



# Convergence, Dynamics and Real-World Applications of C-H Method for Multiple Zeros of Analytic Functions

Rajat Subhra Das\*<sup>1</sup> , Abhimanyu Kumar<sup>2</sup>  and Neha Varma<sup>2</sup> 

<sup>1</sup>Department of Mathematics, Dr. L.K.V.D. College (affiliated to L. N. Mithila University, Darbhanga), Tajpur, Samastipur 848130, Bihar, India

<sup>2</sup>Department of Mathematics, L.N. Mithila University, Darbhanga 846004, Bihar, India

\*Corresponding author: [srajatdas10@gmail.com](mailto:srajatdas10@gmail.com)

**Received:** October 25, 2025    **Revised:** December 15, 2025    **Accepted:** December 22, 2025

**Abstract.** In this study, we focus on the local convergence properties of the C-H combined mean technique applied to analytic functions having simple and multiple zeros. C-H combined mean technique is essentially the average of the two well-known approaches, the Halley's method and the Chebyshev's method. As an outcome, we provide a convergence theorem that guarantees the Q-cubic convergence of the C-H combined mean technique from the very beginning by giving precise domains of starting points together with error estimates. Furthermore, real-life applications demonstrate the robustness and efficiency of the C-H combined mean technique.

**Keywords.** Chebyshev method, Halley method, Multiple zeros, Analytic functions, Error estimates, Local convergence, Basins of attraction

**Mathematics Subject Classification (2020).** 65H04, 12Y05

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## 1. Introduction

An important challenge that emerges from practical issues is determining the zeros of an arbitrary non-linear function  $f : \mathbb{C} \rightarrow \mathbb{C}$ . When trying to get close to the zeros of  $f$ , iteration approaches are quite useful. Among the many iteration techniques described in literature, Halley's, Newton's and Chebyshev's methods are among the most well-known methods.

For multiple zeros, all three approaches converge only linearly; however, for simple zeros, Newton's method achieves second order convergence, whereas both Chebyshev's method and Halley's method attain cubic convergence. For multiple zeros, Chebyshev method (see, Chebyshev [1], Obreshkov [8], Traub [10]) has the following form:

$$z_{k+1} = \begin{cases} z_k - \frac{p^2}{2} \frac{f(z_k)}{f'(z_k)} \left( \frac{3-p}{p} + \frac{f(z_k) f''(z_k)}{f'(z_k) f'(z_k)} \right), & \text{if } f'(z_k) \neq 0, \\ z_k, & \text{otherwise.} \end{cases} \quad (1)$$

For multiple zeros, Halley method (see, Halley [3], Obreshkov [8], Traub [10]) has the following form:

$$z_{k+1} = \begin{cases} z_k - \left( \frac{p+1}{2p} \frac{f'(z_k)}{f'(z_k)} - \frac{1}{2} \frac{f''(z_k)}{f'(z_k)} \right)^{-1}, & \text{if } f'(z_k) \neq 0, \\ z_k, & \text{otherwise.} \end{cases} \quad (2)$$

The theory of analytic functions is fundamental in complex analysis due to its significant theoretical depth and extensive applications in mathematical physics, engineering, and several natural sciences. In 2009, Proinov [9] analyzed Newton method's local convergence when applied to analytic functions with multiple zeros. Ivanov [4] has recently shown a comprehensive study of convergence theorem for the Picard iteration

$$z_{k+1} = Tz_k, \quad k = 0, 1, 2, \dots, \quad (3)$$

In addition, he established a result establishing the local convergence of Halley's method, analyzing its behavior for both simple and multiple zeros. Recently, Kostadinova and Ivanov [5] analyzed convergence behavior and dynamics of Chebyshev's method, presenting local convergence theorem accompanied by error estimates. Marcheiva and Ivanov [7] analyzed Schröder method's local convergence when applied to analytic functions with multiple zeros and formulated theorem describing the admissible sets of initial approximations along with explicit error estimates.

This study investigates the local convergence properties of the C-H method (Das and Kumar [6]), which is defined as follows:

$$z_{k+1} = \begin{cases} z_k - \frac{p^2}{4} \frac{f(z_k)}{f'(z_k)} \left( \frac{3-p}{p} + \frac{f(z_k) f''(z_k)}{f'(z_k) f'(z_k)} \right) - \frac{1}{2} \left( \frac{p+1}{2p} \frac{f'(z_k)}{f'(z_k)} - \frac{1}{2} \frac{f''(z_k)}{f'(z_k)} \right)^{-1}, & \text{if } f(z_k) \text{ and } f'(z_k) \neq 0, \\ z_k, & \text{otherwise,} \end{cases} \quad (4)$$

for simple and multiple polynomial zeros, employing the approach outlined in [4].

