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Research Article

Existence and Uniqueness of Solutions of Volterra Difference Equation of Fractional Order

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Abstract. In this paper, we investigate the existence, uniqueness, and qualitative behavior of solutions to certain fractional-order Volterra-type difference equation involving an iterated sum. The results are established using finite difference inequalities with explicit estimates.

Keywords. Difference equation, Fractional order, Initial value problem, Inequality

Mathematics Subject Classification (2020). 39A05, 39A22, 39A10, 39A99

1. Introduction

The set of natural numbers, including zero, is denoted by $\mathbb{N}_0$, and $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for $a \in \mathbb{Z}$. Let $u(n): \mathbb{N}_0 \to \mathbb{R}$. Consider the following nonlinear Volterra type difference equation with iterated sum and order $\alpha \in (0,1)$:

$$\nabla^\alpha u(n+1) = f(n) + \sum_{s=0}^{n-1} F(n,s,u(s)) + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} H(n,s,\sigma,u(\sigma))\right), \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

for $n \in \mathbb{N}_0$, where $u(n) \in D(\mathbb{N}_0,\mathbb{R})$ is a function, $f \in D(\mathbb{N}_0,\mathbb{R})$, $F: E_1 \times \mathbb{R} \to \mathbb{R}$, $H: E_2 \times \mathbb{R} \to \mathbb{R}$ in which

$$E_1 = \{(n,s) \in \mathbb{N}_0^2 \mid 0 \le s \le n < \infty\},$$

$E_2 = \{(n,s,\sigma) \in \mathbb{N}_0^3 \mid 0 \le \sigma \le s \le n < \infty\}$.

The study of fractional differential equations was initiated earlier, and it has recently been established that many classes of such equations admit unique solutions (Lakshmikantham *et al*. [8]). Although the theory of integro-differential equations has been almost fully developed in parallel with that of differential equations (Lakshmikantham and Rao [7], and Podlubny, [14]), the literature on fractional integro-differential equations is still less developed. Moreover, the advancement in the theory of fractional-order difference equations has been relatively minimal.

By allowing the order of the difference in the usual nth difference expression to be any real or complex number, Diaz and Osler [4] defined the fractional difference. Later, Hirota [6] used Taylor's series to define the fractional order difference operator ∇^α, where α is any real number. By altering Hirota's definition, Nagai [11] selected a different definition for the fractional order difference operator. Deekshitulu and Mohan [2] recently modified Nagai's definition for $0 < \alpha < 1$ so that there is no difference operator in the formula for ∇^α.

In 2010, Deekshitulu and Mohan [2] studied the existence and other properties of special version of equation (1.1) (see Agarwal [1], Deekshitulu and Mohan [3], Gray and Zhang [5], Mohan and Deekshitulu [9,10], Purnima *et al*. [15] and the references cited therein). Authors are motivated by the work of Deekshitulu and Mohan [2,9]. Hence, the equation (1.1) considered in this paper is in the general spirit of the investigations.

The main objective of this paper is to examine the boundedness, uniqueness, and continuous dependence of solutions to the given equations under various assumptions on the associated functions. The analysis primarily employs finite difference inequalities, with explicit estimates (see Deekshitulu and Mohan [2], and Pachpatte [12,13]). We believe that the results, obtained through elementary analysis, offer fundamental insights and may serve as a valuable reference for future research.

2. Preliminaries

For clarity and consistency, the following notations and definitions are employed throughout the paper (more information refer [2]). For all $n_1, n_2 \in \mathbb{N}_0$ and $n_1 > n_2$,

$$\sum_{j=n_1}^{n_2} u(j) = 0, \qquad \prod_{j=n_1}^{n_2} u(j) = 1.$$

In other words, the products and empty sums are taken to be 1 and 0, respectively. If n and $n-1$ are in $\mathbb{N}_0$, then the backward difference operator ∇ for the function $u(n)$, is defined as follows:

$$\nabla u(n) = u(n) - u(n-1).$$

We now present some fundamental definitions and results related to Nabla discrete fractional calculus.

