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Research Article

A Novel Numerical Scheme for the Solution of a System of First-Order Delay Differential Equations

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Abstract. In this paper, an efficient finite difference approach is introduced to solve the system of first-order delay differential equations. The method used for this approach is based on integral identities for the quadrature formula with integral term remainder terms, and using this method a novel difference scheme is constructed. Then, first-order convergence for the method in the discrete maximum norm is proved. Finally, a test problem is presented that is solved using both the proposed and classical Euler methods, which support the theoretical findings. Considering these results, this scheme has greater efficiency and accuracy than the classical Euler scheme, although it has the same convergence rate. So, these results show that the proposed method is reliable, efficient and accurate.

Keywords. System of delay differential equation, Finite difference method, Convergence

Mathematics Subject Classification (2020). 34K06, 65L03, 65L12, 65L20, 65L70

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1. Introduction

Systems of Delay Differential Equations (SDDEs) appear in various scientific areas, for instance, physics, engineering, medicine, biology, and so on. They can model many processes of various problems, such as marine protected areas, B-cell chronic lymphocytic leukemia dynamics, air-to-fuel ratio control systems, etc. (Gilsinn *et al*. [11], Kuang [16], and Rihan [21]). In particular, time-delay systems are modelled, such as the dynamic properties of lasers, vehicular traffic flows, neural networks, mechanisms of biological processes and epidemics, and population dynamics (Briat [5], Chen *et al*. [7], Fridman [10], Nelson and Perelson [18]). For instance, the model of fuel combustion in small rockets is described by the equations:

$$x_1'(t)+(1-\gamma)x_1(t)+\gamma x_1(t-\tau)-x_2(t-\tau)=f_1(t),$$
$$x_2'(t)+\alpha x_1(t)+\alpha k x_2(t)=f_2(t),$$

where $x_1(t)$ is a pressure in the combustion chamber and $x_2(t)$ is a consumption of the fuel (for more details, see Kolmanovskii and Myshkis [15]).

In recent years, many approaches to SDDEs have been proposed, which are analytical or numerical solutions. Researchers studied and improved some methods for getting the exact solution except for classic methods such as the continuous Runge-Kutta method (Bellen and Zennaro [3], and Lenz *et al.* [17]), collocation method with First Boubaker polynomials (Davaeifar and Rashidinia [9]), optimization iterations (He *et al.* [12]), *Chebyshev-Picard Iteration* (CPI) method (Zhou *et al.* [23]), spline approximation method [20]. Also, Szczelina and Zgliczyński [22] developed an algorithm for the approximate solution of SDDEs with several delays, which is based on a piecewise Taylor representation of the solutions in the phase space. On the other hand, Amster and Deboli [2], Berezansky and Braverman [4], Cahlon and Schmidt [6], Hu and Wang [13], Huang *et al.* [14], and Pepe *et al.* [19] investigate the existence, uniqueness, and stability of solutions to SDDEs.

Motivated by the above works, we are interested in solving the following problem of the linear system of delay differential equations:

$$Lu_1 \equiv u_1'(t)+a_1(t)u_1(t)+b_1(t)u_1(t-r)+c_1(t)u_2(t)+d_1(t)u_2(t-r)=f_1(t), \quad t\in I, \tag{1.1}$$
$$Lu_2 \equiv u_2'(t)+a_2(t)u_2(t)+b_2(t)u_2(t-r)+c_2(t)u_1(t)+d_2(t)u_1(t-r)=f_2(t), \quad t\in I, \tag{1.2}$$
$$u_1(t)=\varphi_1(t);\ u_2(t)=\varphi_2(t), \quad t\in I_0, \tag{1.3}$$

where $I=(0,T]=\bigcup_{p=1}^{m} I_p$, $I_p=\{t: r_{p-1}<t\le r_p\}$, $1\le p\le m$ and $r_s=sr$, $0\le s\le m$ (for simplicity, it is supposed that T/r is an integer; i.e., $T=mr$). Moreover, $a_k(t)\ge\alpha_k>0$, $b_k(t)$, $c_k(t)$, $d_k(t)$, $f_k(t)$ and $\varphi_k(t)$ are assumed to be continuous in $\bar{I}=[0,T]$ and $I_0=[-r,0]$, respectively, such that (1.1)-(1.3) has a unique solution $u_k(t)\in C^1(\bar{I})$ satisfying the given initial conditions, r is a constant large delay ($k=1,2$) (more details, see Bellen and Zennaro [3]).

This article presents an efficient numerical approach to solve the equation (1.1)-(1.3). The method is based on a finite difference scheme on a uniform mesh, using appropriate quadrature formulas with weights whose remainder terms are in integral form. The most important advantage of this method over the classical method is that the remaining term contains only the first-order derivative of the unknown function.

The organization of the paper is as follows: In Section 2, some properties for the exact solution of (1.1)-(1.3) and its derivative are given. The difference scheme is constructed on a uniform mesh in Section 3. Next, the error bound and stability analysis are discussed in Section 4. In Section 5, the difference scheme is formulated with an algorithm, and an example is presented to illustrate the theoretical results. In the last section, the main conclusions are summarized.

2. The Behavior of the Exact Solution

In this section, we present a few properties of the solutions to (1.1)-(1.3), which are needed in the following sections for the analysis of appropriate numerical solutions. Moreover, for any continuous function $w(t)$, we use

$$\|w\|_\infty = \|w\|_{\infty,\bar{I}} = \max_{0\le t\le T}|w(t)| \text{ and } \|w\|_{\infty,p} = \|w\|_{\infty,I_p}, \quad 1\le p\le m.$$

C denotes a generic positive constant throughout the paper. Also, $C_p, \widetilde{C}_p, D_p$ and $\widetilde{D}_p$ indicate fixed constants for I_p subintervals.

