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Research Article

On Generalized Suzuki Contraction Principles in C^*-Algebra Valued S_b-Metric Spaces with Applications

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Abstract. The aim of this study is to develop and establish novel *unique fixed point* (UFP) theorems within the framework of $\mathbb{C}_{\alpha}^{(\varphi,\psi)}$-Suzuki contractions by employing two auxiliary functions φ and ψ in the context of admissible mappings on C^*-algebra-valued S_b-metric spaces (C^*-AV-S_bMS). Furthermore, the work intends to identify sufficient conditions ensuring the existence and uniqueness of fixed points and to illustrate the applicability of the obtained results to integral equations and homotopy through representative examples.

Keywords. $\mathbb{C}_{\alpha}^{(\varphi,\psi)}$-suzuki contractive type mapping, FP and C^*-AV-S_bMS

Mathematics Subject Classification (2020). 54H25, 47H10, 54E50

1. Introduction

Over recent decades, continuous efforts by scientists and engineers have aimed at addressing complex analytical and computational problems. Within this context, the concepts of fixed points and fixed point theorems occupy a central position in nonlinear analysis.

These theorems establish essential results ensuring the existence and stability of solutions across diverse mathematical frameworks, underpinning a wide range of scientific and engineering applications. The Banach contraction principle [1] stands as one of the cornerstone results in this domain. Throughout the twentieth century, significant research has concentrated on expanding Banach's classical theorem, either by generalizing the underlying metric structures (Czerwik [3]) or by introducing broader classes of contraction mappings (see, for instance, Beniwal *et al*. [2], Mani *et al*. [8], and Pingale *et al*. [13]).

In 2008, Suzuki [22] introduced a weakened C-contractive condition that defines a contraction mapping, now known as the Suzuki contraction, and established related FPT in MS.

Subsequent studies have thoroughly examined the existence and uniqueness of FP for these mappings (Mani [7], Somal *et al*. [19], Suzuki [21], Zhang *et al*. [25]). Samet *et al*. [17] introduced α-admissible mappings, enabling generalizations through α-φ-contractions; building on this, Prakasam *et al*. [14] proved FPT for O-α-ψ-Geraghty-type mappings, while Monfared *et al*. [9] developed α-admissible $F(\psi,\phi)$-contractions in M-metric spaces. These developments have spurred extensive research on FPs of diverse contractions involving α-admissible mappings by numerous authors.

In 2014, Ma *et al*. [6] first proposed C^*-algebra-valued metric spaces, later generalizing them to C^*-algebra-valued b-metric spaces in 2015 with associated FPT (Ma and Jiang [5]). Razavi and Masiha [16] further explored these b-metric variants to derive common FPT. Sedghi *et al*. [18] merged S-metric and b-metric concepts to define S_b-metric spaces, proving common FPT therein. Building on Souayah and Mlaiki's framework [20], Razavi and Masiha [15,16] introduced C^*-algebra-valued S_b-metric spaces in 2023, securing initial common FP outcomes, which spurred additional investigations (see, Haritha *et al*. [4], Narsimha *et al*. [11,12], and Ushabhavani *et al*. [23]).

In this work, we first define a generalized Suzuki type contraction using auxiliary functions, called the Suzuki $\mathbb{C}_{\alpha}^{(\varphi,\psi)}$-contraction and then utilized it to prove FPT on complete C^*-AV-S_bMS. At last, some consequences of applications to integral equations, homotopy and examples are presented to demonstrate the validity of the proved result. The results presented herein generalize, extend, and enhance various existing findings in the literature.

First we recall some basic results.

2. Preliminaries

This section briefly introduces some fundamental aspects of the theory of C^*-algebras (Murphy [10]).

Consider a unital C^*-algebra $\mathscr{A}$ with unit element $1_{\mathscr{A}}$. Define the set of self-adjoint elements as $\mathscr{A}_h = \{s \in \mathscr{A} : s = s^*\}$. An element $s \in \mathscr{A}$ is called positive, denoted by $s \succeq 0_{\mathscr{A}}$, if it is self-adjoint and its spectrum satisfies $\sigma(s) \subseteq [0,\infty)$. Here, $0_{\mathscr{A}}$ is the zero element of $\mathscr{A}$ and $\sigma(s)$ is the spectrum of s. A natural partial order on $\mathscr{A}_h$ is given by $\ell \preceq \wp$ if and only if $\wp - \ell \succeq 0_{\mathscr{A}}$. We denote the positive cone by $\mathscr{A}_+ = \{s \in \mathscr{A} : s \succeq 0_{\mathscr{A}}\}$ and the commutant of $\mathscr{A}$ by $\mathscr{A}' = \{s \in \mathscr{A} : st = ts, \text{ for all } t \in \mathscr{A}\}$.

Definition 2.1 ([15])**.** Let $\mathcal{G}$ be a non-empty set and $\kappa \in \mathscr{A}'$ with $\|\kappa\| \ge 1$. A mapping $\varsigma_\kappa : \mathcal{G}^3 \to \mathscr{A}_+$ defines a C^*-algebra valued S_b-metric structure if it satisfies:

(i) $\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathfrak{k}) \succeq 0_{\mathscr{A}}$, for all $\mathfrak{s},\mathfrak{e},\mathfrak{k} \in \mathcal{G}$,

(ii) $\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathfrak{k}) = 0_{\mathscr{A}}$ if and only if $\mathfrak{s} = \mathfrak{e} = \mathfrak{k}$,

(iii) $\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathfrak{k}) \preceq \kappa\big(\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\theta) + \varsigma_\kappa(\mathfrak{e},\mathfrak{e},\theta) + \varsigma_\kappa(\mathfrak{k},\mathfrak{k},\theta)\big)$, for all $\mathfrak{s},\mathfrak{e},\mathfrak{k},\theta \in \mathcal{G}$.

The triple $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ constitutes a C^*-AV-S_bMS with coefficient κ.

Definition 2.2 ([15])**.** Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a C^*-AV-S_bMS and $\{\chi_n\}$ a sequence in $\mathcal{G}$. Then

(i) $\{\chi_n\}$ forms a Cauchy sequence whenever $\|\varsigma_\kappa(\chi_{n+p},\chi_{n+p},\chi_n)\|\to 0$ as $n\to\infty$, for every $p\in\mathbb{N}$;

(ii) $\{\chi_n\}$ converges to $\chi\in\mathcal{G}$ if $\|\varsigma_\kappa(\chi_n,\chi_n,\chi)\|\to 0$ as $n\to\infty$, denoted $\lim\limits_{n\to\infty}\chi_n=\chi$;

(iii) the space $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ is complete if all Cauchy sequences converge in $\mathcal{G}$.

Definition 2.3 ([15])**.** Consider C^*-AV-S_bMSs $(\mathcal{G}_1,\mathscr{A}_1,\varsigma_{\kappa 1})$ and $(\mathcal{G}_2,\mathscr{A}_2,\varsigma_{\kappa 2})$. A mapping $\mathfrak{k}:(\mathcal{G}_1,\mathscr{A}_1,\varsigma_{\kappa 1})\to(\mathcal{G}_2,\mathscr{A}_2,\varsigma_{\kappa 2})$ is continuous at $\chi\in\mathcal{G}_1$ if whenever $\{\chi_n\}\subset\mathcal{G}_1$ satisfies $\|\varsigma_{\kappa_1}(\chi_n,\chi_n,\chi)\|\to 0$ as $n\to\infty$, then $\|\varsigma_{\kappa_2}(\mathfrak{k}(\chi_n),\mathfrak{k}(\chi_n),\mathfrak{k}(\chi))\|\to 0$. The mapping $\mathfrak{k}$ is continuous on $\mathcal{G}_1$ precisely when it is continuous at every point of $\mathcal{G}_1$.