## 2. Preliminaries

Let  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  be analytic in a neighborhood of  $\xi \in D$ , which is a zero of  $f$  with multiplicity  $p \geq 1$ , then we define the function  $I : \mathbb{C} \rightarrow \mathbb{R}^+$  by

$$I(z) = \gamma |z - \xi|, \quad (5)$$

where  $\gamma$  is defined by (see, Yakoubsohn [11])

$$\gamma(f, \xi) = \sup_{k > p} \left( \left| \frac{p! f^{(k)}(\xi)}{k! f^{(p)}(\xi)} \right|^{\frac{1}{k-p}} \right). \quad (6)$$

The following definition introduces a wide class of iteration functions associated with the function  $I$  defined by (5).

**Definition 2.1** ([4]). The function  $T : D \subset \mathbb{C} \rightarrow \mathbb{C}$  is called a *gamma iteration function* at the point  $\xi \in \mathbb{C}$  if there exists a quasi-homogeneous function  $\phi : J \rightarrow \mathbb{R}^+$  of degree  $p \geq 0$ , such that for each  $z \in \mathbb{C}$  with  $I(z) \in J$ , it follows that

$$z \in D \text{ and } |T(z) - \xi| \leq \phi(I(z))|z - \xi|, \tag{7}$$

where the function  $I$  is defined by (5). The function  $\phi$  shall be called a *control function* of  $T$ .

The following general convergence theorem shall be used for the proof of our main result:

**Theorem 2.1** ([4]). Let  $T : D \subset \mathbb{C} \rightarrow \mathbb{C}$  be a gamma iteration function at the point  $\xi \in D$  with control function  $\phi$  of exact degree  $p \geq 0$ , and let  $z_0 \in \mathbb{C}$  be an initial guess satisfying

$$I(z_0) \in \{t \in J : \phi(t) < 1\}. \tag{8}$$

Then,  $\xi$  is a unique fixed point of  $T$ , the Picard iteration

$$z_{k+1} = T(z_k) \tag{9}$$

is well-defined and converges to  $\xi$  with  $Q$ -order  $r = p + 1$ . Furthermore, for all  $k \geq 0$ , we have the following error estimates:

$$|z_{k+1} - \xi| \leq \lambda^r |z_k - \xi| \text{ and } |z_k - \xi| \leq \lambda^{S_k(r)} |z_0 - \xi|, \tag{10}$$

where  $\lambda = \phi(I(z_0))$  and  $S_k(r) = \sum_{j=0}^{k-1} r^j$ .

In addition, if  $z_k \neq \xi$  for sufficiently large  $k$ , then we have the following estimate of the asymptotic error constant:

$$\limsup_{k \rightarrow \infty} \frac{|z_{k+1} - \xi|}{|z_k - \xi|^r} \leq \gamma^p \lim_{t \rightarrow 0^+} \frac{\phi(t)}{t^p}. \tag{11}$$

**Definition 2.2** ([4]). Let  $\xi \in \mathbb{C}$  and  $\rho > 0$ . The set

$$B(\xi, \rho) = \{z \in \mathbb{C} : |z - \xi| < \rho\} \tag{12}$$

is said to be a *Q-convergence ball* (with center  $\xi$  and radius  $\rho$ ) of some iterative method if for any initial point  $z_0 \in B(\xi, \rho)$ , the iterative sequence  $(z_k)_{k=0}^\infty$  generated by the method is well defined and converges to  $\xi$  with a  $Q$ -order of at least  $r \geq 1$ .

**Lemma 2.1** ([4]). Let  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ , and let  $\xi \in D$  be a zero of  $f$  with multiplicity  $p \geq 1$ , and suppose  $I(z) < 1$ , where the function  $I : \mathbb{C} \rightarrow \mathbb{R}^+$  is defined by (5). Then, the following estimates hold true:

$$|a| \leq \frac{I(z)}{1 - I(z)}, \quad |b| \leq \frac{(p + 1)I(z) - pI(z)^2}{(1 - I(z))^2}, \quad |l| \leq \frac{2I(z)^2}{(1 - I(z))^3}, \tag{13}$$

$$|c| \leq \frac{p(p - 1)(1 - I(z))^2 - 2(p - 1)I(z) + 2p}{(1 - I(z))^3} \cdot I(z), \tag{14}$$

where the numbers  $a, b, c$ , and  $l$  are defined by

$$a = \sum_{k=p+1}^\infty A_k, \quad b = \sum_{k=p+1}^\infty kA_k, \quad c = \sum_{k=p+1}^\infty k(k - 1)A_k, \quad l = \sum_{k=p+2}^\infty (k - p)(k - p - 1)A_k \tag{15}$$

with  $A_k$  defined by

$$A_k = \left( \frac{p! f^{(k)}(\xi)}{k! f^{(p)}(\xi)} \right) (z - \xi)^{k-p}. \tag{16}$$