Lemma 2.1. *Let* $a_k, b_k, c_k, d_k, f_k \in C(\bar{I})$, *for* $k=1,2$ *and*

$$\lambda_p := \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,p}\|c_2\|_{\infty,p} < 1, \quad 1\le p\le m.$$

Then for the solution $u_k(t)$ *of the* (1.1)-(1.3), *the following estimates hold:*

$$\|u_1\|_{\infty,p} \le C_p, \quad 1\le p\le m, \tag{2.1}$$

$$\|u_2\|_{\infty,p} \le \widetilde{C}_p, \quad 1\le p\le m, \tag{2.2}$$

$$\|u_1'\|_{\infty,p} \le D_p, \quad 1\le p\le m, \tag{2.3}$$

$$\|u_2'\|_{\infty,p} \le \widetilde{D}_p, \quad 1\le p\le m, \tag{2.4}$$

where

$$\begin{aligned}
C_1 &= \{|\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}] \\
&\quad + \alpha_1^{-1}\|c_1\|_{\infty,1}[|\varphi_2(0)| + \alpha_2^{-1}(\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} \\
&\quad + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0})]\}(1-\lambda_1)^{-1}, \\
\widetilde{C}_1 &= \{|\varphi_2(0)| + \alpha_2^{-1}[\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}] \\
&\quad + \alpha_2^{-1}\|c_2\|_{\infty,1}[|\varphi_1(0)| + \alpha_1^{-1}(\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} \\
&\quad + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0})]\}(1-\lambda_1)^{-1}, \\
C_p &= \{|u_1(r_{p-1})| + \alpha_1^{-1}[\|f_1\|_{\infty,p} + \|b_1\|_{\infty,p}C_{p-1} + \|d_1\|_{\infty,1}\widetilde{C}_{p-1}] \\
&\quad + \alpha_1^{-1}\|c_1\|_{\infty,p}[|u_2(r_{j-1})| + \alpha_2^{-1}(\|f_2\|_{\infty,p} + \|b_2\|_{\infty,p}\widetilde{C}_{p-1} \\
&\quad + \|d_2\|_{\infty,p}C_{p-1})]\}(1-\lambda_p)^{-1}, \\
\widetilde{C}_p &= \{|u_2(r_{p-1})| + \alpha_2^{-1}[\|f_2\|_{\infty,p} + \|b_2\|_{\infty,p}\widetilde{C}_{p-1} + \|d_2\|_{\infty,1}C_{p-1}] \\
&\quad + \alpha_2^{-1}\|c_2\|_{\infty,p}[|u_1(r_{j-1})| + \alpha_1^{-1}(\|f_1\|_{\infty,p} + \|b_1\|_{\infty,p}C_{p-1} \\
&\quad + \|d_1\|_{\infty,p}\widetilde{C}_{p-1})]\}(1-\lambda_p)^{-1}, \\
D_1 &= \|f_1\|_{\infty,1} + \|a_1\|_{\infty,1}C_1 + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|c_1\|_{\infty,1}\widetilde{C}_1 + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}, \\
\widetilde{D}_1 &= \|f_2\|_{\infty,1} + \|a_2\|_{\infty,1}\widetilde{C}_1 + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|c_2\|_{\infty,1}C_1 + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}, \\
D_p &= \|f_1\|_{\infty,1} + \|a_1\|_{\infty,1}C_p + \|b_1\|_{\infty,1}C_{p-1} + \|c_1\|_{\infty,1}\widetilde{C}_p + \|d_1\|_{\infty,1}\widetilde{C}_{p-1}, \\
\widetilde{D}_p &= \|f_2\|_{\infty,1} + \|a_2\|_{\infty,1}\widetilde{C}_p + \|b_2\|_{\infty,1}\widetilde{C}_{p-1} + \|c_2\|_{\infty,1}C_p + \|d_1\|_{\infty,1}C_{p-1}.
\end{aligned}$$

Proof. From (1.1), we can write

$$u_1'(t) + a_1(t)u_1(t) = f_1(t) - b_1(t)u_1(t-r) - c_1(t)u_2(t) - d_1(t)u_2(t-r), \tag{2.5}$$

and from (1.2), we can write

$$u_2'(t) + a_2(t)u_2(t) = f_2(t) - b_2(t)u_2(t-r) - c_2(t)u_1(t) - d_2(t)u_1(t-r). \tag{2.6}$$

For $t \in I_p$, from (2.5) we have

$$\begin{aligned}u_1(t) &= u_1(r_{p-1})e^{-\int_{r_{p-1}}^{t} a_1(\tau)d\tau} \\ &\quad + \int_{r_{p-1}}^{t} [f_1(\tau) - b_1(\tau)u_1(\tau - r) - c_1(\tau)u_2(\tau) - d_1(\tau)u_2(\tau - r)]e^{-\int_\tau^t a_1(\eta)d\eta} d\tau,\end{aligned}$$

and from here

$$\begin{aligned}|u_1(t)| &\le |u_1(r_{p-1})|e^{-\int_{r_{p-1}}^{t} \alpha_1 d\tau} \\ &\quad + \int_{r_{p-1}}^{t} [|f_1(\tau)| + |b_1(\tau)|\,|u_1(\tau - r)| + |c_1(\tau)|\,|u_2(\tau)| + |d_1(\tau)|\,|u_2(\tau - r)|]e^{-\int_\tau^t \alpha_1 d\eta} d\tau \\ &\le |u_1(r_{p-1})|e^{-\alpha_1(t - r_{p-1})} \\ &\quad + \int_{r_{p-1}}^{t} [|f_1(\tau)| + |b_1(\tau)|\,|u_1(\tau - r)| + |c_1(\tau)|\,|u_2(\tau)| + |d_1(\tau)|\,|u_2(\tau - r)|]e^{-\alpha_1(t-\tau)} d\tau. \qquad (2.7)\end{aligned}$$