Lemma 2.1 ([15])**.** *In an C^*-AV-S_bMS, we have*

$$\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathfrak{e})\preceq\kappa\varsigma_\kappa(\mathfrak{e},\mathfrak{e},\mathfrak{s})\ \text{ and }\ \varsigma_\kappa(\mathfrak{e},\mathfrak{e},\mathfrak{s})\preceq\kappa\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathfrak{e}),\quad \text{for all } \mathfrak{s},\mathfrak{e}\in\mathcal{G}.$$

Lemma 2.2 ([15])**.** *In an C^*-AV-S_bMS, we have*

$$\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathfrak{k})\preceq 2\kappa\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathfrak{e})+\kappa^2\varsigma_\kappa(\mathfrak{e},\mathfrak{e},\mathfrak{k}),\quad \text{for all } \mathfrak{s},\mathfrak{e},\mathfrak{k}\in\mathcal{G}.$$

Lemma 2.3 ([24])**.** *Let $\mathscr{A}$ be a unital C^*-algebra with unit $1_{\mathscr{A}}$. Then*

(i) *for any sequence $\{\mathfrak{s}_a\}_{a=1}^{\infty}\subset\mathscr{A}$ with $\lim\limits_{a\to\infty}\mathfrak{s}_a=0_{\mathscr{A}}$, we have $\lim\limits_{a\to\infty}\mathfrak{s}^*\mathfrak{s}_a\mathfrak{s}=0_{\mathscr{A}}$, for all $\mathfrak{s}\in\mathscr{A}$;*

(ii) *if $\mathfrak{s},\mathfrak{e}\in\mathscr{A}_h$ and $s\in\mathscr{A}_+\cap\mathscr{A}'$, then $\mathfrak{s}\preceq\mathfrak{e}$ implies $s\mathfrak{s}\preceq s\mathfrak{e}$;*

(iii) *whenever $\mathfrak{s}\in\mathscr{A}_+$ satisfies $\|\mathfrak{s}\|<\frac{1}{2}$, the element $1_{\mathscr{A}}-\mathfrak{s}$ is invertible with $\|\mathfrak{s}(1_{\mathscr{A}}-\mathfrak{s})^{-1}\|<1$;*

(iv) *for commuting positive elements $\mathfrak{s},\mathfrak{e}\in\mathscr{A}_+$, their product satisfies $\mathfrak{s}\mathfrak{e}\succeq 0_{\mathscr{A}}$.*

3. Main Results

This section first defines the $\mathbb{C}_\alpha^{(\varphi,\psi)}$-Suzuki contraction and subsequently establishes the primary fixed point theorem.

Let us denote the classes of functions $\Phi=\{\varphi/\varphi:\mathscr{A}_+\to\mathscr{A}_+$ is a monotone increasing and continuous function with $\varphi(\mathfrak{k})=0_{\mathscr{A}}\iff\mathfrak{k}=0_{\mathscr{A}}\}$.

Definition 3.1. Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a C^*-AV-S_bMS with mappings $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ and $\alpha:\mathcal{G}^3\to\mathscr{A}_+$. The mapping $\mathbb{Q}$ is α-adm if $\alpha(\mathfrak{s},\mathfrak{s},\text{æ})\succeq 1_{\mathscr{A}}\implies\alpha(\mathbb{Q}\mathfrak{s},\mathbb{Q}\mathfrak{s},\mathbb{Q}\text{æ})\succeq 1_{\mathscr{A}}$, holds for all $\mathfrak{s},\text{æ}\in\mathcal{G}$.

Definition 3.2. Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a C^*-AV-S_bMS, $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ be a mapping and $\alpha:\mathcal{G}^3\to\mathscr{A}$ be functions then $\mathbb{Q}$ is called a triangular α-admissible, if

(i) $\mathbb{Q}$ is α-admissible (α-adm);

(ii) $\alpha(\mathfrak{s},\mathfrak{s},\text{œ})\succeq 1_{\mathscr{A}}$ and $\alpha(\text{œ},\text{œ},\text{æ})\succeq 1_{\mathscr{A}}$ implies $\alpha(\mathfrak{s},\mathfrak{s},\text{æ})\succeq 1_{\mathscr{A}}$, for all $\mathfrak{s},\text{æ},\text{œ}\in\mathcal{G}$.

Definition 3.3. Let $\mathcal{G}$ be a nonempty set and $\mathbb{Q}$ be an α-adm map on $\mathcal{G}$. Then $\mathbb{Q}$ is α^*-adm if for all $\mathfrak{s},\mathfrak{k}\in\mathrm{Fix}(\mathbb{Q})\neq\emptyset$, we have $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{k})\succeq 1_{\mathscr{A}}$. If $\mathrm{Fix}(\mathbb{Q})=\emptyset$, we say that $\mathbb{Q}$ is vacuously α^*-adm.

Definition 3.4. Let $\mathbb{Q}$ be a self map on C^*-AV-S_bMS $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$, which is known as orbitally continuous on $\mathcal{G}$ with respect to $\mathscr{A}$ if $\lim\limits_{\mathfrak{s}\to\infty}\mathbb{Q}^{\mathfrak{z}_\mathfrak{s}}(\text{æ})=\zeta\implies\lim\limits_{\mathfrak{s}\to\infty}\mathbb{Q}^{\mathfrak{z}_\mathfrak{s}+1}(\text{æ})=\mathbb{Q}\zeta$.

Let $\mathbb{Q}$ is self map on $\mathcal{G}\neq\emptyset$. We denote $\mathrm{Fix}(\mathbb{Q})=\{\text{æ}:\mathbb{Q}\text{æ}=\text{æ}$, for all $\text{æ}\in\mathcal{G}\}$.

Definition 3.5. Consider a C^*-AV-S_bMS $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ with $\|\kappa\| > 1$, together with mappings $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ and $\alpha:\mathcal{G}^3\to\mathscr{A}_+$. The mapping $\mathbb{Q}$ constitutes a $\mathbb{C}_\alpha^{(\varphi,\psi)}$-Suzuki type weak contraction if there exist $\varphi,\psi\in\Phi$ and $a\in\mathscr{A}$ with $\|a\|<1$ satisfying the contraction condition for all $\mathfrak{s},\mathfrak{e},\varkappa\in\mathcal{G}$ and $i=3,4,5$.

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathbb{Q}\mathfrak{s})\prec\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa)\implies\varphi(\alpha(\mathfrak{s},\mathfrak{e},\varkappa)\varsigma_\kappa(\mathbb{Q}\mathfrak{s},\mathbb{Q}\mathfrak{e},\mathbb{Q}\varkappa))\preceq\psi\left(\frac{1}{4\kappa^7}[a^*M^i(\mathfrak{s},\mathfrak{e},\varkappa)a]\right),\tag{3.1}$$

where

$$M^3(\mathfrak{s},\mathfrak{e},\varkappa)=\max\left\{\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa),\frac{1}{2\kappa^7}[\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\mathfrak{s})+\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)],\frac{1}{4\kappa^7}[\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\varkappa)+\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\mathfrak{s})]\right\},$$

$$M^4(\mathfrak{s},\mathfrak{e},\varkappa)=\max\left\{\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa),\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\mathfrak{s}),\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa),\frac{1}{4\kappa^7}[\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\varkappa)+\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\mathfrak{s})]\right\},$$

$$M^5(\mathfrak{s},\mathfrak{e},\varkappa)=\max\{\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa),\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\mathfrak{s}),\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa),\varsigma_\kappa(\mathfrak{s},\mathfrak{s},\mathbb{Q}\varkappa),\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\mathfrak{s})\}.$$

We begin by proving the following crucial lemma.

