledcirc(A_i)$.

Proof. Parts (i) and (ii) are trivial.

(iii) Let $A_i = \{(\theta, \xi_{A_i}(\theta)\tau_{A_i}(\theta)\zeta_{A_i}(\theta)) : \theta \in U\} \in \text{PFS}(U)$. Then

$$\circledcirc(A_i) = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\}.$$

Again,

$$A_i^c = \{(\theta, \zeta_{A_i}(\theta), \tau_{A_i}(\theta), \xi_{A_i}(\theta)) : \theta \in U\},$$

then we have

$$©(A_i^c) = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i\zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}, \frac{\sum_{i=1}^n w_i\tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i\xi_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\},$$

which implies

$$(\circledcirc(A_i^c))^c = \left\{\left(\theta, \frac{\sum_{i=1}^n w_i \xi_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i \tau_{A_i}(\theta)}{\sum_{i=1}^n w_i} \frac{\sum_{i=1}^n w_i \zeta_{A_i}(\theta)}{\sum_{i=1}^n w_i}\right) : \theta \in U\right\} = \circledcirc(A_i).$$

Therefore, $(\circledcirc(A_i^c))^c = \circledcirc(A_i)$. □

5. Application of the Picture Fuzzy Mean Operators to Pattern Recognition

Multi-attribute decision-making (MADM) problem which aims to determine the best alternative by considering more than one attributes in a selection process. The decision will be performed as per the following steps:

Step 1. Picture Fuzzy Decision Matrix will be formed by the experts considering all the alternatives associated with all the attributes.

Step 2. Find the preference values d_i $(i = 1,2,\dots,5)$ for the operators PFAM, PFGM and PFHM.

Step 3. Find the scores $S(d_i)$ $(i = 1,2,\dots,n)$ (using score function) of the overall picture fuzzy values d_i $(i = 1,2,\dots,n)$.

Step 4. Rank all the alternatives A_i $(i = 1,2,\dots,n)$ in accordance with the values of $S(d_i)$ $(i = 1,2,\dots,n)$ and select the best one(s).

If any two scores of $S(d_i)$ and $S(d_j)$ are equal, then calculate the accuracy degrees $H(d_i)$ and $H(d_j)$ (using the accuracy function) of the overall picture fuzzy values d_i and d_j, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(d_i)$ and $H(d_j)$.

Case Study

Suppose a company wants to recruit a potential salesman in the company's sales department. There are five candidates $C = \{C_1, C_2, C_3, C_4, C_5\}$ for this position and the company considers four attributes $A = \{A_1, A_2, A_3, A_4\}$ from these candidates. These attributes are academic results, communication skills, experience and innovative ideas. According to the decision of the recruitment committee (experts), the following picture fuzzy matrix is formed under the four attributes.

Step 1: Establishing the picture fuzzy decision matrix by the recruitment committee (experts) considering all the five candidates associated with all the four attributes in Table 1.

Table 1. Picture Fuzzy Decision Matrix

	A_1	A_2	A_3	A_4
C_1	(0.7,0.2,0.1)	(0.6,0.2,0.2)	(0.5,0.2,0.3)	(0.4,0.2,0.3)
C_2	(0.5,0.1,0.3)	(0.6,0.2,0.2)	(0.7,0.1,0.2)	(0.4,0.3,0.3)
C_3	(0.5,0.2,0.2)	(0.4,0.2,0.4)	(0.6,0.2,0.2)	(0.3,0.3,0.2)
C_4	(0.4,0.4,0.2)	(0.5,0.3,0.2)	(0.8,0.1,0.1)	(0.6,0.2,0.2)
C_5	(0.3,0.2,0.4)	(0.4,0.4,0.2)	(0.5,0.3,0.2)	(0.7,0.2,0.1)

Step 2: Finding the preference values d_i $(i = 1,2,\dots,5)$ by the operators PFAM, PFGM and PFHM, we have overall preference values d_i as shown in Table 2.

Table 2. Preference values d_i $(i = 1,2,\dots,5)$ for the operators PFAM, PFGM and PFHM

	d_1	d_2	d_3	d_4	d_5
PFAM	(0.55,0.20,0.23)	(0.55,0.18,0.25)	(0.45,0.23,0.25)	(0.58,0.25,0.18)	(0.48,0.28,0.23)
PFGM	(0.54,0.20,0.21)	(0.54,0.17,0.24)	(0.44,0.22,0.24)	(0.56,0.22,0.17)	(0.45,0.26,0.20)
PFHM	(0.53,0.20,0.18)	(0.53,0.14,0.24)	(0.42,0.22,0.23)	(0.54,0.19,0.16)	(0.43,0.25,0.18)

Step 3: Finding the scores $S(d_i)$ $(i = 1,2,\dots,n)$ (using score function) of the overall picture fuzzy values d_i $(i = 1,2,\dots,n)$ in Table 3.