Similarly, for $t \in I_p$, from (2.6) we have

$$\begin{aligned}|u_2(t)| &\le |u_2(r_{p-1})|e^{-\alpha_2(t - r_{p-1})} \\ &\quad + \int_{r_{p-1}}^{t} [|f_2(\tau)| + |b_2(\tau)|\,|u_2(\tau - r)| + |c_2(\tau)|\,|u_1(\tau)| + |d_2(\tau)|\,|u_1(\tau - r)|]e^{-\alpha_2(t-\tau)} d\tau. \qquad (2.8)\end{aligned}$$

For $t \in I_1$, from (2.7) we have

$$\begin{aligned}|u_1(t)| &\le |\varphi_1(0)|e^{-\alpha_1 t} \\ &\quad + \int_0^t [f_1(\tau)| + |b_1(\tau)|\,|\varphi_1(\tau - r)| + |c_1(\tau)|\,|u_2(\tau)| + |d_1(\tau)|\,|\varphi_2(\tau - r)|]e^{-\alpha_1(t-\tau)} d\tau \\ &\le |\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|c_1\|_{\infty,1}\|u_2\|_{\infty,1} + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}](1 - e^{-\alpha_1 t}) \\ &\le |\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|c_1\|_{\infty,1}\|u_2\|_{\infty,1} + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}]. \qquad (2.9)\end{aligned}$$

For $t \in I_1$, from (2.8) the similar relation can be written for (1.2):

$$\|u_2\| \le |\varphi_2(0)| + \alpha_2^{-1}[\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|c_2\|_{\infty,1}\|u_1\|_{\infty,1} + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}]. \qquad (2.10)$$

Considering (2.9) and (2.10) inequalities together, we obtain

$$\begin{aligned}\|u_1\|_{\infty,1} &\le |\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|c_1\|_{\infty,1}|\varphi_2(0)| + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}] \\ &\quad + \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,1}[\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}] \\ &\quad + \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,1}\|c_2\|_{\infty,1}\|u_1\|_{\infty,1} \\ &\le \{|\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|c_1\|_{\infty,1}|\varphi_2(0)| + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}] \\ &\quad + \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,1}[\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}]\} \\ &\quad \times [1 - \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,1}\|c_2\|_{\infty,1}]^{-1} \\ &\equiv C_1,\end{aligned}$$

and

$$\begin{aligned}\|u_2\|_{\infty,1} &\le \{|\varphi_2(0)| + \alpha_2^{-1}[\|f_2\|_{\infty,1} + \|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0} + \|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}] \\ &\quad + \alpha_2^{-1}\|c_2\|_{\infty,1}[|\varphi_1(0)| + \alpha_1^{-1}[\|f_1\|_{\infty,1} + \|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0} + \|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}]]\} \\ &\quad \times [1 - \alpha_1^{-1}\alpha_2^{-1}\|c_1\|_{\infty,1}\|c_2\|_{\infty,1}]^{-1} \\ &\equiv \widetilde{C}_1.\end{aligned}$$

It follows that for $p=1$, inequalities (2.1) and (2.2) are true. Now assume that the inequalities (2.1) and (2.2) are true for $t\in I_j$, $(p=j)$:

$$\begin{aligned}\|u_1\|_{\infty,j} &\le \{|u_1(r_{j-1})|+\alpha_1^{-1}[\|f_1\|_{\infty,j}+\|b_1\|_{\infty,j}C_{j-1}+\|d_1\|_{\infty,j}\widetilde{C}_{j-1}]\\ &\qquad +\alpha_1^{-1}\|c_1\|_{\infty,j}[|u_2(r_{j-1})|+\alpha_2^{-1}(\|f_2\|_{\infty,j}+\|b_2\|_{\infty,j}\widetilde{C}_{j-1}+\|d_2\|_{\infty,j}C_{j-1})]\}(1-\lambda_j)^{-1},\\ \|u_2\|_{\infty,j} &\le \{|u_2(r_{j-1})|+\alpha_2^{-1}[\|f_2\|_{\infty,j}+\|b_2\|_{\infty,j}\widetilde{C}_{j-1}+\|d_2\|_{\infty,j}C_{j-1}]\\ &\qquad +\alpha_2^{-1}\|c_2\|_{\infty,j}\{|u_1(r_{j-1})|+\alpha_1^{-1}[\|f_1\|_{\infty,j}+\|b_1\|_{\infty,j}C_{j-1}+\|d_1\|_{\infty,j}\widetilde{C}_{j-1}]\}\}(1-\lambda_j)^{-1}.\end{aligned}$$

For $t\in I_{j+1}$, $p=j+1$, from (2.7) and (2.8) we can write the following inequalities:

$$|u_1(t)|\le |u_1(r_j)|+\alpha_1^{-1}[\|f_1\|_{\infty,j+1}+\|b_1\|_{\infty,j+1}C_j+\|c_1\|_{\infty,j+1}\|u_2\|_{\infty,j+1}+\|d_1\|_{\infty,j+1}\widetilde{C}_j], \quad (2.11)$$

$$|u_2(t)|\le |u_2(r_j)|+\alpha_2^{-1}[\|f_2\|_{\infty,j+1}+\|b_2\|_{\infty,j+1}\widetilde{C}_j+\|c_2\|_{\infty,j+1}\|u_1\|_{\infty,j+1}+\|d_2\|_{\infty,j+1}C_j]. \quad (2.12)$$

Similarly, to substitute (2.12) in (2.11) and (2.11) in (2.12) we get

$$\begin{aligned}\|u_1\|_{\infty,j+1} &\le \{|u_1(r_j)|+\alpha_1^{-1}[\|f_1\|_{\infty,j+1}+\|b_1\|_{\infty,j+1}C_j+\|d_1\|_{\infty,j+1}\widetilde{C}_j]+\alpha_1^{-1}\|c_1\|_{\infty,j+1}\\ &\qquad \times[|u_2(r_j)|+\alpha_2^{-1}(\|f_2\|_{\infty,j+1}+\|b_2\|_{\infty,j+1}\widetilde{C}_j+\|d_2\|_{\infty,j+1}C_j)]\}(1-\lambda_{j+1})^{-1}\\ &\equiv C_{j+1},\\ \|u_2\|_{\infty,j+1} &\le \{|u_2(r_j)|+\alpha_2^{-1}[\|f_2\|_{\infty,j+1}+\|b_2\|_{\infty,j+1}\widetilde{C}_j+\|d_2\|_{\infty,j+1}C_j]+\alpha_2^{-1}\|c_2\|_{\infty,j+1}\\ &\qquad \times[|u_1(r_j)|+\alpha_1^{-1}(\|f_1\|_{\infty,j+1}+\|b_1\|_{\infty,j+1}C_j+\|d_1\|_{\infty,j+1}\widetilde{C}_j)]\}(1-\lambda_{j+1})^{-1}\\ &\equiv \widetilde{C}_{j+1}.\end{aligned}$$