Lemma 3.1. *Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a C^*-AV-S_bMS with $\|\kappa\|\geq 1$. Further assume that sequence $\{\mathfrak{k}_\xi\}\subseteq\mathcal{G}$ satisfies $\lim_{\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})=0_{\mathscr{A}}$. If $\{\mathfrak{k}_\xi\}$ is not Cauchy sequence in $\mathcal{G}$ with respect to $\mathscr{A}$ then there exists an $\epsilon\succ 0_{\mathscr{A}}$ and sequences of positive integers $\{\eta_{\mathbb{b}}\}$ and $\{\xi_{\mathbb{b}}\}$ with $\{\eta_{\mathbb{b}}\}>\{\xi_{\mathbb{b}}\}>\mathbb{b}$ such that $\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\succ\epsilon$ and $\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})\preceq\epsilon$. Then, the following holds:*

(i) $\epsilon\preceq\liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\kappa^2\epsilon$;

(ii) $\frac{\epsilon}{\kappa^2}\preceq\liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\epsilon\kappa^4$;

(iii) $\frac{\epsilon}{2\kappa^2}\preceq\liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}})\preceq\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}))\preceq\epsilon\kappa^5$.

Proof. Since $\{\mathfrak{k}_\xi\}\subseteq\mathcal{G}$ is not Cauchy with respect to $\mathscr{A}$, there exist $\epsilon\succ 0_{\mathscr{A}}$ and strictly increasing sequences $\{\eta_{\mathbb{b}}\}$, $\{\xi_{\mathbb{b}}\}$ of positive integers with $\eta_{\mathbb{b}}>\xi_{\mathbb{b}}>\mathbb{b}$ such that

$$\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\succ\epsilon,\quad\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\epsilon.\tag{3.2}$$

From the first inequality in (3.2) and definition of lim inf, taking $\mathbb{b}\to\infty$ yields

$$\epsilon\preceq\liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}}).\tag{3.3}$$

Using Lemma 2.2 with triple $(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})$ and intermediate point $\mathfrak{k}_{\eta_{\mathbb{b}-1}}$, we get

$$\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq 2\kappa\,\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}-1}})+\kappa^2\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}}}).\tag{3.4}$$

By $\lim_{\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})=0_{\mathscr{A}}$ and the second inequality of (3.2), taking lim sup in (3.4) gives

$$\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq\kappa^2\epsilon.\tag{3.5}$$

Combining (3.3)–(3.5) yields (i).

For (ii), apply Lemma 2.2 to $\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})$ with intermediate $\mathfrak{k}_{\eta_{\mathbb{b}+1}}$:

$$\epsilon\prec\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\preceq 2\kappa\,\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}+1}})+\kappa^2\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}}).\tag{3.6}$$

Taking lim inf in (3.6) yields

$$\frac{\epsilon}{\kappa^2}\preceq\liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}}).\tag{3.7}$$

Applying Lemma 2.2 with intermediate $\mathfrak{k}_{\eta_{\mathbb{b}}}$ gives

$$\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}}) \preceq 2\kappa\,\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}}}) + \kappa^2\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}}). \tag{3.8}$$

Taking lim sup in (3.8) using the limit assumption and (i) yields

$$\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}}) \preceq \epsilon\kappa^4. \tag{3.9}$$

Combining (3.7)–(3.9) gives (ii).

For (iii), apply Lemma 2.2 twice to estimate $\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})$ via $\mathfrak{k}_{\xi_{\mathbb{b}-1}}$ and then $\mathfrak{k}_{\eta_{\mathbb{b}+1}}$:

$$\begin{aligned}\epsilon &\prec \varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\\ &\preceq 4\kappa^2\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) + 2\kappa^2\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) + \kappa^2\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}}}).\end{aligned} \tag{3.10}$$

Taking lim inf in (3.10) yields

$$\frac{\epsilon}{2\kappa^2} \preceq \liminf_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}). \tag{3.11}$$

Finally, apply Lemma 2.2 to $\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}})$ with intermediate $\mathfrak{k}_{\xi_{\mathbb{b}}}$:

$$\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) \preceq 2\kappa\,\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}}}) + \kappa\,\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}}). \tag{3.12}$$

Taking lim sup in (3.12) using the limit assumption and (ii) gives

$$\limsup_{\mathbb{b}\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) \preceq \epsilon\kappa^5. \tag{3.13}$$

Combining (3.11)–(3.13) yields (iii). □

Theorem 3.1. *Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a complete C^*-AV-S_bMS and $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ be an $\mathbb{C}_\alpha^{(\varphi,\psi)}$-suzuki type weak contraction with $i=5$ or 4 or 3 such that the given hypotheses hold:*

(i) *$\mathbb{Q}$ is a α-adm;*

(ii) *$\exists\ \mathfrak{k}_0\in\mathcal{G}$ such that $\alpha(\mathfrak{k}_0,\mathfrak{k}_0,\mathbb{Q}\mathfrak{k}_0)\succeq 1_{\mathscr{A}}$;*

(iii) *For sequences $\{\mathfrak{k}_\xi\}\subseteq\mathcal{G}$, if $\alpha(\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_{\xi+1})\succeq 1_{\mathscr{A}}$, for all ξ, and $\lim_{\xi\to\infty}\mathfrak{k}_\xi=\mathfrak{k}\in\mathcal{G}$ then $\alpha(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k})\succeq 1_{\mathscr{A}}$;*

(iv) *$\mathbb{Q}$ is continuous or orbitally continuous on $\mathcal{G}$ with respect to $\mathscr{A}$.*

Then $\varkappa\in\mathcal{G}$ constitutes a fixed point (FP) of $\mathbb{Q}$. Moreover, if $\mathbb{Q}$ is α^*-admissible, then $\varkappa\in\mathcal{G}$ serves as the unique fixed point (UFP) of $\mathbb{Q}$. Furthermore, for any initial point $\mathfrak{k}_0\in\mathcal{G}$ satisfying $\mathfrak{k}_{\xi+1}=\mathbb{Q}^{\xi+1}(\mathfrak{k}_0)\neq\mathbb{Q}(\mathfrak{k}_\xi)$ for all $\xi\geq 0$, the iterates converge: $\lim_{\xi\to\infty}\mathbb{Q}^\xi(\mathfrak{k}_0)=\varkappa$.

Proof. Let $\mathfrak{k}_0\in\mathcal{G}$ be such that $\alpha(\mathfrak{k}_0,\mathfrak{k}_0,\mathbb{Q}\mathfrak{k}_0)\succeq 1_{\mathscr{A}}$ and construct a sequence $\{\mathfrak{k}_\xi\}$ by $\mathfrak{k}_{\xi+1}=\mathbb{Q}\mathfrak{k}_\xi=\mathbb{Q}^{\xi+1}\mathfrak{k}_0$ for all $\xi\geq 0$. If $\mathfrak{k}_\xi=\mathfrak{k}_{\xi+1}$, it follows that $\mathbb{Q}\mathfrak{k}_{\xi_0}=\mathfrak{k}_{\xi_0}$ for some $\xi\geq 0$, then $\mathfrak{k}_{\xi_0}$ is a FP of $\mathbb{Q}$.

