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Research Article

Estimates on Coefficient and Logarithmic Coefficient Bounds for a Subclass of Analytic Functions Associated With the Three-Leaf Domain

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Abstract. In this article, we have obtained the estimates of coefficient and logarithmic coefficient bounds for a starlike-type subclass of analytic functions associated with the three-leaf domain. The bounds found here are sharp and comparable to those of starlike functions.

Keywords. Starlike functions, Three-leaf domain, Logarithmic coefficients

Mathematics Subject Classification (2020). 30C45, 30C50

1. Introduction

Geometric function theory in its essence is the study of analytic functions and their geometric implications on the complex plane. Its study traces back to the Riemann mapping theorem (Conway [7]) and it is often studied on the open unit disk $\Delta = \{z \in \mathbb{C}, |z| < 1\}$ rather than the entire complex plane.

Let the family of all the functions f that are analytic on Δ be represented by $\mathcal{A}$ and have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta, \tag{1.1}$$

where $\mathcal{S}$ represents a subclass of $\mathcal{A}$ consisting of functions that are univalent in Δ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$. In geometric function theory, a key problem of analytic functions is their connection with coefficient estimates for these functions. In 1916, Bieberbach conjectured [4] that $|a_n| \le n$, $n = 2, 3, \dots$ which played an important role in research in this field for decades until, in 1984, Louis de Branges [5] proved this result. During 1916-1984, researchers used various techniques and established several coefficient results for various subclasses of $\mathcal{S}$. These coefficient problems in geometric function theory and its subbranch of univalent function theory have significant implications in quantum calculus (Aouf and Mostafa [3], and Mishra *et al*. [14]) which is fundamental to quantum mechanics. It also has significant applications in fluid mechanics (Aleman and Constantin [2], and Morais and Zayed [15]). The subclasses $\mathcal{S}^*$, of starlike functions and the class C, of convex functions are two of the most explored among many. They are defined as below:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \Delta \right\}, \qquad C = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in \Delta \right\}.$$

These classes can also be defined with the help of subordination. We say that, for analytic functions, $f_1(z)$ to be subordinated to $f_2(z)$ in the region $\mathbb{U}$ and denoted mathematically as $f_1(z) \prec f_2(z)$ if a function $w(z)$, a Schwarz function, satisfies the conditions $|w(z)| \le 1$ and $w(0) = 1$, such that $f_1(z) = f_2(w(z))$. Moreover, if $f_2(z)$ is in $\mathcal{S}$, then due to Duren [8] and Goodman [10], the following equivalent conditions will be true

$$f_1(\mathbb{U}) \subseteq f_2(\mathbb{U}) \text{ and } f_1(0) = f_2(0).$$

Thus, one can define the following

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \psi, z \in \Delta \right\}, \tag{1.2a}$$

$$C(\psi) = \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \psi, z \in \Delta \right\}. \tag{1.2b}$$

In (1.2a) and (1.2b), if the right-hand side is changed, several well-known subfamilies will originate. For example, if we put $\psi = \frac{1+Az}{1+Bz}$, we obtain the Janowski-type class of starlike functions (refer Cho *et al*. [6], Goel and Kumar [9], Khan and Abaoud [11], Mendiratta *et al*. [13], Robertson [18], and Ronning [19] for different domains on the right-hand side and different approaches of forming subclasses). In this article, we shall explore when the right-hand side is the function for the three-leaf domain $\psi = 1 + \frac{4}{5}z + \frac{1}{5}z^4$.

The coefficient bounds for starlike functions associated with three-leaf domain was explored by Shi *et al*. [20] which can be written as

$$\mathcal{S}^*_{3\mathcal{L}} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}, \quad z \in \Delta. \tag{1.3}$$

A geometric representation of the three-leaf domain in the complex plane plotted using MATLAB is given in Figure 1.