**Lemma 2.2** ([4]). *Let  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  and  $\xi \in D$  be a zero of  $f$  with multiplicity  $p \geq 1$ . If  $f$  is analytic around  $\xi$ , then we can define  $a$ ,  $b$ , and  $c$  with (15), and the following claims hold true:*

(i) *If  $z \in D$  is not a zero of  $f$  and  $a \neq -1$ , then, we have*

$$\frac{f'(z)}{f(z)} = \frac{p + b}{(1 + a)(z - \xi)}. \tag{17}$$

(ii) *If  $z \in D$  is not a zero of  $f'$  and  $b \neq -p$ , then, we have*

$$\frac{f''(z)}{f'(z)} = \frac{p^2 - p + c}{(p + b)(z - \xi)}. \tag{18}$$

### 3. Local Convergence Analysis of C-H Method

Consider a complex-valued function  $f$  on  $D \subset \mathbb{C}$ , analytic in the vicinity of  $\xi \in D$ , with  $\xi$  being a zero of multiplicity  $p \geq 1$ . This section provides a thorough local convergence analysis of C-H technique

$$z_{k+1} = CH_f(z_k), \quad k = 0, 1, 2, \dots, \tag{19}$$

for analytic functions with simple and multiple zeros. The C-H iteration function  $CH_f$ , associated with the function  $f$ , is a method used to approximate the roots of  $f$ . It is defined by the formula:

$$CH_f(z) = \begin{cases} z - \frac{p^2}{4} \frac{f(z)}{f'(z)} \left( \frac{3-p}{p} + \frac{f(z)}{f'(z)} \frac{f''(z)}{f'(z)} \right) - \frac{1}{2} \left( \frac{p+1}{2p} \frac{f'(z)}{f(z)} - \frac{1}{2} \frac{f''(z)}{f'(z)} \right)^{-1}, & \text{if } f(z) \neq 0 \text{ and } f'(z) \neq 0, \\ z, & \text{otherwise.} \end{cases} \tag{20}$$

Let  $D$  denotes the domain of the function  $CH_f$  given in (20), which is defined below:

$$D = \left\{ z \in F : f(z) \neq 0 \text{ and } f'(z) \neq 0 \implies \frac{p+1}{2p} \frac{f'(z)}{f(z)} - \frac{1}{2} \frac{f''(z)}{f'(z)} \neq 0 \right\}. \tag{21}$$

Let us define a number  $\tau$ , depending on an integer  $p \geq 1$ , as follows:

$$\tau = \frac{2p}{3p + 1 + \sqrt{p^2 + 6p + 1}}. \tag{22}$$

Let  $p \geq 1$ . We define the functions  $h(t)$  and  $g(t)$  as follows:

$$h(t) = [(3p + 1)(p + 1 - pt) + p^2(p - 1)(1 - t)^2]t^2, \tag{23}$$

$$g(t) = (p + 1)(2pt^2 - (3p + 1)t + p)^2 - p^2(2t + (p - 1)(1 - t)^2). \tag{24}$$

The functions  $G(t)$ ,  $H(t)$ , and  $F(t)$  are defined as follows:

$$G(t) = 2(p + 1 - pt)^3 t^3 + p(1 - t)t^2 F(t), \tag{25}$$

$$H(t) = 2(2pt^2 - (3p + 1)t + p)^3, \tag{26}$$

$$F(t) = (6(1 - t) + |p - 3|)(p + 1 - pt) + p(2 - t)[p(p - 1)(1 - t)^2 - 2(p - 1)t + 2p] + 2p(|p - 3|(p + 1 - pt) + 1)(1 - t)^2 + p^2(p - 1)(1 - t)^3 \tag{27}$$

and

$$\phi(t) = \frac{G(t)}{2H(t)} + \frac{h(t)}{2g(t)}. \tag{28}$$

It can be observed that the equation  $2pt^2 - (3p + 1)t + p = 0$  has a unique solution, namely  $\tau$  which lies in  $[0, \frac{1}{2}]$ . Furthermore,  $\phi$  is a exact second-degree quasi-homogeneous function on  $[0, \nu)$ , where  $\nu$  represents the unique zero of  $g$  within  $[0, \tau)$ .

Following lemma presents a very important result of this article.