Now, suppose $\mathfrak{k}_\xi\neq\mathfrak{k}_{\xi+1}$, for all $\xi\geq 0$. Then $\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})\succ 0_{\mathscr{A}}$, for all $\xi\geq 0$. Since, $\mathbb{Q}$ is a α-admissible, then $\alpha(\mathfrak{k}_0,\mathfrak{k}_0,\mathfrak{k}_1)=\alpha(\mathfrak{k}_0,\mathfrak{k}_0,\mathbb{Q}\mathfrak{k}_0)\succeq 1_{\mathscr{A}}$ implies $\alpha(\mathfrak{k}_1,\mathfrak{k}_1,\mathfrak{k}_2)=\alpha(\mathbb{Q}\mathfrak{k}_0,\mathbb{Q}\mathfrak{k}_0,\mathbb{Q}^2\mathfrak{k}_0)\succeq 1_{\mathscr{A}}$. By repeating similar process, we obtain inductively that $\alpha(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})\succeq 1_{\mathscr{A}}$, for all $\xi\geq 0$. Therefore,

$$\alpha(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})=\alpha(\mathbb{Q}^\xi\mathfrak{k}_0,\mathbb{Q}^\xi\mathfrak{k}_0,\mathbb{Q}^{\xi+1}\mathfrak{k}_0)\succeq 1_{\mathscr{A}},\quad \text{for all } \xi\geq 0.$$

Since,

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi)=\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1})\prec\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_{\xi+1}).$$

Now, by eq. (3.1) with $i=5$, we get

$$\begin{aligned}\varphi(\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})) &= \varphi(\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_{\xi},\mathbb{Q}\mathfrak{k}_{\xi},\mathbb{Q}\mathfrak{k}_{\xi+1}))\\ &\preceq \varphi(\alpha(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_{\xi},\mathbb{Q}\mathfrak{k}_{\xi},\mathbb{Q}\mathfrak{k}_{\xi+1}))\\ &\preceq \psi\left(a^*\max\left\{\frac{1}{4\kappa^7}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}),\frac{1}{4\kappa^7}\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2}),\right.\right.\\ &\qquad \left.\left.\frac{1}{4\kappa^7}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+2})\right\}a\right). \end{aligned} \tag{3.14}$$

But

$$\begin{aligned}\frac{1}{4\kappa^7}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+2}) &\preceq \frac{1}{4\kappa^7}[2\kappa\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})+\kappa^2\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})]\\ &\preceq \max\left\{\frac{1}{\kappa^6}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}),\frac{1}{2\kappa^5}\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\right\}.\end{aligned}$$

Hence, from (3.14), we get

$$\varphi(\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})) \preceq \psi\left(a^*\max\left\{\frac{1}{\kappa^6}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}),\frac{1}{2\kappa^5}\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\right\}a\right). \tag{3.15}$$

Since φ and ψ are continuous functions, thus

$$\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2}) \preceq a^*\max\left\{\frac{1}{\kappa^6}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}),\frac{1}{2\kappa^5}\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\right\}a.$$

If $\frac{1}{2\kappa^5}\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})$ is maximum, since $\|a\|<1$ and $\|\kappa\|>1$, then, we have

$$\begin{aligned}\|\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\| &\le \frac{\|a\|^2}{2\|\kappa\|^5}\|\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\|\\ &< \|\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\|\end{aligned}$$

which is contradiction. Hence, $\|\varsigma_\kappa(\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+1},\mathfrak{k}_{\xi+2})\| < \|\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})\|$.

Similarly, by proceeding as above, we get $\|\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})\| < \|\varsigma_\kappa(\mathfrak{k}_{\xi-1},\mathfrak{k}_{\xi-1},\mathfrak{k}_{\xi})\|$. Thus, we get a sequence of strictly non-increasing functions such that for any $\delta \succeq 0_{\mathscr{A}}$, we have $\lim\limits_{\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})=\delta$. Suppose $\delta \succ 0_{\mathscr{A}}$, then on taking $\lim\limits_{\xi\to\infty}$ in eq. (3.15), we get

$$\varphi(\delta) \preceq \psi\left(a^*\frac{\delta}{\kappa^6}a\right)$$

implies

$$\|\delta\| \le \frac{\|a\|^2}{\|\kappa\|^6}\|\delta\| < \|\delta\|$$

which is a contradiction and hence,

$$\lim_{\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}) = 0_{\mathscr{A}}.$$

Next, we will show that $\{\mathfrak{k}_\xi\}$ is a *Cauchy Sequence* (CS) in $\mathcal{G}$ with regard to $\mathscr{A}$. Assume on contrary that the sequence $\{\mathfrak{k}_\xi\}$ is not CS, then for an $\epsilon \succ 0_{\mathscr{A}}$ there exists sub-sequences of positive integers $\eta_{\mathbb{b}} > \xi_{\mathbb{b}} > \mathbb{b}$ such that

$$\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}}) \succ \epsilon \quad\text{and}\quad \varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}}) \preceq \epsilon.$$

Also for this $\epsilon \succ 0_{\mathscr{A}}$, the convergence of sequence $\{\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1})\}$ implies that there exists some $N_0 \in \mathbb{N}$ such that $\varsigma_\kappa(\mathfrak{k}_{\xi},\mathfrak{k}_{\xi},\mathfrak{k}_{\xi+1}) \prec \epsilon$. For all $\xi \ge N_0$, let $N_1 = \max\{\eta_i, N_0\}$. Thus, for all

$\eta_{\mathbb{b}} > \xi_{\mathbb{b}} \geq N_1$, we have

$$\begin{aligned}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) &\preceq \epsilon \prec \varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}})\\ &\preceq 2\kappa\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})+\kappa\varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})\\ &\prec 2\kappa\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})+\kappa\epsilon\end{aligned}$$

since $\epsilon \succ 0_{\mathscr{A}}$ is arbitrary, and thus $\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}) \prec \varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})$.

Since, $\alpha(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}}) \succeq 1_{\mathscr{A}}$. Now, by eq. (3.1) with $i=5$, we get

$$\begin{aligned}\varphi(\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}},\mathfrak{k}_{\xi_{\mathbb{b}}})) &= \varphi(\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_{\eta_{\mathbb{b}}},\mathbb{Q}\mathfrak{k}_{\eta_{\mathbb{b}}},\mathbb{Q}\mathfrak{k}_{\xi_{\mathbb{b}-1}}))\\ &\preceq \varphi(\alpha(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}})\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_{\eta_{\mathbb{b}}},\mathbb{Q}\mathfrak{k}_{\eta_{\mathbb{b}}},\mathbb{Q}\mathfrak{k}_{\xi_{\mathbb{b}-1}}))\\ &\preceq \psi\Bigg(\frac{1}{4\kappa^7}\Bigg[a^*\max\Big\{\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}-1}}),\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}+1}}),\\ &\qquad \varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}}}),\varsigma_\kappa(\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\eta_{\mathbb{b}}},\mathfrak{k}_{\xi_{\mathbb{b}}}),\\ &\qquad \varsigma_\kappa(\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\xi_{\mathbb{b}-1}},\mathfrak{k}_{\eta_{\mathbb{b}+1}})\Big\}a\Bigg]\Bigg).\end{aligned}$$

Taking $\lim\limits_{\mathbb{b}\to\infty}$, and on using Lemma 3.1 in above equality, we obtain

$$\begin{aligned}\varphi\left(\frac{\epsilon}{\kappa^2}\right) &\preceq \psi\left(\frac{1}{4\kappa^7}[a^*\max\{\epsilon,0,0,\epsilon\kappa^2,\epsilon\kappa^5\}a]\right)\\ &\preceq \psi\left(a^*\frac{\epsilon}{4\kappa^2}a\right).\end{aligned}$$

Since φ and ψ are continuous functions, $\left\|\frac{\epsilon}{\kappa^2}\right\| \leq \frac{\|a\|^2\|\epsilon\|}{4\|\kappa\|^2} < \left\|\frac{\epsilon}{\kappa^2}\right\|$ which is again a contradiction. Thus our assumption is wrong. Hence, $\{\mathfrak{k}_\xi\}$ is a CS. Since $\mathcal{G}$ is a complete C^*-AV-S_bMS so the sequence $\{\mathfrak{k}_\xi\}$ converges along with all its sub-sequences to some point say $\varkappa$, i.e.,

$$\lim_{\xi\to\infty}\mathbb{Q}\mathfrak{k}_\xi = \varkappa = \lim_{\xi\to\infty}\mathfrak{k}_{\xi+1}$$

and

$$\lim_{\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\varkappa) = \varsigma_\kappa(\varkappa,\varkappa,\varkappa) = \lim_{\eta,\xi\to\infty}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathfrak{k}_\eta) = 0. \tag{3.16}$$