Similarly, Sokół [21] and Nandeesh *et al*. [16] defined a subclass related to starlike functions which when associated with the three-leaf domain is given by

$$\mathcal{S}\mathcal{S}^*_{3\mathcal{L}} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4, 0 \le \alpha < 1 \right\}, \quad z \in \Delta. \tag{1.4}$$

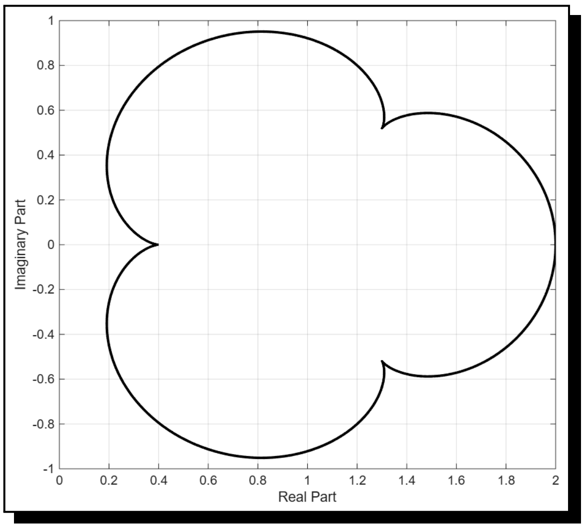

Figure 1

The introduction of a parameter has its effect on the geometry of the image of the function. Note that, when the parameter $a = 0$, the class (1.4) reduces to $\mathcal{S}^*$ which contains the functions which maps unit disk to a starlike domain with respect to the origin. This allows us to compare the bounds obtained for the functions in (1.4) to the bounds obtained for functions in (1.3) as discussed in [20].

Our main purpose in this article is to obtain sharp bounds for the first four coefficients and sharp estimates for logarithmic coefficients for the class defined in (1.4).

2. Preliminaries

Let $\mathcal{P}$ be the subclass of mappings p that are analytic in Δ with $\mathrm{Re}(p(z)) > 0$ and its series form is as follows:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \Delta. \tag{2.1}$$

Lemma 2.1 ([20]). *If $p(z) \in \mathcal{P}$ and it is of the form* (2.1)*, then*

$$\begin{aligned} &|c_n| \le 2, && \text{for } n \ge 1, \\ &|c_{n+k} - \delta c_n c_k| \le 2, && \text{for } 0 \le \delta \le 1 \end{aligned} \tag{2.2}$$

and for $\xi \in \mathbb{C}$,

$$|c_2 - \xi c_1^2| \le 2\max\{1, |2\xi - 1|\} \tag{2.3}$$

and for real λ,

$$|c_3 - \lambda c_2^2| \le \begin{cases} -4\lambda + 2, & \text{if } \lambda \le 0, \\ 2, & \text{if } 0 \le \lambda \le 1, \\ 4\lambda - 2, & \text{if } \lambda \ge 1. \end{cases} \tag{2.4}$$

Lemma 2.2 ([20]). *Let $p \in \mathcal{P}$ have the representation of the form* (2.1)*; then, real numbers α, β and γ,*

$$|\alpha c_1^3 - \beta c_1 c_2 + \gamma c_3| \le 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|. \tag{2.5}$$

Lemma 2.3 ([20])**.** *Let* m, n, l *and* r *satisfy the inequalities* $0 < m < 1$, $0 < r < 1$ *and*

$$8r(1-r)[(mn-2l)^2+(m(r+m)-n)^2]+m(1-m)(n-2rm)^2 \le 4m^2(1-m)^2r(1-r).$$

If $p \in \mathcal{P}$ *and has power series* (2.1)*, then*

$$\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4 \right| \le 2.$$

Lemma 2.4 ([20])**.** *Let* $h \in \mathcal{P}$ *have the series expansion of the form* (2.1)*. Then, for* $x, y \in \overline{\mathbb{D}} = \mathbb{D} \cup \{1\}$,

$$2c_2 = c_1^2 + x(4-c_2^1), \tag{2.6}$$

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x_2 + 2(4-c_1^2)(1-|x|^2)y. \tag{2.7}$$