Definition 1 ([2]). The extended binomial coefficient $\binom{a}{n}$, where $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, is defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)}, & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ 0, & \text{if } n < 0. \end{cases} \tag{2.1}$$

Definition 2 ([5]). For any complex numbers α and β, we define $\binom{\alpha}{\beta}$ as follows:

$$\binom{\alpha}{\beta} = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}, & \text{when } \alpha \text{ and } \alpha-\beta \text{ are neither zero nor negative integers,} \\ 1, & \text{when } \alpha=\beta=0, \\ 0, & \text{when } \alpha=0, \beta \text{ is neither zero nor negative integer,} \\ \text{undefined}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Remark. Let α and β be any two complex numbers. If α,β, and $\alpha-\beta$ are neither zero nor negative integers, then

$$(\alpha+\beta)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_{n-k} (\beta)_k, \tag{2.3}$$

for any positive integer n.

In 2003, Nagai [11] introduced the following definition for fractional order difference operator.

Definition 3. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1<\alpha \le m$. The difference operator ∇ of order α, with step length ϵ, is defined as

$$\nabla^{\alpha} u(n) = \begin{cases} \nabla^{\alpha-m}[\nabla^m u(n)] = \epsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \nabla^m u(n-j), & \text{if } \alpha>0, \\ u(n), & \text{if } \alpha=0, \\ \epsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} u(n-j), & \text{if } \alpha<0. \end{cases} \tag{2.4}$$

Studying the properties of the solution becomes challenging because the definition of $\nabla^{\alpha} u(n)$ given by Nagai [11] includes an ∇ operator and the term $(-1)^j$ inside the summation index. To circumvent this, Deekshitulu and Mohan [2,9] provided the following definition for $\varepsilon=m=1$.

Definition 4. The fractional sum operator of order α is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u(j). \tag{2.5}$$

The following definition of the fractional order difference operator of order α,

$$\nabla^{\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla(n-j) = \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0). \tag{2.6}$$

Remark. Assume that u, $v:\mathbb{N}_0^+ \to \mathbb{R}$ and $\alpha,\beta \in \mathbb{R}$ are constants such that $0<\alpha,\beta,\alpha+\beta<1$, and c,d are also constants. Then

(1) $\nabla^{\alpha}\nabla^{\beta} u(n) = \nabla^{\alpha+\beta} u(n)$,

(2) $\nabla^{\alpha}[(cu(n)+dv(n)] = c\nabla^{\alpha} u(n) + d\nabla^{\alpha} v(n)$,

(3) $\nabla^{-\alpha}\nabla^{\alpha} u(n) = u(n) - u(0)$,

(4) $\nabla^{\alpha}\nabla^{-\alpha} u(n) = u(n)$,

(5) $\nabla^{\alpha} u(0) = 0$ and $\nabla^{\alpha} u(1) = u(1) - u(0) = \nabla u(1)$.

3. Existence of Solution

The following theorem establishes the existence of a solution to equations (1.1)-(1.2).

Theorem 1. *There exists a solution* $u(n)$ *of the initial value problem* (1.1)-(1.2).

Proof. The existence of a solution to the Volterra-type difference equation with an iterated sum is straightforward, since the solution can be represented as a recurrence relation involving the values of the unknown function at earlier arguments. It follows from the definition of the fractional sum operator and the initial condition. Hence, considering equation (2.5) and replacing $u(n)$ by $\nabla^\alpha u(n)$, we obtain

$$\nabla^{-\alpha}[\nabla^\alpha u(n)] = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^\alpha u(j)],$$

or

$$u(n) - u(0) = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} [\nabla^\alpha u(j)],$$

or

$$u(n) = u(0) + \sum_{j=0}^{n-1} \binom{n-j+\alpha-2}{n-j-1} [\nabla^\alpha u(j+1)], \tag{3.1}$$

or

$$u(n) = u_0 + \sum_{j=0}^{n-1} B(n-1,\alpha;j) \left[f(j) + \sum_{s=0}^{j-1} F(j,s,u(s)) + \sum_{s=0}^{j-1} \left(\sum_{\sigma=0}^{s-1} H(j,s,\sigma,u(\sigma)) \right) \right], \tag{3.2}$$

where $B(n,\alpha;j) = \binom{n-j+\alpha-1}{n-j}$, for $0 \le j \le n$. The recurrence relation above indicates that (1.1)-(1.2) has a solution. □

4. Uniqueness of Solution

We now prove that the solutions to the fractional order difference equations (1.1)-(1.2) unique. For this, we need the following results.

Lemma 1 ([9])**.** *For* $n \in \mathbb{N}_0$,

$$\sum_{j=0}^{n} B(n,\alpha;j) = \binom{n+\alpha}{n}.$$

For more clarity, we present some basic finite difference inequalities which play crucial role to establish the fractional difference inequalities.