**Lemma 3.1.** *Let  $f$  be a function from  $D \subset \mathbb{C}$  to  $\mathbb{C}$ , analytic in a neighborhood of a point  $\xi \in D$ , and suppose that  $\xi$  is a zero of  $f$  of multiplicity  $p \geq 1$ . Then, the C-H Combined Mean iteration function (20) is a  $\gamma$ -iteration function at  $\xi$ , with control function  $\phi$  given by (28).*

*Proof.* Consider the function  $I$  which is defined in (5) and the number  $\tau$  which is defined in (22). If  $z \in D$  satisfies  $I(z) < \nu < \tau$ , then

$$2pI(z)^2 - (3p + 1)I(z) + p > 0 \implies H(I(z)) > 0, \quad I(z) < \nu \text{ and } g(I(z)) > 0, \tag{29}$$

where the function  $H$  is given by (26). We now proceed to prove that

$$|CH_f(z) - \xi| \leq \phi(I(z))|z - \xi|, \tag{30}$$

where  $\phi$  is given by (28). Assume that  $f(z) \neq 0$ , and the quantities  $a, b, c$ , and  $l$  be defined in (15). Therefore, using triangle inequality along with the second inequality of (13) and (29), we get

$$|p + b| \geq p - |b| \geq \frac{2pI(z)^2 - (3p + 1)I(z) + p}{(1 - I(z))^2} > 0, \tag{31}$$

$$|1 + a| \geq 1 - |a| \geq \frac{1 - 2I(z)}{1 - I(z)} > 0. \tag{32}$$

This implies that  $b \neq -p$  and  $a \neq -1$ , and hence, by Lemma 2.2, we conclude that  $f'(z) \neq 0$ . It is now required to prove (30). Lemma 2.2 and the formulation of C-H iteration function (20) allow us to derive

$$\begin{aligned} CH_f(z) - \xi &= z - \xi - \frac{p^2(1+a)(z-\xi)}{4(p+b)} \left( \frac{3-p}{p} + \frac{(1+a)(p^2-p+c)}{(p+b)^2} \right) \\ &\quad - \frac{p(1+a)(p+b)(z-\xi)}{(p+1)(p+b)^2 - p(1+a)(p^2-p+c)} \\ &= \sigma(z - \xi), \end{aligned} \tag{33}$$

where

$$\begin{aligned} \sigma &= \frac{2(p+b)^3 - p(3-p)(1+a)(p+b)^2 - p^2(1+a)^2(p^2-p+c)}{4(p+b)^3} \\ &\quad + \frac{1}{2} \frac{(p+1)(p+b)^2 - p(1+a)(p^2-p+c) - 2p(1+a)(p+b)}{(p+1)(p+b)^2 - p(1+a)(p^2-p+c)} \\ &= \frac{2b^3 + p(6 + (p-3)(1+a))b^2 + p^2[2(p-3)ab - (2+a)ac - p(p-1)a^2 - l]}{4(p+b)^3} \\ &\quad + \frac{1}{2} \frac{(p+1)b^2 - pac - 2pab + 2p^2b - p^2(p+1)a - pc}{(p+1)(p+b)^2 - p(1+a)(p^2-p+c)} \\ &= \frac{2b^3 + p(6 + (p-3)(1+a))b^2 + p^2[2(p-3)ab - (2+a)ac - p(p-1)a^2 - l]}{4(p+b)^3} \\ &\quad + \frac{1}{2} \frac{(p+1)b^2 - pac - 2pab + pl}{(p+1)(p+b)^2 - p(1+a)(p^2-p+c)}. \end{aligned} \tag{34}$$

From above and applying triangle inequality together with the estimates (13) and (31), we get

$$|\sigma| \leq \frac{2|b|^3 + p(6 + |p-3||1+a|)|b|^2 + p^2[2|p-3||a||b| + |2+a||a||c| + p(p-1)|a|^2 + |l|]}{4(p - |b|)^3}$$

$$\begin{aligned}
 & + \frac{1}{2} \frac{(p+1)|b|^2 + p|a||c| + 2p|a||b| + p|l|}{(p+1)|p+b|^2 - p(1+|a|)(p^2 - p + |c|)} \\
 & \leq \frac{2(p+1 - pI(z))^3 I(z)^3 + p(1 - I(z))I(z)^2 F(I(z))}{2H(I(z))} \\
 & + \frac{(p+1)[(p+1)I(z) - pI(z)^2]^2 + pI(z)^2[p(p-1)(1 - I(z))^2 + 2p(1 - I(z)) + 2I(z)]}{2g(I(z))} \\
 & + \frac{2p[(p+1)I(z)^2 - pI(z)^3](1 - I(z)) + 2pI(z)^2(1 - I(z))}{2g(I(z))} \\
 & = \frac{G(I(z))}{2H(I(z))} + \frac{2p[(p+1)I(z)^2 - pI(z)^3](1 - I(z)) + 2pI(z)^2(1 - I(z))}{2g(I(z))} \\
 & = \frac{G(I(z))}{2H(I(z))} + \frac{h(I(z))}{2g(I(z))} \\
 & = \phi(I(z)). \tag{35}
 \end{aligned}$$