Lemma 2.5 ([20])**.** *Let* $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ *be of the form* (1.1)*, then*

$$|a_2| \le \frac{4}{5}, \tag{2.8}$$

$$|a_3| \le \frac{2}{5}, \tag{2.9}$$

$$|a_4| \le \frac{4}{15}, \tag{2.10}$$

$$|a_5| \le \frac{1}{5}. \tag{2.11}$$

All these bounds are sharp for the functions defined below, respectively,

$$f_1(z) = z\exp\left(\int_0^z \left(\frac{4}{5} + \frac{1}{5}t^3\right)dt\right) = z + \frac{4}{5}z_2 + \cdots, \tag{2.12}$$

$$f_2(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t + \frac{1}{5}t^7\right)dt\right) = z + \frac{2}{5}z^3 + \cdots, \tag{2.13}$$

$$f_3(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t_2 + \frac{1}{5}t^{11}\right)dt\right) = z + \frac{4}{15}z^4 + \cdots, \tag{2.14}$$

$$f_4(z) = z\exp\left(\int_0^z \left(\frac{4}{5}t^3 + \frac{1}{5}t^{15}\right)dt\right) = z + \frac{1}{5}z^5 + \cdots. \tag{2.15}$$

Lemma 2.6 ([20])**.** *If* $f(z) \in \mathcal{S}^*_{3\mathcal{L}}$ *is of the form given in* (1.1)*, then*

(i) $$|\gamma_1| \le \tfrac{4}{5}, \tag{2.16}$$

(ii) $$|\gamma_2| \le \tfrac{1}{5}. \tag{2.17}$$

The results are sharp for the functions given in (2.12) *and* (2.13)*, respectively.*

3. Main Results

Coefficient Bounds

Theorem 3.1. *If* $f \in \mathcal{SS}^*_{3\mathcal{L}}$ *is of the form given in* (1.1)*, then*

(i) $|a_2| \le \frac{4}{5(1-a)}$,

(ii) $|a_3| \le \frac{1}{5(1-a)} 2\max\left\{1, \left|\frac{13-5a}{50(1-a)}\right|\right\}$.

The results are sharp.

Proof. (i) Since $f \in \mathcal{SS}^*_{3\mathcal{L}}$, there exists an analytic function $w(z), |w(z)| \leq 1$ and $w(0) = 0$, such that

$$\frac{zf'(z)}{(1-a)f(z)+af'(z)} = 1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4.$$

Denote

$$\Psi(w(z)) = 1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4$$

and

$$k(z) = 1+c_1z+c_2z_2+\cdots = \frac{1+w(z)}{1-w(z)}.$$

Obviously, the function $k(z)\in\mathcal{P}$ and $w(z) = \frac{k(z)-1}{k(z)+1}$. This gives

$$w(z) = \frac{k(z)-1}{k(z)+1} = \frac{c_1z+c_2z_2+c_3z^3+\cdots}{2+c_1z+c_2z_2+c_3z^3+\cdots}$$

and

$$\begin{aligned}1+\frac{4}{5}w(z)+\frac{1}{5}(w(z))^4 &= 1+\frac{2}{5}c_1z+\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right)z_2+\left(\frac{1}{10}c_1^3-\frac{2}{5}c_2c_1+\frac{2}{5}c_3\right)z^3\\ &\quad+\left(-\frac{3}{80}c_1^4+\frac{3}{10}c_1^2c_2-\frac{2}{5}c_3c_1-\frac{1}{5}c_2^2+\frac{2}{5}c_4\right)z^4+\cdots \end{aligned} \tag{3.1}$$

while,

$$\begin{aligned} zf'(z) &= (1+a_2z+(2a_3-a_2^2)z_2+(a_2^3-3a_2a_3+3a_4)z^3\\ &\quad+(-a_2^4+4a_2^2a_3-4a_2a_4-2a_2^3+4a_5)z^4+\cdots)\\ &\quad\cdot((1-a)f(z)+af'(z)). \end{aligned} \tag{3.2}$$

Equating coefficients of z in (3.2) gives

$$2a_2 = a_2+\frac{2c_1}{5}+aa_2$$

$$\implies \quad a_2 = \frac{2c_1}{5(1-a)}. \tag{3.3}$$

Using Lemma 2.1, one can see that

$$|a_2| \leq \frac{4}{5(1-a)}. \tag{3.4}$$

(ii) Equating coefficients of z^2 in (3.2) gives

$$3a_3 = a_3+\frac{c_2}{5}-\frac{c_1^2}{5}+2aa_3+\frac{2a_2c_1}{5}+\frac{2aa_2c_1}{5}$$

$$\implies \quad a_3 = \frac{1}{5(1-a)}\left(c_2-\left(\frac{4}{25(1-a)}+\frac{1}{10}\right)c_1^2\right).$$