We can rewrite equations (1.1) and (1.2) as follows:

$$\begin{aligned}u_1'(t)&=f_1(t)-a_1(t)u_1(t)-b_1(t)u_1(t-r)-c_1(t)u_2(t)-d_1(t)u_2(t-r),\\ u_2'(t)&=f_2(t)-a_2(t)u_2(t)-b_2(t)u_2(t-r)-c_2(t)u_1(t)-d_2(t)u_1(t-r).\end{aligned}$$

Then, we have

$$\begin{aligned}|u_1'(t)|&\le |f_1(t)|+|a_1(t)|\,|u_1(t)|+|b_1(t)|\,|u_1(t-r)|+|c_1(t)|\,|u_2(t)|+|d_1(t)|\,|u_2(t-r)|,\\ |u_2'(t)|&\le |f_2(t)|+|a_2(t)|\,|u_2(t)|+|b_2(t)|\,|u_2(t-r)|+|c_2(t)|\,|u_1(t)|+|d_2(t)|\,|u_1(t-r)|.\end{aligned}$$

From here, for $p=1$, we obtain

$$\begin{aligned}\|u_1'\|_{\infty,1}&\le \|f_1\|_{\infty,1}+\|a_1\|_{\infty,1}\|u_1\|_{\infty,1}+\|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0}+\|c_1\|_{\infty,1}\|u_2\|_{\infty,1}+\|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}\\ &\le \|f_1\|_{\infty,1}+\|a_1\|_{\infty,1}C_1+\|b_1\|_{\infty,1}\|\varphi_1\|_{\infty,0}+\|c_1\|_{\infty,1}\widetilde{C}_1+\|d_1\|_{\infty,1}\|\varphi_2\|_{\infty,0}\\ &\equiv D_1,\\ \|u_2'\|_{\infty,1}&\le \|f_2\|_{\infty,1}+\|a_2\|_{\infty,1}\widetilde{C}_1+\|b_2\|_{\infty,1}\|\varphi_2\|_{\infty,0}+\|c_2\|_{\infty,1}C_1+\|d_2\|_{\infty,1}\|\varphi_1\|_{\infty,0}\\ &\equiv \widetilde{D}_1.\end{aligned}$$

We prove (2.3) and (2.4) by following similar steps. Thus, the proof is complete. □

3. The Mesh and Difference Scheme

3.1 The Mesh

Let ω_{N_0} be a uniform mesh on $\bar{I}$:

$$\omega_{N_0}=\{t_i=ih,\ 1\le i\le N_0,\ h=T/N_0=r/N\},$$

which contains N mesh points at each subintervals I_p $(1 \le p \le m)$:

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \le i \le pN\},$$

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N_p},$$

and

$$\bar{\omega}_{N_0} = \omega_{N_0} \cup \{t_0 = 0\}.$$

For any mesh function $g(t)$, we use $g_i = g(t_i)$ and moreover y_{ki} to denote approximations of $u_k(t)$ at t_i (for $k = 1,2$) and

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \quad \|g\|_{\infty,p} = \|g\|_{\infty,\omega_{N_p}} := \max_{(p-1)N+1 \le i \le pN} |g_i|, \quad 1 \le p \le m.$$

3.2 The Difference Scheme

To obtain an approximation for the problem, we multiply the equations (1.1) and (1.2) by the basis functions $\Phi_{ki}(t)$ (for $k = 1,2$), respectively, and then integrate over (t_{i-1}, t_i). We then obtain the following identity:

$$h^{-1} \int_{t_{i-1}}^{t_i} L u_k(t) \Phi_{ki}(t) dt = h^{-1} \int_{t_{i-1}}^{t_i} f_k(t) \Phi_{ki}(t) dt, \quad 1 \le i \le N_0. \tag{3.1}$$

The basis functions

$$\Phi_{ki}(t) = e^{-\int_t^{t_i} a_k(s) ds}, \quad t_{i-1} < t < t_i, \tag{3.2}$$

which are the solutions of the following problems:

$$\begin{cases} -\frac{d}{dt}\Phi_{ki}(t) + a_k(t)\Phi_{ki}(t) = 0, & t_{i-1} < t < t_i, \\ \Phi_{ki}(t_i) = 1. \end{cases}$$

If we rewrite (3.1) for $t \in (t_{i-1}, t_i)$ and $(k = 1,2)$, we have

$$\begin{aligned} &h^{-1} \int_{t_{i-1}}^{t_i} u_k'(t)\Phi_{ki}(t) dt + h^{-1} \int_{t_{i-1}}^{t_i} a_k(t) u_k(t) \Phi_{ki}(t) dt + h^{-1} \int_{t_{i-1}}^{t_i} b_k(t) u_k(t-r) \Phi_{ki}(t) dt \\ &+ h^{-1} \int_{t_{i-1}}^{t_i} c_k(t) u_{3-k}(t) \Phi_{ki}(t) dt + h^{-1} \int_{t_{i-1}}^{t_i} d_k(t) u_{3-k}(t-r) \Phi_{ki}(t) dt \\ &\quad = h^{-1} \int_{t_{i-1}}^{t_i} f_k(t) \Phi_{ki}(t) dt. \end{aligned} \tag{3.3}$$