Now, we prove that $\mathbb{Q}$ has a FP $\varkappa$ in $\mathcal{G}$. Now, we claim that

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi) \prec \varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\varkappa) \ \text{ or } \ \frac{1}{2\kappa}\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}^2\mathfrak{k}_\xi) \prec \varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi,\varkappa), \quad \text{for all } \xi\in\mathbb{N}. \tag{3.17}$$

Again, suppose that $\eta\in\mathbb{N}$ such that

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta) \succeq \varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\varkappa) \ \text{ and } \ \frac{1}{2\kappa}\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}^2\mathfrak{k}_\eta) \succeq \varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta,\varkappa). \tag{3.18}$$

On using property (iii) of Definition 2.1, we have

$$\begin{aligned}\varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta) &\preceq 2\kappa\varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\varkappa)+\kappa\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta,\varkappa)\\ &\preceq \varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta)+\frac{1}{2}\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta,\mathbb{Q}^2\mathfrak{k}_\eta)\\ &\preceq \frac{3}{2}\varsigma_\kappa(\mathfrak{k}_\eta,\mathfrak{k}_\eta,\mathbb{Q}\mathfrak{k}_\eta).\end{aligned}$$

This is a contradiction, and hence we have our claim, and thus inequality (3.17) is true, for all $\xi\in\mathbb{N}$. Inequality (3.17) implies that (from eq. (3.1)) with $i=5$ and condition (iii) in hypothesis of theorem, we have

$$\begin{aligned}\varphi(\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\varkappa)) &\preceq \varphi(\alpha(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\varkappa)\varsigma_\kappa(\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi,\mathbb{Q}\varkappa))\\ &\preceq \psi\left(\frac{1}{4\kappa^7}[a^*M^i(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\varkappa)a]\right)\\ &\preceq \psi\left(\frac{1}{4\kappa^7}\left[a^*\max\left\{\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\varkappa),\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathbb{Q}\mathfrak{k}_\xi),\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa),\right.\right.\right.\\ &\qquad\left.\left.\left.\varsigma_\kappa(\mathfrak{k}_\xi,\mathfrak{k}_\xi,\mathbb{Q}\varkappa),\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\mathfrak{k}_\xi)\right\}a\right]\right).\end{aligned} \tag{3.19}$$

Taking $\lim\limits_{\xi\to\infty}$ in above inequality, and using the fact that the maps φ and ψ are continuous, we have

$$\varphi(\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)) \preceq \psi\left(\frac{1}{4\kappa^7}\left[a^*\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)a\right]\right)$$

since $\|a\|<1$ and $\|\kappa\|>1$. It follows that

$$\begin{aligned}\|\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)\| &\le \frac{\|a\|^2}{4\|\kappa\|^7}\|\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)\|\\ &< \|\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)\|\end{aligned}$$

which is a contradiction and only possible $\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)=0_{\mathscr{A}}$. Hence, $\varkappa$ is a FP of $\mathbb{Q}$. Now, assume $\mathbb{Q}$ is orbitally continuous on $\mathcal{G}$ with respect to $\mathscr{A}$, then $\mathfrak{k}_{\xi+1}=\mathbb{Q}\mathfrak{k}_\xi=\mathbb{Q}(\mathbb{Q}^\xi\mathfrak{k}_0)\to\mathbb{Q}\varkappa$ as $\xi\to\infty$. By completeness property, we obtain $\mathbb{Q}\varkappa=\varkappa$. Therefore, $\mathrm{Fix}(\mathbb{Q})\neq\emptyset$. Since, $\mathbb{Q}$ is α^*-adm, so that, for all $\varkappa,\varkappa'\in\mathrm{Fix}(\mathbb{Q})$, we have $\alpha(\varkappa,\varkappa,\varkappa')\succeq 1_{\mathscr{A}}$.

Assume, $\varsigma_\kappa(\mathbb{Q}\varkappa,\mathbb{Q}\varkappa,\mathbb{Q}\varkappa')=\varsigma_\kappa(\varkappa,\varkappa,\varkappa')\succ 0_{\mathscr{A}}$. Then, we have

$$\frac{1}{2\kappa}\varsigma_\kappa(\varkappa,\varkappa,\mathbb{Q}\varkappa)=0_{\mathscr{A}}\prec\varsigma_\kappa(\varkappa,\varkappa,\varkappa').$$

Then from eq. (3.1) with $i=5$, we have

$$\begin{aligned}\varphi(\varsigma_\kappa(\varkappa,\varkappa,\varkappa')) &= \varphi(\varsigma_\kappa(\mathbb{Q}\varkappa,\mathbb{Q}\varkappa,\mathbb{Q}\varkappa'))\\ &\preceq \varphi(\alpha(\varkappa,\varkappa,\varkappa')\varsigma_\kappa(\mathbb{Q}\varkappa,\mathbb{Q}\varkappa,\mathbb{Q}\varkappa'))\\ &\preceq \psi\left(\frac{1}{4\kappa^6}[a^*\varsigma_\kappa(\varkappa,\varkappa,\varkappa')a]\right).\end{aligned}$$

Using the fact that $\|a\|<1$ and the maps φ and ψ are continuous, we have

$$\begin{aligned}\|\varsigma_\kappa(\varkappa,\varkappa,\varkappa')\| &\le \frac{\|a\|^2}{4\|\kappa\|^6}\|\varsigma_\kappa(\varkappa,\varkappa,\varkappa')\|\\ &< \|\varsigma_\kappa(\varkappa,\varkappa,\varkappa')\|\end{aligned}$$

which is a contradiction. Hence, $\varkappa$ in $\mathcal{G}$ is a UFP of $\mathbb{Q}$.

By substituting $i=3$ or $i=4$ into eq. (3.1) and proceeding along similar lines as in the previous part of the proof, we conclude that $\mathbb{Q}$ possesses the UFP. □

Corollary 3.2. *Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a complete C^*-AV-S_bMS, and let $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ be a self-map satisfying*

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathbb{Q}\mathfrak{s})\prec\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa)\implies\varphi(2\kappa^4\varsigma_\kappa(\mathbb{Q}\mathfrak{s},\mathbb{Q}\mathfrak{e},\mathbb{Q}\varkappa))\preceq\psi(a^*\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa)a),$$

where $\varphi,\psi\in\Phi$ and $a\in\mathscr{A}$ with $\|a\|<1$. Then, $\mathbb{Q}$ has a UFP in $\mathcal{G}$.

Proof. The result follows along similar lines as Theorem 3.1, by taking $\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa)$ in place of $M^i(\mathfrak{s},\mathfrak{e},\varkappa)$ and setting $\alpha(\mathfrak{s},\mathfrak{e},\varkappa) = 1_{\mathscr{A}}$ in Theorem 3.1. $\square$

Corollary 3.3. *Let $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ be a complete C^*-AV-S_bMS, and let $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ be a self-map such that*

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathbb{Q}\mathfrak{s}) \prec \varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa) \implies \varsigma_\kappa(\mathbb{Q}\mathfrak{s},\mathbb{Q}\mathfrak{e},\mathbb{Q}\varkappa) \preceq a^*\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa)a,$$

where $a\in\mathscr{A}$ and $\|a\|<1$. Then, the mapping $\mathbb{Q}$ possesses a UFP in $\mathcal{G}$.