Applying triangle inequality, we get

$$|a_3| \leq \frac{1}{5(1-a)}\left|c_2-\left(\frac{13-5a}{50(1-a)}\right)c_1^2\right|.$$

Applying Lemma 2.1, we obtain the bound

$$|a_3| \leq \frac{1}{5(1-a)}2\max\left\{1,\left|\frac{13-5a}{50(1-a)}\right|\right\}. \tag{3.5}$$

□

Remark 3.2. If we put $a = 0$ in both the results (3.4) and (3.5), we get $|a_2| \le \frac{4}{5}$ and $|a_3| \le \frac{2}{5}$ which is true for starlike functions mentioned in Lemma 2.5 and hence the result is sharp for the functions in (2.12) and (2.13), respectively.

Theorem 3.3. *If* $f \in \mathcal{SS}^*_{3\mathcal{L}}$ *is of the form given in* (1.1)*, then*

$$|a_4| \le \frac{2}{125(1-a)^3}\left(\left|\frac{27a^2-12a+1}{2}\right| + \left|\frac{9a^2-74a+17}{3}\right| + \left|\frac{a^2-16a+63}{6}\right|\right).$$

The result is sharp.

Proof. Equating coefficients of z^3 in (3.2) and simplifying the same gives

$$a_4 = \frac{a_2c_1^2}{15a-15} - \frac{c_1^3}{30a-30} - \frac{2c_3}{15a-15} - \frac{2a_2c_2}{15a-15} - \frac{2a_3c_1}{15a-15} + \frac{2c_1c_2}{15a-15} - \frac{2aa_2c_2}{15a-15} - \frac{4aa_3c_1}{15a-15} + \frac{aa_2c_1^2}{15a-15}.$$

Substituting for a_2, a_3 in terms of c_1, c_2 and c_3, we get

$$a_4 = \frac{2c_1c_2}{15a-15} - \frac{c_1^3}{30a-30} - \frac{2c_1^3}{\sigma_3} - \frac{2c_1^3}{\sigma_2} - \frac{8c_1^3}{\sigma_1} - \frac{2ac_1^3}{\sigma_3} - \frac{4ac_1^3}{\sigma_2} - \frac{24ac_1^3}{\sigma_1} - \frac{16a^2c_1^3}{\sigma_1} - \frac{2c_3}{15a-15} + \frac{4c_1c_2}{\sigma_3} + \frac{4c_1c_2}{\sigma_2} + \frac{4ac_1c_2}{\sigma_3} + \frac{8ac_1c_2}{\sigma_2},$$

where

$$\sigma_1 = 750a^3 - 2250a^2 + 2250a - 750,$$
$$\sigma_2 = 150a^2 - 300a + 150,$$
$$\sigma_3 = 75a^2 - 150a + 75.$$

Rearranging and simplifying the terms, we get

$$|a_4| \le 2\left|\left(\frac{27a_2-12a+1}{250(a-1)^3}\right)c_1^3 - \left(\frac{2(9a-2)}{75(a-1)^2}\right)c_1c_2 + \left(\frac{2}{15(a-1)}\right)c_3\right|.$$

Using Lemma 2.2, we get

$$\begin{aligned} |a_4| &\le 2\left(\left|-\frac{27a^2-12a+1}{250(a-1)^3}\right| + \left|-\frac{9a^2-74a+17}{375(a-1)^3}\right| + \left|-\frac{a^2-16a+63}{750(a-1)^3}\right|\right) \\ &= \frac{2}{125(1-a)^3}\left(\left|\frac{27a^2-12a+1}{2}\right| + \left|\frac{9a^2-74a+17}{3}\right| + \left|\frac{a^2-16a+63}{6}\right|\right). \end{aligned} \tag{3.6}$$

□

Remark 3.4. If we put $a = 0$ in the result (3.6), we get $|a_4| \le \frac{4}{15}$ which is true for starlike functions mentioned in Lemma 2.5 and hence the result is sharp for the function given in (2.14).