If we consider the formulas (2.1) and (2.2) from [1] in the interval (t_{i-1}, t_i) (3.3), we obtain the following exact relation:

$$\begin{aligned} \ell u_{ki} \equiv\ & A_{ki} u_{k\bar{t},i} + B_{ki} u_{(3-k)\bar{t},i} + C_{ki} u_{k\bar{t},i-N} + D_{ki} u_{(3-k)\bar{t},i-N} + E_{ki} u_{ki}, \\ & + G_{ki} u_{k(i-N)} + H_{ki} u_{(3-k)i} + K_{ki} u_{(3-k)(i-N)} = F_{ki} + R_i^{(k)}, \quad 1 \le i \le N_0, \end{aligned} \tag{3.4}$$

with

$$A_{ki} = h^{-1} \int_{t_{i-1}}^{t_i} \Phi_{ki}(t) dt + h^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) a_k(t) \Phi_{ki}(t) dt,$$

$$B_{ki} = h^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) c_k(t) \Phi_{ki}(t) dt,$$

$$C_{ki} = h^{-1} \int_{t_{i-1}}^{t_i} (t - t_i) b_k(t) \Phi_{ki}(t) dt,$$

$$
\begin{aligned}
D_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} (t-t_i)d_k(t)\Phi_{ki}(t)dt,\\
E_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} a_k(t)\Phi_{ki}(t)dt,\\
G_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} b_k(t)\Phi_{ki}(t)dt,\\
H_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} c_k(t)\Phi_{ki}(t)dt,\\
K_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} d_k(t)\Phi_{ki}(t)dt,\\
F_{ki} &= h^{-1}\int_{t_{i-1}}^{t_i} f_k(t)\Phi_{ki}(t)dt,
\end{aligned}
$$

and the remainder terms

$$
\begin{aligned}
R_i^{(k)} = {} & h^{-1}\int_{t_{i-1}}^{t_i} dt b_k(t)\Phi_{ki}(t)\int_{t_{i-1}}^{t_i} u_k'(\xi-r)K_0(t,\xi)d\xi\\
&+h^{-1}\int_{t_{i-1}}^{t_i} dt c_k(t)\Phi_{ki}(t)\int_{t_{i-1}}^{t_i} u_{3-k}'(\xi)K_0(t,\xi)d\xi\\
&+h^{-1}\int_{t_{i-1}}^{t_i} dt d_k(t)\Phi_{ki}(t)\int_{t_{i-1}}^{t_i} u_{3-k}'(\xi-r)K_0(t,\xi)d\xi,
\end{aligned}
\tag{3.5}
$$

$$
K_0(t,\xi)=T_0(t-\xi)-h^{-1}(t-t_{i-1}),\quad T_0(\delta)=1,\ \delta\ge 0;\ T_0(\delta)=0,\ \delta<0.
$$

Eventually, we propose the following difference scheme for the approximate solution of (1.1)-(1.3):

$$
\begin{aligned}
\ell y_{ki} \equiv {} & A_{ki}y_{k\bar{t},i}+B_{ki}y_{(3-k)\bar{t},i}+C_{ki}y_{k\bar{t},i-N}+D_{ki}y_{(3-k)\bar{t},i-N}+E_{ki}y_{ki}\\
&+G_{ki}y_{ki-N}+H_{ki}y_{(3-k)i}+K_{ki}y_{(3-k)i-N}=F_{ki},\quad 1\le i\le N_0,
\end{aligned}
\tag{3.6}
$$

$$
\ell y_{k0}\equiv \varphi_{k0}, \tag{3.7}
$$

where y_{ki} $(k=1,2)$ indicate that the solution of (1.1)-(1.3) system is at mesh point t_i.

In addition, we come up with the implicit Euler method to the approximate solution of system (1.1)-(1.3), alternatively [3]:

$$
\ell y_{ki}\equiv y_{k\bar{t},i}+a_{ki}y_{ki}+b_{ki}y_{ki-N}+c_{ki}y_{(3-k)i}+d_{ki}y_{(3-k)i-N}=f_{ki},\quad 1\le i\le N_0, \tag{3.8}
$$

$$
\ell y_{k0}\equiv \varphi_{k0}, \tag{3.9}
$$

for $k=1,2$.

4. Convergence Analysis

To investigate the convergence of the presented method, we define error functions as $z_{ki}=y_{ki}-u_{ki}$, $1\le i\le N_0$, which is the solution of the following discrete problem:

$$
\ell z_{ki}\equiv R_i^{(k)},\quad 1\le i\le N_0, \tag{4.1}
$$

$$
z_{k0}=0, \tag{4.2}
$$

for $k=1,2$.

Lemma 4.1. *If $a_k, b_k, c_k, d_k, f_k \in C(I)$, then for the truncation errors, we arrive at*

$$\|R^{(k)}\|_{\infty,\omega} \le CN^{-1},$$

for $k = 1,2$.

Proof. From (3.5), we can write

$$\begin{aligned}|R_i^{(k)}| \le\ & h^{-1}\int_{t_{i-1}}^{t_i} dt|b_k(t)||\Phi_{ki}(t)|\int_{t_{i-1}}^{t_i}|u_k'(\xi-r)|K_0(t,\xi)d\xi \\ &+h^{-1}\int_{t_{i-1}}^{t_i} dt|c_k(t)||\Phi_{ki}(t)|\int_{t_{i-1}}^{t_i}|u_{3-k}'(\xi)|K_0(t,\xi)d\xi \\ &+h^{-1}\int_{t_{i-1}}^{t_i} dt|d_k(t)||\Phi_{ki}(t)|\int_{t_{i-1}}^{t_i}|u_{3-k}'(\xi-r)|K_0(t,\xi)d\xi,\end{aligned}$$

and by virtue of Lemma 2.1 and from (3.2), we obtain

$$|R_i^{(k)}| \le Ch.$$

□

Lemma 4.2. *Let z_{ki} be the solution* (4.1)-(4.2) *holds true and*

$$\mu_p := \alpha_1^{-1}\alpha_2^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p})(\|B_2\|_{\infty,p}+\|H_2\|_{\infty,p}) < 1, \quad 1 \le p \le m.$$