Table 3. The scores $S(d_i)$ $(i = 1,2,\dots,5)$ of the overall picture fuzzy values d_i $(i = 1,2,\dots,5)$

	$S(d_1)$	$S(d_2)$	$S(d_3)$	$S(d_4)$	$S(d_5)$
PFAM	0.52	0.58	0.43	0.65	0.53
PFGM	0.53	0.47	0.42	0.61	0.51
PFHM	0.55	0.43	0.41	0.57	0.50

Step 4: Ranking all the candidates C_i $(i = 1,2,3,4,5)$ in accordance with the values of $S(d_i)$ $(i = 1,2,3,4,5)$ and selecting the best one(s) in Table 4.

Table 4. Ranking all the candidates C_i $(i = 1,2,3,4,5)$ in accordance with the values of $S(d_i)$ $(i = 1,2,3,4,5)$ found in Table 3

Operators	Ranking	Best Candidates
PFAM	$C_4 \succ C_2 \succ C_5 \succ C_1 \succ C_3$	C_4
PFGM	$C_4 \succ C_1 \succ C_5 \succ C_2 \succ C_3$	C_4
PFHM	$C_4 \succ C_1 \succ C_5 \succ C_2 \succ C_3$	C_4

Hence, the candidate C_4 is the best and selected for the potential salesman in that company.

6. Conclusions

Picture fuzzy set is the generalization of intuitionistic fuzzy set and fuzzy set and capable to successfully handle the uncertain data. In this article, several types of operators such as difference operator, necessity operator, possibility operator, closure operator, interior operator and some mean operators for picture fuzzy sets are defined. Also, some related properties of these operators are established. Finally, a decision making problem is discussed to demonstration how these operators work in real life situation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. T. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, 1st edition, Physica Heidelberg, xviii + 324 pages (1999), DOI: 10.1007/978-3-7908-1870-3.

[2] N. M. Chau, N. T. Lan and N. X. Thao, A new similarity measure of picture fuzzy sets and application in the fault diagnosis of steam turbine, *International Journal of Mathematical Sciences and Computing* **6**(5) (2020), 47 – 55, DOI: 10.5815/ijmsc.2020.05.05.

[3] B. C. Cuong, Picture fuzzy sets, *Journal of Computer Science and Cybernetics* **30**(4) (2014), 409 – 420, DOI: 10.15625/1813-9663/30/4/5032.

[4] B. C. Cuong and V. Kreinovich, Picture fuzzy sets – A new concept for computational intelligence problems, in: *Third World Congress on Information and Communication Technologies* (WICT2013, Hanoi, Vietnam, 2013), pp. 1 – 6 (2013), DOI: 10.1109/WICT.2013.7113099.

[5] N. V. Dinh and N. X. Thao, Some measures of picture fuzzy sets and their application in multi-attribute decision making, *International Journal of Mathematical Sciences and Computing* **4**(3) (2018), 23 – 41, DOI: 10.5815/ijmsc.2018.03.03.

[6] S. Dogra and M. Pal, Picture fuzzy matrix and its application, *Soft Computing* **24** (2020), 9413 – 9428, DOI: 10.1007/s00500-020-05021-4.

[7] P. Dutta and S. Ganju, Some aspects of picture fuzzy set, *Transactions of A. Razmadze Mathematical Institute* **172**(2) (2018), 164 – 175, DOI: 10.1016/j.trmi.2017.10.006.

[8] A. H. Ganie, S. Singh and P. K. Bhatia, Some new correlation coefficients of picture fuzzy sets with applications, *Neural Computing and Applications* **32** (2020), 12609 – 12625, DOI: 10.1007/s00521-020-04715-y(0123456789.

[9] H. Garg, Some picture fuzzy aggregation operators and their applications to multicriteria decision-making, *Arabian Journal for Science and Engineering* **42** (2017), 5275 – 5290, DOI: 10.1007/s13369-017-2625-9.

[10] M. K. Hasan, Md. Y. Ali, A. Sultana and N. K. Mitra, Some picture fuzzy mean operators and their applications in decision-making, *Journal of Fuzzy Extension and Applications* **3**(4) (2022), 349 – 361.

[11] C. Jana, T. Senapati, M. Pal and R. R. Yager, Picture fuzzy Dombi aggregation operators: Application to MADM process, *Applied Soft Computing* **74** (2019), 99 – 109, DOI: 10.1016/j.asoc.2018.10.021.

[12] R. Kadian and S. Kumar, A new picture fuzzy divergence measure based on Jensen-Tsallis information measure and its application to multicriteria decision making, *Granular Computing* **7** (2022), 113 – 126, DOI: 10.1007/s41066-021-00254-6.

[13] M. J. Khan, P. Kumam, W. Deebani, W. Kumam and Z. Shah, Bi-parametric distance and similarity measures of picture fuzzy sets and their applications in medical diagnosis, *Egyptian Informatics Journal* **22**(2) (2021), 201 – 212, DOI: 10.1016/j.eij.2020.08.002.