Theorem 2 ([12])**.** *Let* $u(n)$, $a(n)$, *and* $b(n)$ *be real non-negative functions defined on* $\mathbb{N}_0$ *and*

$$\Delta u(n) \le a(n)u(n) + b(n)$$

for $n \in \mathbb{N}_0$. *Then,*

$$u(n) \le u(0) \prod_{j=0}^{n-1} [1+a(j)] + \sum_{j=0}^{n-1} b(j) \prod_{k=j+1}^{n-1} [1+a(k)],$$

for $n \in \mathbb{N}_0$.

Theorem 3 ([3]). *Let $u(n)$, $a(n)$, and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$. If*

$$u(n) \le u(0) + \sum_{j=0}^{n-1} [a(j)u(j) + b(j)]$$

for $n \in \mathbb{N}_0$, then

$$\begin{aligned} u(n) &\le u(0) \prod_{j=0}^{n-1} [1 + a(j)] + \sum_{j=0}^{n-1} b(j) \prod_{k=j+1}^{n-1} [1 + a(k)] \\ &\le u(0) \exp\left(\sum_{j=0}^{n-1} a(j)\right) + \sum_{j=0}^{n-1} b(j) \exp\left(\sum_{k=j+1}^{n-1} a(k)\right). \end{aligned}$$

The following corollary is proved by Pachpatte ([12, p. 12]).

Corollary 1. *Let $u(n)$ and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$, and $c \ge 0$ (a constant). If*

$$u(n) \le c + \sum_{j=0}^{n-1} [b(j)u(j)],$$

for $n \in \mathbb{N}_0$, then

$$\begin{aligned} u(n) &\le c \prod_{j=0}^{n-1} [1 + b(j)] \\ &\le c \exp\left(\sum_{j=0}^{n-1} b(j)\right). \end{aligned}$$

Finite fractional difference inequalities which provides explicit bounds on the unknown functions and analysis of various problems in the theory of finite fractional difference equations. So, on similar line of discrete inequalities mentioned above, we present the finite fractional inequalities.

Theorem 4 ([3]). *Let $u(n)$, $a(n)$, and $b(n)$ be real valued non-negative functions defined on $\mathbb{N}_0$. If for $n \in \mathbb{N}_0$, $0 < \alpha < 1$,*

$$\nabla^{\alpha} u(n+1) \le a(n)u(n) + b(n),$$

then

$$u(n) \le u(0) \prod_{j=0}^{n-1} [1 + B(n-1, \alpha; j)a(j)] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} [1 + B(n-1, \alpha; k)a(k)].$$

Corollary 2. *Let $u(n)$, $a(n)$, and $b(n)$ be real valued non-negative functions defined on $\mathbb{N}_0$. If for $0 < \alpha < 1$, $n \in \mathbb{N}_0$,*

$$u(n) \le u(0) + \sum_{j=0}^{n-1} B(n-1, \alpha; j)[a(j)u(j) + b(j)],$$

then

$$u(n) \le u(0) \prod_{j=0}^{n-1} [1 + B(n-1, \alpha; j)a(j)] + \sum_{j=0}^{n-1} B(n-1, \alpha; j)b(j) \prod_{k=j+1}^{n-1} [1 + B(n-1, \alpha; k)a(k)].$$

for $n \in \mathbb{N}_0$.

In the literature, some authors used Corollary 2 to study the various properties of solutions of finite fractional difference equations. But direct use of this corollary leads to some flaws. So, we present the following fractional inequality to address the issue raised due to the use of Corollary 2.

Theorem 5. *Let $u(n)$, $a(n)$, and $b(n)$ be real non-negative functions defined on $\mathbb{N}_0$, and $c \ge 0$ (a constant). If for $0 < \alpha < 1$ and $n \in \mathbb{N}_0$,*

$$u(n) \le c + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[a(j)u(j)+b(j)], \tag{4.1}$$

then

$$u(n) \le c\prod_{j=0}^{n-1}[1+B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}[1+B(n-1,\alpha;k)a(k)], \tag{4.2}$$

for $n \in \mathbb{N}_0$.