This, along with (33), proves (30) and this completes the proof. □

**Theorem 3.1.** Let  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  be a complex-valued function that is analytic in a neighborhood of a point  $\xi \in D$ , where  $\xi$  is a zero of  $f$  of multiplicity  $p \geq 1$ . Assume that the initial approximation  $z_0 \in D$  satisfies

$$h(I(z_0)) < g(I(z_0)) \text{ and } \Psi(I(z_0)) > 0, \tag{36}$$

where

$$I(z) = \gamma|z - \xi|, \quad \gamma = \sup_{k > p} \left| \frac{p!h^{(k)}(\xi)}{k!h^{(p)}(\xi)} \right|^{1/(k-p)}, \tag{37}$$

and  $h, g$ , and  $\Psi = H - G$  are auxiliary real functions.

Hence, the sequence  $\{z_k\}$  obtained from the combined iteration is given by

$$z_{k+1} = CH_f(z_k), \tag{38}$$

is properly defined and converges cubically (in the sense of Q-convergence) to  $\xi$ .

Moreover,

(1) The error estimates below are valid for all nonnegative integers  $k$ :

$$|z_{k+1} - \xi| \leq \lambda^{3^k} |z_k - \xi|, \tag{39}$$

$$|z_k - \xi| \leq \lambda^{(3^k - 1)/2} |z_0 - \xi|, \tag{40}$$

where  $\lambda = \phi(I(z_0))$ , and  $\phi(t)$  is given in (28).

(2) Moreover, if  $z_k \neq \xi$ ,  $k$  being sufficiently large, then the asymptotic error constant satisfies the following:

$$\limsup_{k \rightarrow \infty} \frac{|z_{k+1} - \xi|}{|z_k - \xi|^3} \leq \gamma^2 \frac{1}{2} \left( \frac{(p+1)^3 - p(p-1)}{2p^3} + \frac{p(3p^2 + 1) + (p+1)(p+6 + |p-3|(2p+1))}{2p^2} \right). \tag{41}$$

*Proof.* Assume  $h(I(z_0)) < g(I(z_0))$  and  $\Psi(I(z_0)) = H(I(z_0)) - G(I(z_0)) > 0$ . Then  $h(I(z_0)) < g(I(z_0))$  implies  $I(z_0) < \nu$  (the unique zero of  $g$  in  $[0, \tau)$ ), while  $\Psi(I(z_0)) > 0$  yields  $H(I(z_0)) > G(I(z_0))$  and hence  $I(z_0) < \tau$ , so in particular  $I(z_0) \in J = [0, \nu) \cap [0, \tau)$  and  $\phi(I(z_0)) < 1$ . By Lemma 3.1 the map  $CH_f$  is a  $\gamma$  iteration function at  $\xi$ . Therefore, the hypothesis of theorem is satisfied,

and applying it gives that  $z_{k+1} = CH_f(z_k)$  is defined, remains in  $J$ , and converges to  $\xi$  with  $Q$ -order 3, with the error bounds. This establishes the claim. □

**Theorem 3.2.** *Let  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function with a simple zero  $\xi \in D$ . Assume the initial approximation  $z_0 \in D$  satisfies*

$$I(z_0) < R, \tag{42}$$

where

$$I(z) = \gamma|z - \xi|, \quad \gamma = \sup_{k>1} \left| \frac{f^{(k)}(\xi)}{k!f'(\xi)} \right|^{1/(k-1)}, \tag{43}$$

and  $R$  be the smallest positive root of the following equation

$$7t^6 - 40t^5 + 89t^4 - 96t^3 + 55t^2 - 21t - 2 = 0. \tag{44}$$

(Numerically,  $R \approx 0.1294$ .)

Then, the combined method defined by

$$z_{k+1} = CH_f(z_k), \tag{45}$$

is properly defined and converges cubically (in the sense of  $Q$ -convergence) to  $\xi$ .