[14] S. Khan, S. Abdullah and S. Ashraf, Picture fuzzy aggregation information based on Einstein operations and their application in decision making, *Mathematical Sciences* **13**(3) (2019), 213 – 229, DOI: 10.1007/s40096-019-0291-7.

[15] M. Luo and H. Long, Picture fuzzy geometric aggregation operators based on a trapezoidal fuzzy number and its application, *Symmetry* **13**(1) (2021), 119, DOI: 10.3390/sym13010119.

[16] M. Luo and Y. Zhang, A new similarity measure between picture fuzzy sets and its application, *Engineering Applications of Artificial Intelligence* **96** (2020), 103956, DOI: 10.1016/j.engappai.2020.103956.

[17] P. Meksavang, H. Shi, S.-M. Lin and H.-C. Liu, An extended picture fuzzy VIKOR approach for sustainable supplier management and its application in the beef industry, *Symmetry* **11**(4) (2019), 468, DOI: 10.3390/sym11040468.

[18] X. T. Nguyen, Evaluating water reuse applications under uncertainty: A novel picture fuzzy multi criteria decision making method, *International Journal of Information Engineering and Electronic Business* **10**(6) (2018), 32 – 39, DOI: 10.5815/ijieeb.2018.06.04.

[19] X. Peng and J. Dai, Algorithm for picture fuzzy multiple attribute decision-making based on new distance measure, *International Journal for Uncertainty Quantification* **7**(2) (2017), 177 – 187, DOI: 10.1615/Int.J.UncertaintyQuantification.2017020096.

[20] M. Qiyas, S. Abdullah, S. Ashraf and M. Aslam, Utilizing linguistic picture fuzzy aggregation operators for multiple-attribute decision-making problems, *International Journal of Fuzzy Systems* **22** (2020), 310 – 20, DOI: 10.1007/s40815-019-00726-7.

[21] A. Si, S. Das and S. Kar, An approach to rank picture fuzzy numbers for decision making problems, *Decision Making: Applications in Management and Engineering* **2**(2) (2019), 54 – 64, DOI: 10.31181/dmame1902049s.

[22] I. Silambarasan, Some algebraic properties of picture fuzzy sets, *Bulletin of the International Mathematical Virtual Institute* **11**(3) (2021), 429 – 442.

[23] P. Singh, Correlation coefficients for picture fuzzy sets, *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology* **28**(2) (2015), 591 – 604, DOI: 10.3233/IFS-141338.

[24] S. Singh and A. H. Ganie, Applications of a picture fuzzy correlation coefficient in pattern analysis and decision-making, *Granular Computing* **7** (2022), 353 – 367, DOI: 10.1007/s41066-021-00269-z.

[25] L. H. Son, Measuring analogousness in picture fuzzy sets: From picture distance measures to picture association measures, *Fuzzy Optimization and Decision Making* **16** (2016), 359 – 378, DOI: 10.1007/s10700-016-9249-5.

[26] L. H. Son, P. V. Viet and P. V. Hai, Picture inference system: A new fuzzy inference system on picture fuzzy set, *Applied Intelligence* **46** (2017), 652 – 669, DOI: 10.1007/s10489-016-0856-1.

[27] N. X. Thao, Similarity measures of picture fuzzy sets based on entropy and their application in MCDM, *Pattern Analysis and Applications* **23**(2020), 1203 – 1213, DOI: 10.1007/s10044-019-00861-9.

[28] C. Tian, J.-J. Peng, S. Zhang, W.-Y. Zhang and J.-Q. Wang, Weighted picture fuzzy aggregation operators and their applications to multi-criteria decision-making problems, *Computers & Industrial Engineering* **137** (2019), 106037, DOI: 10.1016/j.cie.2019.106037.

[29] R. Wang, Methods for MADM with picture fuzzy Muirhead mean operators and their application for evaluating the financial investment risk, *Symmetry* **11**(1) (2019), 6, DOI: 10.3390/sym11010006.

[30] C. Wang, X. Zhou, H. Tu and S. Tao, Some geometric aggregation operators based on picture fuzzy sets and their applications in multiple attribute decision making, *Italian Journal of Pure and Applied Mathematics* **37** (2017), 477 – 492, URL: https://ijpam.uniud.it/online_issue/201737/44-WangZhouTuTao.pdf.

[31] G. Wei, Picture fuzzy cross-entropy for multiple attribute decision making problems, *Journal of Business Economics and Management* **17**(4) (2016), 491 – 502, DOI: 10.3846/16111699.2016.1197147.

[32] G. Wei, Picture fuzzy hamacher aggregation operators and their application to multiple attribute decision making, *Fundamenta Informaticae* **157**(3) (2018), 271 – 320, DOI: 10.3233/FI-2018-1628.

[33] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.