Proof. The conclusion is obtained by following arguments similar to those in Corollary 3.2, using the choices $\varphi(\mathfrak{k}) = \psi(\mathfrak{k}) = \mathfrak{k}$ for simplification. $\square$

Example 3.6. Let $\mathscr{A} = \mathbb{R}$ with usual addition and multiplication, regarded as a trivial C^*-algebra, and let $\mathcal{G} = \{1,2,3\}$, a finite non-empty set. Define the function $\varsigma_\kappa:\mathcal{G}^3\to\mathbb{R}_+$ by

$$\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\mathfrak{k}) = |s-e|+|e-k|+|k-s|.$$

This satisfies the properties of a C^*-AV-S_bMS with coefficient $\|\kappa\| = 2 > 1$. Define the mapping $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ by $\mathbb{Q}(s) = 1$, for all $s\in\mathcal{G}$. Define $\alpha:\mathcal{G}^3\to\mathbb{R}_+$ by $\alpha(\mathfrak{s},\mathfrak{e},\mathfrak{k}) = 1$.

Define $\varphi,\psi:\mathbb{R}_+\to\mathbb{R}_+$ by $\varphi(x) = x$, $\quad\psi(x) = \frac{1}{2}x$. Take $a = \frac{1}{2}$, which satisfies $\|a\| = 0.5 < 1$. Hence, by Theorem 3.1, $\mathbb{Q}$ has a UFP $\varkappa = 1$, and for any $\mathfrak{k}_0\in\mathcal{G}$, $\lim\limits_{n\to\infty}\mathbb{Q}^n\mathfrak{k}_0 = 1$.

Example 3.7. Let $\mathcal{G} = [0,\infty)$, $\mathscr{A} = M_2(\mathbb{R})$, the algebra of 2×2 real matrices, with the operator norm $\|\cdot\|$. Define the metric function $d:\mathcal{G}^2\to[0,\infty)$ as $d(\mathfrak{s},\varkappa) = (\mathfrak{s}-\varkappa)^2$, making $(\mathcal{G},d)$ a b-MS with constant $\kappa = 2$.

Define $\varsigma_\kappa:\mathcal{G}^3\to M_2(\mathbb{R})$ by $\varsigma_\kappa(\mathfrak{s},\mathfrak{e},\varkappa) = \begin{bmatrix} d(\mathfrak{s},\mathfrak{e})+d(\mathfrak{e},\varkappa)+d(\varkappa,\mathfrak{s}) & 0 \\ 0 & d(\mathfrak{s},\mathfrak{e})+d(\mathfrak{e},\varkappa)+d(\varkappa,\mathfrak{s}) \end{bmatrix}$.

This makes $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ a complete C^*-AV-S_bMS with $\|\kappa\| = 2\geq 1$. Define $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ as $\mathbb{Q}(\mathfrak{s}) = \frac{b\mathfrak{s}}{3+\mathfrak{s}}$, where $b\in(0,1/6]$. Define $\varphi,\psi\in\Phi$ entry-wise by $\varphi(\mathfrak{k}) = \frac{1}{1+\|\mathfrak{k}\|}$, $\psi(\mathfrak{k}) = \frac{1}{2}\mathfrak{k}$. Choose $a = \frac{1}{2}I_2$, with $\|a\| = 1/2 < 1$. Hence, by Corollary 3.2 $\mathbb{Q}$ has a UFP at $\mathfrak{s} = 0$.

4. Applications

4.1 Application to Nonlinear Integral Equations

Let $\mathcal{E} = [0,1]$ be a measurable set with finite measure and $\mathcal{G} = L^\infty(\mathcal{E})$. Consider the space $B(L^2(\mathcal{E}))$ of bounded linear operators on the Hilbert space $L^2(\mathcal{E})$. Equip $\mathcal{G}$ with the C^*-algebra-valued S_b-metric $\varsigma_\kappa:\mathcal{G}\times\mathcal{G}\times\mathcal{G}\to B(L^2(\mathcal{E}))$ defined by $\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa) = \mathbb{M}_{(|\mathfrak{z}-\varkappa|+|\mathfrak{e}-\varkappa|)^p}$, where $\mathbb{M}_h$ denotes the multiplication operator by the function $h\in L^\infty(\mathcal{E})$ on $L^2(\mathcal{E})$, $p>1$, and $\kappa = 2^{2(p-1)}$. Consider the nonlinear integral equation $\mathfrak{z}(s) = \int_0^1 w(s,t)\mathfrak{f}(t,\mathfrak{z}(t))dt$, $s\in\mathcal{E}$, subject to these conditions:

(i) The kernel $w:\mathcal{E}\times\mathcal{E}\to\mathbb{R}_+$ is measurable with $W = \sup\limits_{s\in\mathcal{E}}\int_0^1 w(s,t)dt < \infty$.

(ii) The function $\mathfrak{f}:\mathcal{E}\times\mathbb{R}\to\mathbb{R}$ satisfies the Lipschitz condition $|\mathfrak{f}(t,a)-\mathfrak{f}(t,b)|\leq L|a-b|$, for all $t\in\mathcal{E}$, $a,b\in\mathbb{R}$, where $L\in(0,1/W)$.

Define the operator $\mathbb{Q}:\mathcal{G}\to\mathcal{G}$ by $\mathbb{Q}\mathfrak{z}(s) := \int_0^1 w(s,t)\mathfrak{f}(t,\mathfrak{z}(t))dt$.

We explicitly choose the control functions $\varphi,\psi\in\Phi$ as linear scalings on $B(L^2(\mathcal{E}))_+$ as $\varphi(X) = c_1X$, $\quad\psi(X) = c_2X$, where $0<c_1<c_2<1$. Let $a = \sqrt{LW}\cdot 1_{B(L^2(\mathcal{E}))}$, so $\|a\| = \sqrt{LW} < 1$.

Theorem 4.1. *Under the above conditions, the operator* $\mathbb{Q}$ *satisfies the contractive condition of Corollary* 3.2*:*

$$\frac{1}{2\kappa}\varsigma_\kappa(\mathfrak{z},\mathfrak{z},\mathbb{Q}\mathfrak{z}) \prec \varsigma_\kappa(\mathfrak{z},\mathfrak{z},\varkappa) \implies \varphi(2\kappa^4\varsigma_\kappa(\mathbb{Q}\mathfrak{z},\mathbb{Q}\mathfrak{z},\mathbb{Q}\varkappa)) \preceq \psi(a^*\varsigma_\kappa(\mathfrak{z},\mathfrak{z},\varkappa)a).$$

Consequently, $\mathbb{Q}$ *has a unique fixed point* $\mathfrak{z}^* \in \mathcal{G}$*, which solves the integral equation.*

Proof. For any $\mathfrak{z},\varkappa \in \mathcal{G}$, and $s \in \mathcal{E}$, apply the Lipschitz condition and kernel bound:

$$|\mathbb{Q}\mathfrak{z}(s)-\mathbb{Q}\varkappa(s)| \leq L\int_0^1 w(s,t)|\mathfrak{z}(s)-\varkappa(s)|dt.$$

Using the C^*-algebra valued S_b-metric definition, $\varsigma_\kappa(\mathbb{Q}\mathfrak{z},\mathbb{Q}\mathfrak{z},\mathbb{Q}\varkappa) = \mathbb{M}_{(|\mathbb{Q}\mathfrak{z}-\mathbb{Q}\varkappa|+|\mathbb{Q}\mathfrak{z}-\mathbb{Q}\varkappa|)^p}$. Then, we have $\|\varsigma_\kappa(\mathbb{Q}\mathfrak{z},\mathbb{Q}\mathfrak{z},\mathbb{Q}\varkappa)\| \leq (LW)^p\|\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa)\|$.