Moreover,

(1) The error estimate

$$|z_{k+1} - \xi| \leq \phi(E(z_k))|z_k - \xi|, \tag{46}$$

holds,  $\phi(t)$  is given by

$$\phi(t) = \frac{1}{2} \left( \frac{2(2-t)t^2}{4t^4 - 16t^3 + 20t^2 - 9t + 1} + \frac{(2-t)^3t^3 - (1-t)(2t^3 - 12t^2 + 23t - 15)t^2}{(2t^2 - 4t + 1)^3} \right). \tag{47}$$

(2) The sequence satisfies the quantitative estimate

$$|z_k - \xi| \leq \lambda^{(3^k - 1)/2} |z_0 - \xi|, \tag{48}$$

where  $\lambda = \phi(I(z_0)) < 1$ .

(3) Moreover, when  $z_k \neq \xi$ ,  $k$  being sufficiently large, then the asymptotic error constant can be estimated as follows:

$$\limsup_{k \rightarrow \infty} \frac{|z_{k+1} - \xi|}{|z_k - \xi|^3} \leq \frac{1}{2}(4 + 15)\gamma^2 = \frac{19}{2}\gamma^2 = 9.5\gamma^2. \tag{49}$$

*Proof.* For  $p = 1$ , the conditions (42) and (36) are equivalent, hence the hypothesis of Theorem 3.2 is satisfied by the assumption of Theorem 3.1. Therefore, Theorem 3.1 applies directly, yielding that the iteration is defined and remains in the convergence domain and converges  $Q$ -cubically to the simple zero  $\xi$ , having stated a priori/a posteriori estimates and asymptotic bound. This completes the proof. □

**Remark 3.1** (Comparison Favoring the Combined Method). From Table 1, it is evident that the combined method exhibits significantly favorable convergence behavior compared to both Halley’s and Chebyshev’s methods, depending on the multiplicity  $p$ .

For  $p = 1$ , the convergence radius of the combined method ( $R_{\text{combined}} \approx 0.1294$ ) is notably larger than that of Chebyshev’s method ( $R \approx 0.119$ ), providing a wider convergence domain while maintaining cubic convergence.

For moderate multiplicities ( $p = 2, 3$ ), although the convergence radius of Chebyshev’s method slightly exceeds that of the combined method, the combined method consistently maintains a much larger convergence radius than Halley’s method.

As  $p$  increases further ( $p \geq 4$ ), the performance gap between the combined and Halley methods becomes increasingly pronounced. Halley’s convergence radii diminish rapidly, while the combined method sustains relatively wider convergence domains. For example, at  $p = 5$ , the combined method maintains a convergence radius around 0.1310, whereas Halley’s method drops to a much smaller radius of approximately 0.073.

Thus, for larger multiplicities, the combined method robustly outperforms Halley’s method in terms of convergence domain size, ensuring better stability and practical applicability. Consequently, the combined method offers an effective balance between convergence speed and robustness across a broad range of multiplicities, making it a highly advantageous choice for solving problems involving multiple zeros of analytic functions.

**Table 1.** Comparison of convergence Radii  $R$ ,  $R_H$ , and  $R_{\text{combined}}$  for  $p = 1$  to  $p = 10$

| $p$ | $R$ (Chebyshev’s method) | $R_H$ (Halley’s method) | $R_{\text{C-H}}$ (C-H method) |
|-----|--------------------------|-------------------------|-------------------------------|
| 1   | 0.119                    | 0.143                   | 0.1294                        |
| 2   | 0.154                    | 0.121                   | 0.1373                        |
| 3   | 0.169                    | 0.100                   | 0.1388                        |
| 4   | 0.163                    | 0.084                   | 0.1354                        |
| 5   | 0.156                    | 0.073                   | 0.1310                        |
| 6   | 0.149                    | 0.064                   | 0.1266                        |
| 7   | 0.144                    | 0.056                   | 0.1229                        |
| 8   | 0.139                    | 0.051                   | 0.1195                        |
| 9   | 0.134                    | 0.046                   | 0.1163                        |
| 10  | 0.130                    | 0.042                   | 0.1134                        |

#### 4. Convergence and Dynamics in a Real-World Applications

We devoted this section to the application of the C-H method for approximating the roots of polynomial equations of higher degree that originate from practical issues and compare it with the Chebyshev and Halley method.