Then

$$\|z_k\|_{\infty,p} \le C\{\|R_1\|_{\infty,p}+\|R_2\|_{\infty,p}\}, \quad \text{for } k = 1,2.$$

Proof. (4.1) can be written as follows:

$$\begin{aligned}\ell z_{ki} \equiv\ & A_{ki}z_{k\bar t,i}+B_{ki}z_{(3-k)\bar t,i}+C_{ki}z_{k\bar t,i-N}+D_{ki}z_{(3-k)\bar t,i-N}+E_{ki}z_{ki} \\ &+G_{ki}z_{k(i-N)}+H_{ki}z_{(3-k)i}+K_{ki}z_{(3-k)i-N} = R_{ki}, \quad 1 \le i \le N_0.\end{aligned}$$

For $k = 1$:

$$A_{1i}z_{1\bar t,i}+E_{1i}z_{1i} = Q_{1i},$$

where $Q_{1i} = R_{1i}-B_{1i}z_{2\bar t,i}-C_{1i}z_{1\bar t,i-N}-D_{1i}z_{2\bar t,i-N}-G_{1i}z_{1(i-N)}-H_{1i}z_{2i}-K_{1i}z_{2(i-N)}$. Applying the maximum principle [8], we have

$$\begin{aligned}|z_1|_p &\le |z_{1(p-1)}|+\alpha_1^{-1}\|Q_1\|_{\infty,p} \\ &\le |z_{1(p-1)}|+\alpha_1^{-1}\|R_1\|_{\infty,p}+\alpha_1^{-1}(\|C_1\|_{\infty,p}+\|G_1\|_{\infty,p})|z_1|_{p-1} \\ &\quad +\alpha_1^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p})\|z_2\|_{\infty,p}+\alpha_1^{-1}(\|D_1\|_{\infty,p}+\|K_1\|_{\infty,p})|z_2|_{p-1} \\ &\le C+\alpha_1^{-1}\|R_1\|_{\infty,p}+\alpha_1^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p})\|z_2\|_{\infty,p}.\end{aligned}$$

By writing the same relation for $k = 2$, we have

$$|z_2|_p \le C+\alpha_2^{-1}\|R_2\|_{\infty,p}+\alpha_2^{-1}(\|B_2\|_{\infty,p}+\|H_2\|_{\infty,p})\|z_1\|_{\infty,p}$$

If we consider this inequality in $|z_1|_p$, we obtain the following inequality:

$$\begin{aligned}|z_1|_p &\le C+\alpha_1^{-1}\|R_1\|_{\infty,p}+\alpha_1^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p}) \\ &\quad \times(C+\alpha_2^{-1}\|R_2\|_{\infty,p}+\alpha_2^{-1}(\|B_2\|_{\infty,p}+\|H_2\|_{\infty,p})\|z_1\|_{\infty,p}) \\ &\le C+\alpha_1^{-1}\|R_1\|_{\infty,p}+\alpha_1^{-1}\alpha_2^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p})\|R_2\|_{\infty,p} \\ &\quad +\alpha_1^{-1}(\|B_1\|_{\infty,p}+\|H_1\|_{\infty,p})\alpha_2^{-1}(\|B_2\|_{\infty,p}+\|H_2\|_{\infty,p})\|z_1\|_{\infty,p}.\end{aligned}$$

From here, considering $\mu_p < 1$, the proof is complete for $k = 1$. By following similar steps, the estimate for z_{2i} is achieved. □

Considering Lemma 4.1 and Lemma 4.2, the following theorem gives the main convergence result of this paper.

Theorem 4.1. *Let* u_k *and* y_k *(for* $k=1,2$*) be the solution of* (1.1)-(1.3) *and* (3.6)-(3.7)*, respectively. Then* $\|y_k-u_k\|_{\infty,\bar{\omega}_{N_0}} \le CN^{-1}$.

5. Numerical Results

In this section, we give some illustrative numerical results to support the theoretical findings and demonstrate the advantages of the constructed method. To this end, we consider a test problem solved by the *Presented Method* (PM) and the classical *Euler Method* (EM). The results obtained from solving this problem are given in tables and figures.

Example 5.1. We consider the test problem:

$$\begin{aligned}
&u_1'(t)+u_1(t)+2u_1(t-1)-u_2(t)+u_2(t-1)=2e^{t-1}+e^{-t+1}, \quad t\in(0,2],\\
&u_2'(t)+4u_2(t)-2u_2(t-1)+2u_1(t)-4u_1(t-1)=-4e^{t-1}-2e^{-t+1}, \quad t\in(0,2],\\
&u_1(t)=e^t;\ u_2(t)=e^{-t}, \quad -1\le t\le 0,
\end{aligned}$$

whose exact solution is given by

$$u_1(t)=\begin{cases} 3e^{-2t}-2e^{-3t}, & 0<t\le 1,\\ \delta_1 e^{-2(t-1)}+\delta_2 e^{-3(t-1)}+\cosh(t-1), & 1<t\le 2,\end{cases}$$

$$u_2(t)=\begin{cases} -3e^{-2t}+4e^{-3t}, & 0<t\le 1,\\ \delta_1 e^{-2(t-1)}-2\delta_2 e^{-3(t-1)}+2\sinh(t-1), & 1<t\le 2,\end{cases}$$

$$\delta_1=3(e^{-2}-1),\ \delta_2=2(1-e^{-3}).$$

We define the exact error $E_i^{(k;N)}$ and the computed maximum pointwise error $E^{(k;N)}$ for any N as follows: $E_i^{(k;N)}=|y_{ki}-u_{ki}|$, $E^{(k;N)}=\max\limits_{0\le i\le N} E_i^{(k;N)}$, $k=1,2$.