Proof. Define a function $z(n)$ by the right hand side of (4.2), that is

$$z(n) = c + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[a(j)u(j)+b(j)], \quad \text{for } n \in \mathbb{N}_0. \tag{4.3}$$

Then, $z(0) = c, u(n) \le z(n)$ and

$$\nabla^\alpha z(n+1) = a(n)u(n) + b(n), \quad \text{for } n \in \mathbb{N}_0. \tag{4.4}$$

As $u(n) \le z(n)$, the equation (4.4) becomes

$$\nabla^\alpha z(n+1) \le a(n)z(n) + b(n), \quad \text{for } n \in \mathbb{N}_0 \tag{4.5}$$

with $z(0) = c$, and $0 < \alpha < 1$.
Now, application of Theorem 4 to (4.5) yields,

$$z(n) \le z(0)\prod_{j=0}^{n-1}[1+B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}[1+B(n-1,\alpha;k)a(k)],$$

which implies

$$z(n) \le c\prod_{j=0}^{n-1}[1+B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}[1+B(n-1,\alpha;k)a(k)], \tag{4.6}$$

for $n \in \mathbb{N}_0$. Hence, using (4.6) in $u(n) \le z(n)$, we get

$$u(n) \le c\prod_{j=0}^{n-1}[1+B(n-1,\alpha;j)a(j)] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}[1+B(n-1,\alpha;k)a(k)]. \tag{4.7}$$

for $n \in \mathbb{N}_0$. This is the required inequality. □

The following theorem deals with uniqueness of the solution to fractional order difference equations.

Theorem 6. *Suppose that the functions F, H in equation* (1.1)-(1.2) *satisfy the conditions*

$$\sum_{s=0}^{n-1} |F(n,s,u(s)) - F(n,s,v(s))| \le L_1|u(n)-v(n)|, \tag{4.8}$$

$$\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} |H(n,s,\sigma,u(\sigma)) - H(n,s,\sigma,v(\sigma))|\right) \le L_2|u(n)-v(n)|, \tag{4.9}$$

where L_1 *and* L_2 *are non-negative constants. Then, the initial value problem* (1.1)-(1.2) *has a unique solution.*

Proof. Let $v(n)$ and $w(n)$ be any two solutions of (1.1)-(1.2) satisfying $v(0) = w(0) = u_0$. Then, recalling recurrence relation for solution and hypotheses, for an arbitrary $\epsilon > 0$, it follows that

$$\begin{aligned}
|v(n)-w(n)| &= \left|\sum_{j=0}^{n-1} B(n-1,\alpha;j)\left[\sum_{s=0}^{j-1}(F(j,s,v(s))-F(j,s,w(s)))\right.\right.\\
&\quad \left.\left.+\sum_{s=0}^{j-1}\sum_{\sigma=0}^{s-1}(H(j,s,\sigma,v(\sigma))-H(j,s,\sigma,w(\sigma)))\right]\right|\\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\sum_{s=0}^{j-1}|F(j,s,v(s))-F(j,s,w(s))|\\
&\quad +\sum_{j=0}^{n-1} B(n-1,\alpha;j)\sum_{s=0}^{j-1}\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,v(\sigma))-H(j,s,\sigma,w(\sigma))|\\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)L_1|v(j)-w(j)| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)L_2|v(j)-w(j)|\\
&= \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1+L_2)|v(j)-w(j)|\\
&< \epsilon + \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1+L_2)|v(j)-w(j)|.
\end{aligned}$$

Let $u(n) = |v(n)-w(n)|$. Then, the above inequality implies

$$u(n) < \epsilon + \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1+L_2)u(j). \tag{4.10}$$

Hence, by application of Theorem 5 to (4.10), we get

$$\begin{aligned}
u(n) &< \epsilon \prod_{j=0}^{n-1}[1+B(n-1,\alpha;j)(L_1+L_2)]\\
&< \epsilon \exp\left[\sum_{j=0}^{n-1}(L_1+L_2)B(n-1,\alpha;j)\right].
\end{aligned} \tag{4.11}$$

Using Lemma 1 in (4.11), we obtain

$$u(n) < \epsilon \exp\left[(L_1+L_2)\binom{n+\alpha-1}{n-1}\right]. \tag{4.12}$$

Since the arbitrary nature of ϵ, inequality (4.12) conclude that $u(n) \to 0$ as $n \to \infty$ and hence, we have $v(n) = w(n)$. This proves the uniqueness of the solutions. □

5. Boundedness of Solution

The following theorem shows boundedness of solution to the problem (1.1)-(1.2).