**Example 4.1.** This polynomial equation

$$f(z) = z^5 - z + 1 \tag{50}$$

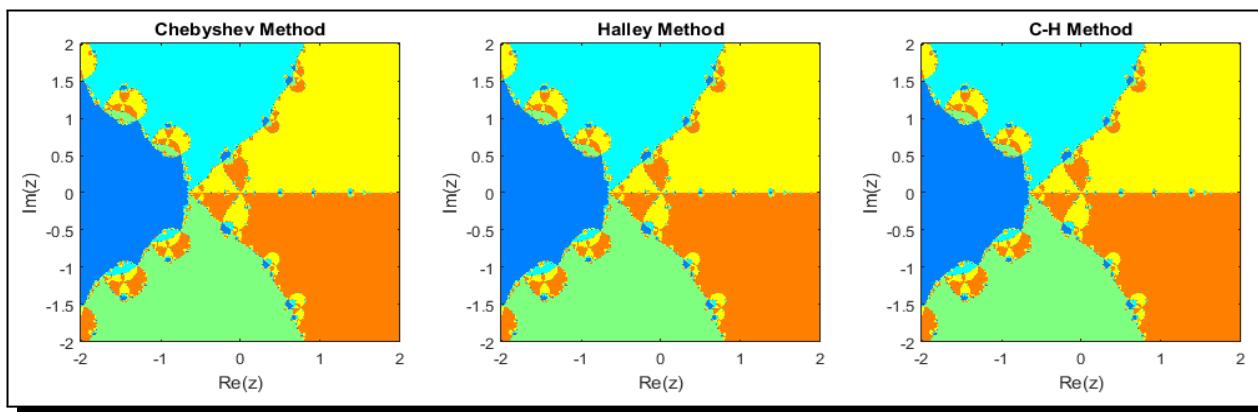
is a quintic equation, and it arises in complex analysis and dynamical systems. The roots of the polynomial  $f(z) = z^5 - z + 1$  are:  $z_1 \approx -1.1673 + 0.0000i$ ,  $z_2 \approx -0.1812 + 1.0840i$ ,  $z_3 \approx -0.1812 - 1.0840i$ ,  $z_4 \approx 0.7649 + 0.3525i$ ,  $z_5 \approx 0.7649 - 0.3525i$ .

In Table 2, we have shown the error estimation and no of iteration of Chebyshev, Halley and C-H methods.

Figure 1 illustrates the resulting basins of attraction, showcasing the dynamic behavior and robustness of Chebyshev, Halley and C-H methods across the complex plane.

**Table 2.** Comparison of methods for the real root of  $f(z) = z^5 - z + 1$

| Method    | Real root | Iterations ( $k$ ) | Error estimate            |
|-----------|-----------|--------------------|---------------------------|
| Chebyshev | -1.1673   | 3                  | $9.87213 \times 10^{-26}$ |
| Halley    | -1.1673   | 2                  | $1.55605 \times 10^{-11}$ |
| C-H       | -1.1673   | 3                  | $1.35949 \times 10^{-26}$ |



**Figure 1.** Basins of attraction for the combined method applied to  $f(z) = z^5 - z + 1$

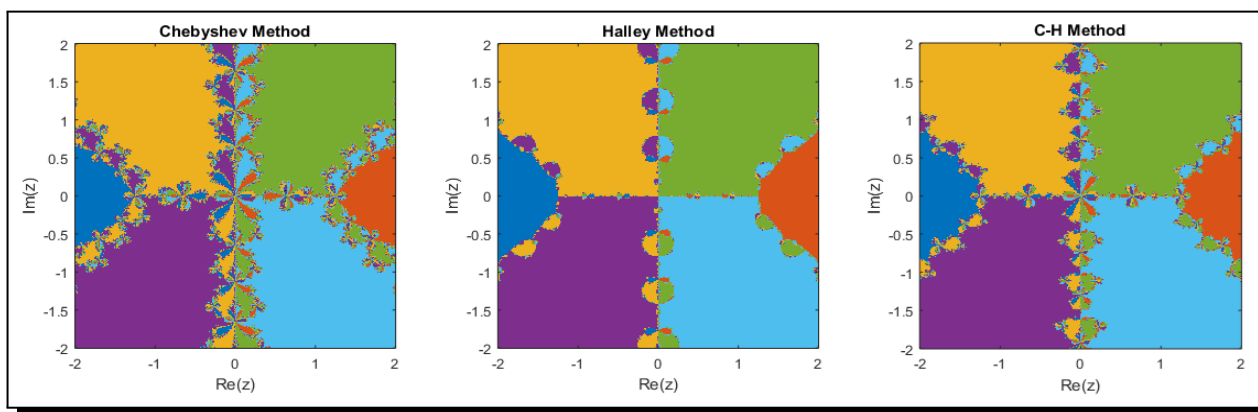
**Example 4.2.** Consider the sextic polynomial

$$f(z) = z^6 - 3z^4 + 2z^2 - 1, \tag{51}$$

which models vibrational characteristics in symmetric mechanical systems. This function has several real and complex roots. We focus on the positive real root, approximately  $z_1 \approx 1.1580$ . In Table 3, we have shown the error estimation and no of iteration of Chebyshev, Halley and C-H methods.