Multiplying by $2\kappa^4$ and applying φ: $\|\varphi(2\kappa^4\varsigma_\kappa(\mathbb{Q}\mathfrak{z},\mathbb{Q}\mathfrak{e},\mathbb{Q}\varkappa))\| = c_1 2\kappa^4(LW)^p\|\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa)\|$. On the right side, $\|\psi(a^*\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa)a)\| = c_2\|a\|^2\|\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa)\| = c_2(LW)\|\varsigma_\kappa(\mathfrak{z},\mathfrak{e},\varkappa)\|$. Choosing c_1, c_2, p, and κ such that $c_1 2\kappa^4(LW)^p \leq c_2(LW)$ guarantees the contractive inequality.
Thus, all conditions of Corollary 3.2 hold, and $\mathbb{Q}$ has a unique fixed point. □

4.2 Application to Homotopy

In this section, we investigate whether homotopy could have a unique solution.

Theorem 4.2. *Let* $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ *be a complete* C^**-AV-*S_b*MS with* $\|\kappa\| > 1$. *Suppose* $\mathbb{U} \subseteq \overline{\mathbb{U}}$ *where* $\mathbb{U}$ *is open and* $\overline{\mathbb{U}}$ *is closed in* $\mathcal{G}$. *Consider a homotopy* $\mathrm{T}: \overline{\mathbb{U}} \times [0,1] \to \mathcal{G}$ *satisfying:*

(τ_0) $\mathrm{T}(\nu,s) \neq \nu$*, for all* $\nu \in \partial\mathbb{U}$*,* $s \in [0,1]$*,*

(τ_1) $\exists\ \nu,\varkappa \in \overline{\mathbb{U}}$*,* $\mathfrak{b} \in \mathscr{A}$ *with* $\|\mathfrak{b}\| < 1$ *such that*

$$\frac{1}{2\kappa}\varsigma_\kappa(\nu,\nu,\mathrm{T}(\nu,s)) \prec \varsigma_\kappa(\nu,\nu,\varkappa) \implies 2\kappa\varsigma_\kappa(\mathrm{T}(\nu,s),\mathrm{T}(\nu,s),\mathrm{T}(\varkappa,s)) \preceq \mathfrak{v}^*\varsigma_\kappa(\nu,\nu,\varkappa)\mathfrak{v},$$

(τ_2) $\exists\ M \succeq 0_{\mathscr{A}}$ *such that* $\varsigma_\kappa(\mathrm{T}(\nu,s),\mathrm{T}(\nu,s),\mathrm{T}(\nu,t)) \leq \|M\||s-t|$*, for all* $\nu \in \overline{\mathbb{U}}$*,* $s,t \in [0,1]$.

Then $\mathrm{T}(\cdot,0)$ *has a FP if and only if* $\mathrm{T}(\cdot,1)$ *does.*

Proof. Consider the set $\mathscr{G} = \{\mathfrak{k} \in [0,1] : \exists\ \nu \in \mathbb{U}$ such that $\nu = \mathrm{T}(\nu,\mathfrak{k})\}$.

Since $\mathrm{T}(\cdot,0)$ admits a FP in $\mathbb{U}$, it follows that $0 \in \mathscr{G}$, so $\mathscr{G} \neq \emptyset$. We establish $\mathscr{G} = [0,1]$ by showing $\mathscr{G}$ is both open and closed in $[0,1]$, implying $\mathrm{T}(\cdot,1)$ has a FP in $\mathbb{U}$.

First, verify $\mathscr{G}$ is closed. Let $\{\mathfrak{k}_\xi\}_{\xi=1}^\infty \subset \mathscr{G}$ with $\mathfrak{k}_\xi \to \mathfrak{k} \in [0,1]$. For each ξ, there exists $\nu_\xi \in \mathbb{U}$ satisfying $\nu_{\xi+1} = \mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)$. The proof succeeds if some $\xi \in \mathbb{N}$ exists such that $\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)) = 0_{\mathscr{A}}$.

Since $\frac{1}{2\kappa}\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)) \prec \varsigma_\kappa(\nu_\xi,\nu_\xi,\nu_{\xi+1})$ implies that

$$\begin{aligned}\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_{\xi+1})) &\preceq 2\kappa\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_\xi)) \\ &\quad + \kappa\varsigma_\kappa(\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_{\xi+1}),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_{\xi+1}),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_\xi)). \\ &\preceq 2\kappa\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_\xi)) + \|M\||\mathfrak{k}_{\xi+1}-\mathfrak{k}_\xi|.\end{aligned}$$

Letting $\xi \to \infty$, we get

$$\begin{aligned}\lim_{\xi\to\infty}\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_{\xi+2}) &\preceq \lim_{\xi\to\infty} 2\kappa\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_\xi)) + 0_{\mathscr{A}} \\ &\preceq \lim_{\xi\to\infty}\mathfrak{v}^*\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu_{\xi+1})\mathfrak{v} \\ &\ \ \vdots\end{aligned}$$

$$\preceq \lim_{\xi\to\infty}(\mathfrak{v}^*)^{\xi+1}\varsigma_\kappa(\nu_0,\nu_0,\nu_1)(\mathfrak{v})^{\xi+1}$$

which, together with the property, if æ, œ $\in \mathscr{A}_h$ and æ $\preceq$ œ implies u^*æ$u \preceq u^*$œu, yields that for each $\xi \in N\cup\{0\}$, put $\Delta_\xi = \varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_{\xi+2})$, we have

$$0_{\mathscr{A}} \preceq \lim_{\xi\to\infty}\Delta_\xi = \lim_{\xi\to\infty}\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_{\xi+2}) \preceq \lim_{\xi\to\infty}(\mathfrak{v}^*)^{\xi+1}\Delta_0(\mathfrak{v})^{\xi+1}.$$

Since $\|\mathfrak{v}\|<1$, it follows that $\lim_{\xi\to\infty}\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_{\xi+2}) = 0_{\mathscr{A}}$.

Next, we show that $\{\nu_\xi\}$ is a CS with respect to $\mathscr{A}$. Now for any $\zeta\in\mathbb{N}$ and $\xi\in\mathbb{N}$ using condition (iii) of Definition 2.1, we have

$$\begin{aligned}\varsigma_\kappa(\nu_{\xi+\zeta},\nu_{\xi+\zeta},\nu_\xi) &\preceq 2\kappa\varsigma_\kappa(\nu_{\xi+\zeta},\nu_{\xi+\zeta},\nu_{\xi+\zeta-1}) + 2\kappa^3\varsigma_\kappa(\nu_{\xi+\zeta-1},\nu_{\xi+\zeta-1},\nu_{\xi+\zeta-2})\\ &\quad + 2\kappa^5\varsigma_\kappa(\nu_{\xi+\zeta-2},\nu_{\xi+\zeta-2},\nu_{\xi+\zeta-3}) + \cdots + 2\kappa^\zeta\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_\xi)\\ &\preceq 2\kappa\sum_{i=\xi}^{\xi+\zeta-1}\kappa^{\zeta-i+\xi}\varsigma_\kappa(\nu_i,\nu_i,\nu_{i+1}).\end{aligned}$$

Letting $\xi\to\infty$, we get

$$\begin{aligned}\lim_{\xi\to\infty}\varsigma_\kappa(\nu_{\xi+\zeta},\nu_{\xi+\zeta},\nu_\xi) &\preceq \lim_{\xi\to\infty}2\kappa\sum_{i=\xi}^{\xi+\zeta-1}\kappa^{\zeta-i+\xi}\varsigma_\kappa(\nu_i,\nu_i,\nu_{i+1})\\ &\preceq \lim_{\xi\to\infty}2\kappa\sum_{i=\xi}^{\xi+\zeta-1}\kappa^{\zeta-i+\xi}(\mathfrak{v}^\star)^i\Delta_0^{\frac12}\Delta_0^{\frac12}(\mathfrak{v})^i\\ &\preceq \lim_{\xi\to\infty}2\kappa\|\Delta_0^{\frac12}\|^2\frac{\|\kappa\|^{\zeta+1}\|\mathfrak{v}\|^{2\xi}}{\|\kappa\|-\|\mathfrak{v}\|}1_{\mathscr{A}} = 0_{\mathscr{A}}.\end{aligned}$$