Theorem 3.5. *If* $f \in \mathcal{SS}^*_{3\mathcal{L}}$ *is of the form given in* (1.1)*, then*

$$|a_5| \le 2,$$

whenever

$$\frac{a}{5120000(9a-3)((19a-13)(21a-11))^2((27a-7)^2(a-1))^4} \le 1,$$

where

$$\alpha = 40215013987689a^{10} - 393780373627932a^9 + 1136252487629655a^8 - 1684778583278592a^7 + 1520882312587146a^6 - 896241133248984a^5 + 353980536315902a^4 - 93430032343904a^3 + 15905499288189a^2 - 1590187806380a + 71501975867.$$

Proof. Equating coefficients of z^4 in (3.2) and simplifying the same gives

$$a_5 = \frac{c_2^2}{20a-20} - \frac{2c_4}{20a-20} + \frac{3c_1^4}{320a-320} + \frac{a_3c_1^2}{20a-20} - \frac{a_2c_1^3}{40a-40} - \frac{3c_1^2c_2}{40a-40} - \frac{2a_2c_3}{20a-20} - \frac{2a_3c_2}{20a-20} - \frac{2a_4c_1}{20a-20} + \frac{2c_1c_3}{20a-20} - \frac{2aa_2c_3}{20a-20} - \frac{4aa_3c_2}{20a-20} - \frac{6aa_4c_1}{20a-20} + \frac{2a_2c_1c_2}{20a-20} + \frac{2aa_3c_1^2}{20a-20} - \frac{aa_2c_1^3}{40a-40} + \frac{2aa_2c_1c_2}{20a-20}.$$

Substituting for a_2, a_3 and a_4 in terms of c_1, c_2, c_3 and c_4 we get

$$a_5 = \frac{16c_1^4}{(15000(a-1)^4)} - \frac{2c_4}{20a-20} + \frac{c_2^2}{20a-20} + \frac{3c_1^4}{320a-320} + \frac{4c_2^2}{(200(a-1)^2)} + \frac{3c_1^4}{(200(a-1)^2)} + \frac{2c_1^4}{(600(a-1)^2)} + \frac{4c_1^4}{(1000(a-1)^3)} + \frac{4c_1^4}{(1500(a-1)^3)} + \frac{4c_1^4}{(3000(a-1)^3)} - \frac{3c_1^2c_2}{40a-40} + \frac{8ac_2^2}{(200(a-1)^2)} + \frac{4ac_1^4}{(200(a-1)^2)} + \frac{6ac_1^4}{(600(a-1)^2)} - \frac{4c_1^2c_2}{(100(a-1)^2)} - \frac{4c_1^2c_2}{(200(a-1)^2)} - \frac{4c_1^2c_2}{(300(a-1)^2)} + \frac{12ac_1^4}{(1000(a-1)^3)} + \frac{16ac_1^4}{(1500(a-1)^3)} + \frac{20ac_1^4}{(3000(a-1)^3)} - \frac{8c_1^2c_2}{(1000(a-1)^3)} - \frac{8c_1^2c_2}{(1500(a-1)^3)} - \frac{8c_1^2c_2}{(3000(a-1)^3)} + \frac{96ac_1^4}{(15000(a-1)^4)} + \frac{8a^2c_1^4}{(1000(a-1)^3)} + \frac{12a^2c_1^4}{(1500(a-1)^3)} + \frac{24a^2c_1^4}{(3000(a-1)^3)} + \frac{2c_1c_3}{20a-20} + \frac{176a^2c_1^4}{(15000(a-1)^4)} + \frac{96a^3c_1^4}{(15000(a-1)^4)} + \frac{4c_1c_3}{(100(a-1)^2)} + \frac{4c_1c_3}{(300(a-1)^2)} - \frac{16a^2c_1^2c_2}{(1000(a-1)^3)} - \frac{24a^2c_1^2c_2}{(1500(a-1)^3)} - \frac{48a^2c_1^2c_2}{(3000(a-1)^3)} + \frac{4ac_1c_3}{(100(a-1)^2)} + \frac{12ac_1c_3}{(300(a-1)^2)} - \frac{4ac_1^2c_2}{(100(a-1)^2)} - \frac{8ac_1^2c_2}{(200(a-1)^2)} - \frac{12ac_1^2c_2}{(300(a-1)^2)} - \frac{24ac_1^2c_2}{(1000(a-1)^3)} - \frac{32ac_1^2c_2}{(1500(a-1)^3)} - \frac{40ac_1^2c_2}{(3000(a-1)^3)}.$$