The data in Tables 1-4 contain numerical results obtained from Example 5.1 for different values of N and mesh points. In Figures 1 and 2, we plot the maximum pointwise errors (in log scale) for this problem. These two figures also demonstrate that these errors are bounded by $O(N^{-1})$, which is proved in Theorem 4.1. Furthermore, in Table 5, we compare the presented method with the classical Euler method. The numerical data in this table indicate that the presented method has better numerical accuracy and efficiency than the classical method.

6. Conclusion

In this study, we have attained an effective difference scheme for the numerical solution of the initial value problem of a linearly coupled differential equation system with a constant large delay parameter r. We prove that the scheme converges to first order in the discrete maximum norm. We considered a test problem, which was solved using the presented method and the classical method (Euler). We presented the comparison of the maximum error values obtained from both methods in Table 5 and illustrated in Figures 1-2. Looking at these tables and figures, it is seen that the presented method is more efficient than the classical method, even though they have the same order of convergence ($O(N^{-1})$). Theoretical results in this paper represent ongoing studies in further research, such as systems of neutral-type delay differential equations and systems of delay integro-differential equations.

Table 1. The numerical results y_{1i} on $(0,2]$ for Example 5.1 (PM)

t_i	u_{1i}	y_{1i} ($N=64$)	$E_i^{(1;64)}$	y_{1i} ($N=128$)	$E_i^{(1;128)}$
0.125	0.9618238	0.9618639	4.011E-5	0.9618338	1.003E-5
0.250	0.8748589	0.8749194	6.048E-5	0.8748740	1.512E-5
0.375	0.7677947	0.7678619	6.717E-5	0.7678115	1.679E-5
0.500	0.6573780	0.6574431	6.508E-5	0.6573943	1.627E-5
0.625	0.5528045	0.5528623	5.784E-5	0.5528190	1.446E-5
0.750	0.4585920	0.4586401	4.802E-5	0.4586040	1.200E-5
0.875	0.3764423	0.3764796	3.731E-5	0.3764516	9.328E-6
1.000	0.3064317	0.3064585	2.676E-5	0.3064384	6.689E-6
1.125	0.2937603	0.2937278	3.251E-5	0.2937522	8.130E-6
1.250	0.3557737	0.3557017	7.206E-5	0.3557557	1.802E-5
1.375	0.4628022	0.4627080	9.417E-5	0.4627787	2.355E-5
1.500	0.5973912	0.5972881	1.031E-4	0.5973654	2.577E-5
1.625	0.7500017	0.7498986	1.030E-4	0.7499759	2.576E-5
1.750	0.9161884	0.9160906	9.777E-5	0.9161639	2.444E-5
1.875	1.0947665	1.0946763	9.006E-5	1.0947439	2.251E-5
2.000	1.2866383	1.2865564	8.189E-5	1.2866179	2.047E-5

Table 2. The numerical results y_{1i} on $(0,2]$ for Example 5.1 (EM)

t_i	u_{1i}	y_{1i} ($N=64$)	$E_i^{(1;64)}$	y_{1i} ($N=128$)	$E_i^{(1;128)}$
0.125	0.9618238	0.9590151	2.808E-3	0.9603837	1.440E-3
0.250	0.8748589	0.8725952	2.264E-3	0.8736958	1.163E-3
0.375	0.7677947	0.7673395	4.552E-4	0.7675528	2.420E-4
0.500	0.6573780	0.6589237	1.546E-3	0.6581533	7.753E-4
0.625	0.5528045	0.5560544	3.250E-3	0.5544440	1.640E-3
0.750	0.4585920	0.4630781	4.486E-3	0.4608564	2.264E-3
0.875	0.3764423	0.3816883	5.246E-3	0.3790889	2.647E-3
1.000	0.3064317	0.3120265	5.595E-3	0.3092518	2.820E-3
1.125	0.2937603	0.3029936	9.233E-3	0.2984362	4.676E-3
1.250	0.3557737	0.3648288	9.055E-3	0.3603536	4.580E-3
1.375	0.4628022	0.4701154	7.313E-3	0.4664904	3.688E-3
1.500	0.5973912	0.6025919	5.201E-3	0.6000022	2.611E-3
1.625	0.7500017	0.7532979	3.296E-3	0.7516444	1.643E-3
1.750	0.9161884	0.9180229	1.835E-3	0.9170902	9.019E-4
1.875	1.0947665	1.0956353	8.689E-4	1.0951806	4.141E-4
2.000	1.2866383	1.2870027	3.643E-4	1.2867992	1.609E-4

Table 3. The numerical results y_{2i} on $(0,2]$ for Example 5.1 (PM)

t_i	u_{2i}	y_{2i} ($N=64$)	$E_i^{(2;64)}$	y_{2i} ($N=128$)	$E_i^{(2;128)}$
0.125	0.4127548	0.4127805	2.578E-5	0.4127612	6.443E-6
0.250	0.0698742	0.0698999	2.576E-5	0.0698807	6.438E-6
0.375	−0.1184898	−0.1184725	1.724E-5	−0.1184855	4.308E-6
0.500	−0.2111177	−0.2111094	8.274E-6	−0.2111156	2.067E-6
0.625	−0.2460945	−0.2460925	2.023E-6	−0.2460940	5.046E-7
0.750	−0.2477936	−0.2477944	7.739E-7	−0.2477938	1.942E-7
0.875	−0.2315628	−0.2315633	4.496E-7	−0.2315629	1.126E-7
1.000	−0.2068576	−0.2068553	2.300E-6	−0.2068570	5.752E-7
1.125	−0.2713230	−0.2712894	3.362E-5	−0.2713146	8.410E-6
1.250	−0.4652925	−0.4652168	7.563E-5	−0.4652735	1.892E-5
1.375	−0.7338209	−0.7337116	1.093E-4	−0.7337935	2.732E-5
1.500	−1.0454211	−1.0452922	1.289E-4	−1.0453889	3.223E-5
1.625	−1.3836807	−1.3835454	1.353E-4	−1.3836469	3.383E-5
1.750	−1.7417846	−1.7416531	1.314E-4	−1.7417517	3.286E-5
1.875	−2.1189797	−2.1188589	1.208E-4	−2.1189495	3.019E-5
2.000	−2.5183298	−2.5182234	1.063E-4	−2.5183032	2.658E-5