**Table 3.** Comparison of methods for real root of  $f(z) = z^6 - 3z^4 + 2z^2 - 1$

| Method    | Real root | Iterations ( $k$ ) | Error estimate         |
|-----------|-----------|--------------------|------------------------|
| Chebyshev | 1.1580    | 3                  | $2.43 \times 10^{-23}$ |
| Halley    | 1.1580    | 2                  | $8.76 \times 10^{-11}$ |
| C-H       | 1.1580    | 3                  | $1.97 \times 10^{-24}$ |



**Figure 2.** Basins of attraction for  $f(z) = z^6 - 3z^4 + 2z^2 - 1$

Figure 2 illustrates the resulting basins of attraction, showcasing the dynamic behavior and robustness of Chebyshev, Halley and C-H methods across the complex plane.

**Example 4.3.** The steady-state concentration  $C$  (ppm) of a toxic contaminant in a river network with branching channels and seasonal cycles is modeled by the octic polynomial:

$$f(z) = (z - 1)^3(z + 2)^2(z - i)(z + i)(z - 4). \tag{52}$$

Roots of the equation are given by:

$$z_1 = 1(\text{triple root}), \quad z_2 = -2(\text{double root}), \quad z_{3,4} = \pm i, \quad z_5 = 4,$$

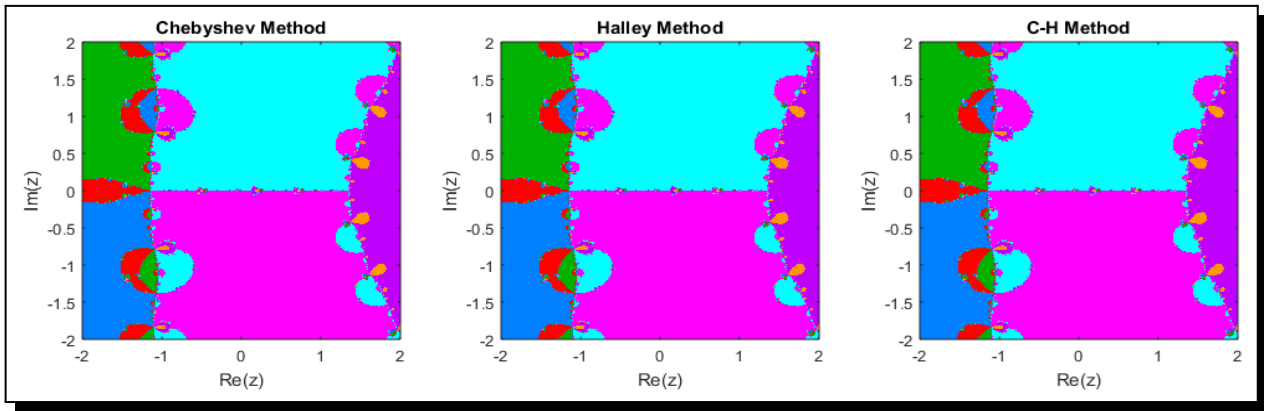
where  $z = 1$  represents a critical saturation threshold,  $z = -2$  indicates seasonal depletion cycles, and  $z = \pm i$  model oscillatory diffusion patterns.

In Table 4, we have shown the error estimation and no of iteration of Chebyshev, Halley and C-H methods.

**Table 4.** C-H method versus classical methods for the real root  $z = 1$  (multiplicity  $p = 3$ )

| Method       | Real root | Iterations ( $k$ ) | Error estimate        |
|--------------|-----------|--------------------|-----------------------|
| Chebyshev    | 1.0       | 5                  | $3.8 \times 10^{-16}$ |
| Halley       | 1.0       | 6                  | $7.1 \times 10^{-10}$ |
| C-H Combined | 1.0       | 4                  | $5.2 \times 10^{-24}$ |

Figure 3 illustrates the resulting basins of attraction, showcasing the dynamic behavior and robustness of Chebyshev, Halley and C-H methods across the complex plane.



**Figure 3.** Basins of attraction for  $f(z) = (z - 1)^3(z + 2)^2(z - i)(z + i)(z - 4)$

## 5. Conclusion

The C-H combined mean method, integrating Halley’s and Chebyshev’s schemes, achieves *Q-cubic convergence* for multiple-type zeros of any analytic functions, surpassing classical linear convergence limitations. A local convergence theorem defines precise initial domains (*Q-convergence balls*) with a error bounds. For higher multiplicities ( $p \geq 4$ ), it outperforms Halley’s method in stability (e.g.,  $R_{C-H} \approx 0.1310$  vs.  $R_H \approx 0.073$  at  $p = 5$ ). Numerical tests on real-world polynomials yield errors below  $10^{-24}$  in 3-4 iterations, while basins of attraction confirm faster convergence.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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