Rearranging and simplifying the terms, we get

$$|a_5| \le |lc_1^4 + c_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4|, \tag{3.7}$$

where

$$l = -\frac{2791a^3 - 2109a^2 + 261a + 81}{4000(a-1)^3},$$

$$r = -\frac{9a-3}{10(a-1)},$$

$$m = -\frac{27a-7}{15(a-1)},$$
$$n = \frac{729a^2-414a+53}{300(a-1)^2}.$$

Using Lemma 2.3, we get

$$|a_5| \le 2$$

whenever, the ratio of $L = 8r(1-r)((mn-2l)^2+(m(r+m)-n)^2)+m(1-m)(n-2rm)^2$ to $R = 4m^2(1-m)^2r(1-r)$ is less than 1. By calculating the ratio $\frac{L}{R}$, we obtain the result. □

Remark 3.6. Put $a = 0$ in the results (3.7), we get $|a_5| = 0.0604 < 2$. However, the result is not sharp.

Logarithmic Coefficients

Logarithmic coefficients for a function $f(z)$ in a class are the coefficients of the following function

$$\log\left(\frac{f(z)}{z}\right) = 2(\gamma_1 z + \gamma_2 z_2 + \cdots).$$

Many authors have worked on obtaining the logarithmic coefficients for a class of analytic functions (see Adegani *et al.* [1], Lecko and Partyka [12], and Ponnusamy [17]). In this subsection, we establish estimates for the first three logarithmic coefficients.

Theorem 3.7. *If* $f \in \mathcal{SS}^*_{3\mathcal{L}}$ *is of the form given in* (1.1)*, then*

(i) $|\gamma_1| \le \frac{4}{5(1-a)}$, (3.8)

(ii) $|\gamma_2| \le \frac{1}{5(1-a)} \max\left\{1, \left|\frac{4a}{5(1-a)}\right|\right\}$. (3.9)

The results are sharp.

Proof. For a function $f \in \mathcal{S}$ of the form (1.1), its first two logarithmic coefficients are given by

$$\gamma_1 = \frac{a_2}{2},$$
$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right).$$

We know from calculations (3.8) and (3.9) in Theorem 3.7, for a function $f \in \mathcal{SS}^*_{3\mathcal{L}}$,

$$a_2 = \frac{2c_1}{5(1-a)}$$

and

$$a_3 = \frac{c_1^2}{10a-10} - \frac{2c_2}{10a-10} + \frac{4c_1^2}{50a^2-100a+50} + \frac{4ac_1^2}{50a^2-100a+50}.$$

Therefore,

$$|\gamma_1| = \left|\frac{a_2}{2}\right| = \left|\frac{c_1}{5(1-a)}\right| \le \frac{2}{5(1-a)},$$
$$|\gamma_2| = \left|\frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right)\right|$$
$$= \left|\frac{c_1^2}{2(10a-10)} - \frac{c_1^2}{(5a-5)^2} - \frac{c_2}{10a-10} + \frac{2c_1^2}{50a^2-100a+50} + \frac{2ac_1^2}{50a^2-100a+50}\right|.$$

Simplifying the RHS term, we get

$$|\gamma_2| = \frac{1}{10(1-a)}\left|c_2 - \left(\frac{-(9a-5)}{10(1-a)}\right)c_1^2\right|.$$

Using Lemma 2.1, we obtain

$$|\gamma_2| \le \frac{1}{10(1-a)} 2\max\left\{1, \left|2\left(\frac{-(9a-5)}{10(1-a)}\right)-1\right|\right\}.$$

Simplifying the RHS we obtain the required result. □

Remark 3.8. If we put $a = 0$ in the results (3.8)-(3.9), we get $|\gamma_2| \le \frac{1}{5}$ which is true for starlike functions mentioned in [20].

Remark 3.9. In [20], the bound for the third logarithmic coefficient is estimated as $\frac{6}{25}$ in which there is an error in the calculation. The correct estimation is given in the following theorem.