Table 4. The numerical results y_{2i} on $(0,2]$ for Example 5.1 (EM)

t_i	u_{2i}	y_{2i} ($N=64$)	$E_i^{(2;64)}$	y_{2i} ($N=128$)	$E_i^{(2;128)}$
0.125	0.4127548	0.4273300	1.458E-2	0.4201556	7.401E-3
0.250	0.0698742	0.0883813	1.851E-2	0.0792486	9.374E-3
0.375	−0.1184898	−0.1012170	1.727E-2	−0.1097641	8.726E-3
0.500	−0.2111177	−0.1971859	1.393E-2	−0.2041018	7.016E-3
0.625	−0.2460945	−0.2359904	1.010E-2	−0.2410260	5.069E-3
0.750	−0.2477936	−0.2412186	6.575E-3	−0.2445135	3.280E-3
0.875	−0.2315628	−0.2279013	3.662E-3	−0.2297539	1.809E-3
1.000	−0.2068576	−0.2054256	1.432E-3	−0.2061704	6.872E-4
1.125	−0.2713230	−0.2787097	7.387E-3	−0.2751257	3.803E-3
1.250	−0.4652925	−0.4737964	8.504E-3	−0.4696458	4.353E-3
1.375	−0.7338209	−0.7402019	6.381E-3	−0.7370757	3.255E-3
1.500	−1.0454211	−1.0488038	3.382E-3	−1.0471381	1.717E-3
1.625	−1.3836807	−1.3843399	6.593E-4	−1.3840061	3.254E-4
1.750	−1.7417846	−1.7404680	1.317E-3	−1.7411032	6.814E-4
1.875	−2.1189797	−2.1165491	2.431E-3	−2.1177322	1.247E-3
2.000	−2.5183298	−2.5155890	2.741E-3	−2.5169255	1.404E-3

Table 5. Comparison of $E^{(k;N)}$ for both methods on $(0,2]$ for Example 5.1

N	$E^{(1;N)}$(PM)	$E^{(1;N)}$(EM)	$E^{(2;N)}$(PM)	$E^{(2;N)}$(EM)
32	4.156E-4	1.845E-2	5.410E-4	3.618E-2
64	1.039E-4	9.462E-3	1.353E-4	1.855E-2
128	2.598E-5	4.791E-3	3.384E-5	9.395E-3
256	6.495E-6	2.411E-3	8.459E-6	4.728E-3
512	1.624E-6	1.209E-3	2.115E-6	2.372E-3
1024	4.059E-7	6.056E-4	5.287E-7	1.188E-3

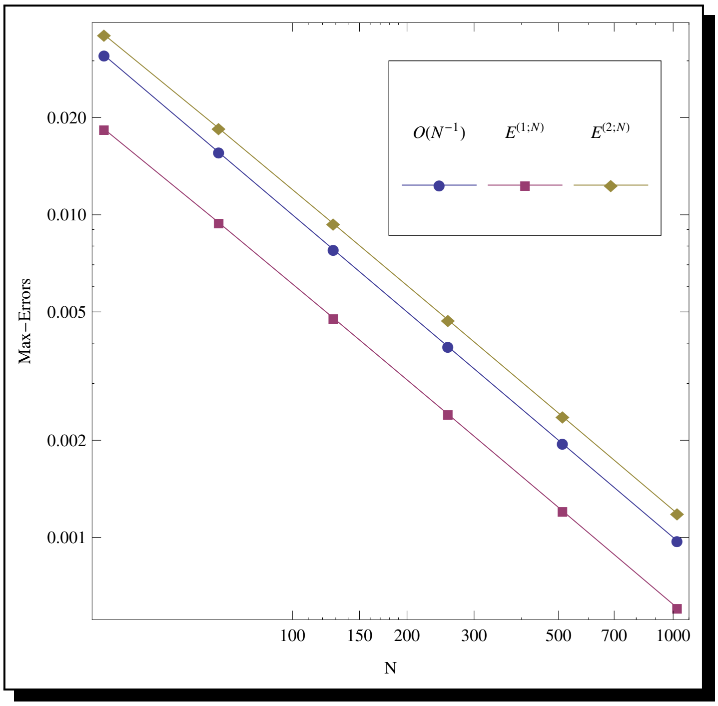

Figure 1. Maximum point-wise errors of log-log plot for Example 5.1 (EM)

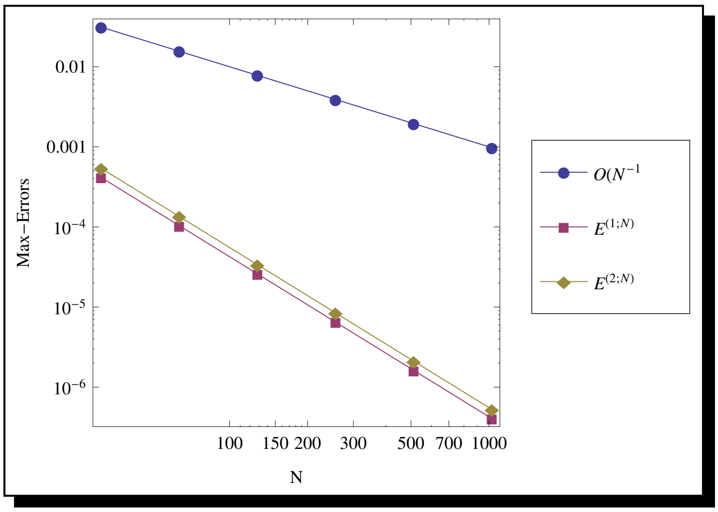

Figure 2. Maximum point-wise errors of log-log plot for Example 5.1 (PM)

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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