Hence, the sequence $\{\nu_\xi\}$ is a C^*-AV-S_b-CS with respect to $\mathscr{A}$. The sequence $\{\nu_\xi\}\to\nu\in(\mathcal{G},\mathscr{A},\varsigma_\kappa)$ comes from the completeness of $(\mathcal{G},\mathscr{A},\varsigma_\kappa)$. $\lim_{\xi\to\infty}\nu_{\xi+1} = \nu = \lim_{\xi\to\infty}\nu_\xi$. We can prove $\mathrm{T}(\nu,\mathfrak{k}) = \nu$. Now, we claim that

$$\frac{1}{2\kappa}\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)\prec\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu) \ \text{ or } \ \frac{1}{2\kappa}\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_{\xi+1})\prec\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu).$$

Suppose above inequality is not true for every $\xi\in\mathbb{N}$. Therefore for some $\xi\ge1$, we have

$$\frac{1}{2\kappa}\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)\succeq\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu) \ \text{ or } \ \frac{1}{2\kappa}\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\mathrm{T}(\nu_{\xi+1},\mathfrak{k}_{\xi+1})\succeq\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu).$$

On using property (iii) of Definition 2.1, we have

$$\begin{aligned}\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)) &\preceq 2\kappa\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu) + \kappa\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\nu)\\ &\preceq \varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi) + \frac12\varsigma_\kappa(\nu_{\xi+1},\nu_{\xi+1},\nu_{\xi+2}).\end{aligned}$$

Letting $\xi\to\infty$, we get

$$\begin{aligned}\lim_{\xi\to\infty}\|\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi))\| &\preceq \lim_{\xi\to\infty}\|\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi))\| + \lim_{\xi\to\infty}\frac{\|\mathfrak{v}\|^2}{2}\|\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi))\|\\ &\preceq \lim_{\xi\to\infty}\left(1+\frac{\|\mathfrak{v}\|^2}{2}\right)\|\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi))\|\end{aligned}$$

which is a contradiction, and hence we have our claim, and so above inequality is true, for all $\xi\in\mathbb{N}$. Since, $\frac{1}{2\kappa}\varsigma_\kappa(\nu_\xi,\nu_\xi,\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi)\prec\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu)$, from (τ_1), we have

$$\varsigma_\kappa(\nu,\nu,\mathrm{T}(\nu,\mathfrak{k}))\preceq\lim_{\xi\to\infty}2\kappa\varsigma_\kappa(\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu_\xi,\mathfrak{k}_\xi),\mathrm{T}(\nu,\mathfrak{k}))$$

$$\preceq \lim_{\xi\to\infty} \mathfrak{v}^*\varsigma_\kappa(\nu_\xi,\nu_\xi,\nu)\mathfrak{v} = 0_{\mathscr{A}}.$$

Accordingly, $\mathrm{T}(\nu,\mathfrak{k}) = \nu$ indicates that $\varsigma_\kappa(\nu,\nu,\mathrm{T}(\nu,\mathfrak{k})) = 0_{\mathscr{A}}$. So, $\mathfrak{k}$ in $\mathscr{G}$. It is obvious that $\mathscr{G}$ is closed in $[0,1]$. Let $\mathscr{G}$ be $\mathfrak{k}_0$. Then, ν_0 exists in $\mathbb{U}$ such that $\nu_0 = \mathrm{T}(\nu_0,\mathfrak{k}_0)$. Because $\mathbb{U}$ is open, $\delta > 0$ must exist for $B_{\varsigma_\kappa}(\nu_0,\delta) \subseteq \mathbb{U}$. Select the value of $\mathfrak{k} \in (\mathfrak{k}_0-\epsilon,\mathfrak{k}_0+\epsilon)$ such that $|\mathfrak{k}-\mathfrak{k}_0| \le \frac{1}{\|M\|^\xi} < \epsilon$. Consequently, for $\overline{B_b(\nu_0,\delta)} = \{\nu \in \mathcal{G} : \|\varsigma_\kappa(\nu,\nu,\nu_0)\| \le \delta + \|\varsigma_\kappa(\nu_0,\nu_0,\nu_0)\|\}$.

Since, $\frac{1}{2\kappa}\varsigma_\kappa(\nu,\nu,\mathrm{T}(\nu_0,\mathfrak{k}_0)) \prec \varsigma_\kappa(\nu,\nu,\nu_0)$ then from (τ_1), we have

$$\begin{aligned}\kappa\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu_0,\mathfrak{k}_0)) &\preceq 2\kappa\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu_0,\mathfrak{k}_0))\\ &\preceq \mathfrak{v}^*\varsigma_\kappa(\nu,\nu,\nu_0)\mathfrak{v}.\end{aligned}$$

Now from above inequality, we see that

$$\begin{aligned}\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}),\nu_0)) &= \varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu_0,\mathfrak{k}_0))\\ &\preceq 2\kappa\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}_0)) + \kappa\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu,\mathfrak{k}_0),\mathrm{T}(\nu_0,\mathfrak{k}_0))\\ &\preceq 2\kappa\|M\|\,|\mathfrak{k}-\mathfrak{k}_0| + \mathfrak{v}^*\varsigma_\kappa(\nu,\nu,\nu_0)\mathfrak{v}\\ &\preceq \frac{2\kappa}{\|M\|^{\xi-1}} + \mathfrak{v}^*\varsigma_\kappa(\nu,\nu,\nu_0)\mathfrak{v}.\end{aligned}$$

Taking the limit as $\xi \to \infty$ yields

$$\|\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}),\nu_0)\| \le \|\mathfrak{v}\|^2\|\varsigma_\kappa(\nu,\nu,\nu_0)\|.$$

Since $\|\mathfrak{v}\| < 1$, it follows that

$$\begin{aligned}\|\varsigma_\kappa(\mathrm{T}(\nu,\mathfrak{k}),\mathrm{T}(\nu,\mathfrak{k}),\nu_0)\| &< \|\varsigma_\kappa(\nu,\nu,\nu_0)\|\\ &\le \delta + \|\varsigma_\kappa(\nu_0,\nu_0,\nu_0)\|.\end{aligned}$$

Therefore, for each fixed $\mathfrak{k} \in (\mathfrak{k}_0-\epsilon,\mathfrak{k}_0+\epsilon)$, the operator $\mathrm{T}(\cdot,0)$ maps the closed ball $\overline{B_b(\nu_0,\delta)}$ into itself. Hence, all conditions of Theorem 4.2 are satisfied, guaranteeing the existence of a FP of $\mathrm{T}(\cdot,0)$ in $\overline{\mathbb{U}}$. Since this FP must lie in $\mathbb{U}$, it follows that $(\mathfrak{k}_0-\epsilon,\mathfrak{k}_0+\epsilon) \subseteq \mathscr{G}$, showing that $\mathscr{G}$ is open in $[0,1]$. A parallel argument can establish that $\mathscr{G}$ is closed. □

5. Conclusion

This paper presents UFPT for α-admissible self-mappings in C^*-AV-S_bMS by introducing $\mathbb{C}_\alpha^{(\varphi,\psi)}$-Suzuki contractions with two control functions φ and ψ. The results provide sufficient conditions guaranteeing existence and uniqueness of FPs, extending known theorems in these settings. Applications to nonlinear integral equations and homotopy problems demonstrate the robustness and versatility of the approach, highlighting potential uses in nonlinear analysis and operator theory.

Nomenclature

C^*-AV-S_bMS	:	C^*-Algebra-Valued S_b-Metric Spaces
UFP	:	Unique Fixed Point
UFPT	:	Unique Fixed Point Theorem
FPs	:	Fixed Points
FP	:	Fixed Point
CS	:	Cauchy Sequence

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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