Theorem 3.10. *If $f \in \mathcal{S}^*$ is of the form given in* (1.1)*, then*

$$|\gamma_3| \le \frac{2}{15}.$$

Proof. For $f \in \mathcal{S}^*$ of the form (1.1), we have

$$a_2 = \frac{2c_1}{5},$$

$$a_3 = \frac{c_2}{5} - \frac{c_1^2}{50},$$

$$a_4 = \frac{c_1^3}{250} - \frac{4c_2c_1}{75} + \frac{2c_3}{15}.$$

Since $\gamma_3 = \left(\frac{1}{2}\right)\left(a_4 - a_2a_3 + \left(\frac{1}{3}\right)a_2^3\right)$, substituting for a_2, a_3 and a_4 and simplifying the same we have,

$$|\gamma_3| = \left|\frac{c_1^3}{60} - \frac{c_2c_1}{15} + \frac{c_3}{15}\right|.$$

Using Lemma 2.2, we have

$$|\gamma_3| \le 2\left(\frac{1}{60} + \frac{1}{30} + \frac{1}{60}\right) = \frac{2}{15}.$$ □

Theorem 3.11. *If $f \in \mathcal{SS}^*_{3\mathcal{L}}$ is of the form given in* (1.1)*, then*

$$|\gamma_3| \le \frac{1}{750(1-a)^3}|(a-5)_2 + (9a-5)_2 + (9a_2 - 50a + 25)|. \tag{3.10}$$

Proof. Since $\gamma_3 = \left(\frac{1}{2}\right)\left(a_4 - a_2a_3 + \left(\frac{1}{3}\right)a_2^3\right)$, we have

$$\begin{aligned}\gamma_3 = {} & \frac{c_1^3}{\sigma_5} - \frac{c_1^3}{60a-60} - \frac{c_3}{15a-15} - \frac{c_1^3}{\sigma_3} - \frac{c_1^3}{\sigma_2} + \frac{4c_1^3}{\sigma_4} - \frac{4c_1^3}{375a^3 - 1125a^2 + 1125a - 375} \\ & - \frac{4c_1^3}{\sigma_1} - \frac{ac_1^3}{\sigma_3} - \frac{2ac_1^3}{\sigma_2} + \frac{4ac_1^3}{\sigma_4} - \frac{12ac_1^3}{\sigma_1} - \frac{8a^2c_1^3}{\sigma_1} + \frac{c_1c_2}{15a-15} - \frac{2c_1c_2}{\sigma_5} + \frac{2c_1c_2}{\sigma_3} \\ & + \frac{2c_1c_2}{\sigma_2} + \frac{2ac_1c_2}{\sigma_3} + \frac{4ac_1c_2}{\sigma_2},\end{aligned}$$

where

$$\sigma_1 = 750a^3 - 2250a^2 + 2250a - 750,$$
$$\sigma_2 = 150a^2 - 300a + 150,$$
$$\sigma_3 = 75a^2 - 150a + 75,$$
$$\sigma_4 = 250a^3 - 750a^2 + 750a - 250,$$
$$\sigma_5 = 50a^2 - 100a + 50.$$

Simplifying this, we get

$$|\gamma_3| = \left|\left(\frac{(9a-5)^2}{1500(1-a)^3}\right)c_1^3 - \left(-\frac{9a-5}{75(1-a)^2}\right)c_1c_2 + \left(\frac{1}{15(1-a)}\right)c_3\right|.$$

Using Lemma 2.2 and after simplifying the RHS, we obtain the following

$$\begin{aligned}|\gamma_3| &\le \left|\frac{(a-5)^2}{750(1-a)^3}\right| + \left|\frac{9a^2-50a+25}{375(1-a)^3}\right| + \left|\frac{(9a-5)^2}{750(1-a)^3}\right| \\ &= \frac{1}{750(1-a)^3}|(a-5)^2 + (9a-5)^2 + 2(9a^2-50a+25)|.\end{aligned}$$

□

Remark 3.12. If we put $a = 0$ in the result (3.10), we get $|\gamma_3| \le \frac{2}{15}$ which is true for starlike functions proved in Theorem 3.10.

4. Conclusion

This work is an example of generalization of known class of starlike functions using a class defined by Sokół, as one can see that allowing $a = 0$ in the Sokół's class gives the class of starlike functions associated with the three-leaf domain. Several authors in the past have associated various subclasses with various geometric domains in the right-hand side with the aim of generalizing classes. Two examples of such domains are the four-leaf domain and lemniscate of Bernoulli. Authors can look forward to considering these domains in their work. One can also look forward to applying fractional derivative operators and building new subclasses and obtaining the sharp bounds as further work.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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