Theorem 7. *Suppose that the functions* F, H *in equations* (1.1)-(1.2) *satisfy the conditions* (4.8) *and* (4.9) *respectively. If* $u : \mathbb{N}_0 \to \mathbb{R}$ *is a solution of the Volterra difference equation with iterated* *sum* (1.1)-(1.2)*, then*

$$|u(n)| \le |u_0| \prod_{j=0}^{n-1} [1+(L_1+L_2)B(n-1,\alpha;j)] + \sum_{j=0}^{n-1} (L_3+L_4+|f(j)|)B(n-1,\alpha;j) \prod_{k=j+1}^{n-1} [1+(L_1+L_2)B(n-1,\alpha;k)], \tag{5.1}$$

for $n \in \mathbb{N}_0$.

Proof. From the equation (3.2) and hypotheses, we estimate

$$\begin{aligned}|u(n)| &\le |u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j)| + \sum_{s=0}^{j-1} |F(j,s,u(s))-F(j,s,0)| + \sum_{s=0}^{j-1}|F(j,s,0)| \\ &\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,u(\sigma))-H(j,s,\sigma,0)| + \sum_{\sigma=0}^{s-1}|H(j,s,\sigma,0)|\right)\Bigg] \\ &\le |u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j)| + \sum_{s=0}^{j-1} |F(j,s,u(s))-F(j,s,0)| + \sum_{s=0}^{j-1}|F(j,s,0)| \\ &\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,u(\sigma))-H(j,s,\sigma,0)| + \sum_{\sigma=0}^{s-1}|H(j,s,\sigma,0)|\right)\Bigg] \\ &\le |u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[|f(j)|+L_1|u(j)|+L_2|u(j)|+L_3+L_4] \\ &\le |u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[(L_1+L_2)|u(j)|+(|f(j)|+L_3+L_4)],\end{aligned} \tag{5.2}$$

where

$$L_3 = \max_{0\le j\le n-1}\left(\sum_{s=0}^{j-1}|F(j,s,0)|\right) \quad \text{and} \quad L_4 = \max_{0\le j\le n-1}\left(\sum_{s=0}^{j-1}\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,0)|\right).$$

Using Theorem 5 to the inequality (5.2), we get

$$|u(n)| \le |u_0| \prod_{j=0}^{n-1} [1+(L_1+L_2)B(n-1,\alpha;j)] + \sum_{j=0}^{n-1} (L_3+L_4+|f(j)|)B(n-1,\alpha;j) \prod_{k=j+1}^{n-1} [1+(L_1+L_2)B(n-1,\alpha;k)], \tag{5.3}$$

which is the required result. □

6. Continuous Dependence

In this section, we shall deal with continuous dependence of the problem (1.1)-(1.2) on the initial data, function induced therein and also on parameters.

6.1 Dependence on Initial Data

We first discuss dependence of solution on given initial data.

Theorem 8. *Suppose that equations* (4.8)-(4.9) *hold. If* $v(n)$ *and* $w(n)$ *are solutions of* (1.1)-(1.2) *with initial data* $v(0) = v_0$ *and* $w(0) = w_0$ *respectively, then*

$$|v(n) - w(n)| \le |v_0 - w_0| \exp\left[(L_1 + L_2)\binom{n+\alpha-1}{n-1}\right].$$

Proof. By using the fact that $v(n)$ and $w(n)$ are solutions of (1.1)-(1.2). Hence, by hypotheses and looking at the proof of Theorem 6, we have

$$|v(n) - w(n)| \le |v_0 - w_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1 + L_2)|v(j) - w(j)|. \tag{6.1}$$

Using Theorem 5 to the inequality (6.1), we obtain

$$\begin{aligned} |v(n) - w(n)| &\le |v_0 - w_0| \prod_{j=0}^{n-1} [1 + (L_1 + L_2)B(n-1,\alpha;j)] \\ &\le |v_0 - w_0| \exp\left[\sum_{j=0}^{n-1} (L_1 + L_2)B(n-1,\alpha;j)\right]. \end{aligned} \tag{6.2}$$

Using Lemma 1 in (6.2), we obtain

$$|v(n) - w(n)| \le |v_0 - w_0| \exp\left[(L_1 + L_2)\binom{n+\alpha-1}{n-1}\right]. \tag{6.3}$$

This demonstrates how the equation's solutions rely continuously on the initial data. □

6.2 Dependence on Function

Consider the equations (1.1)-(1.2) and the corresponding equation

$$\nabla^\alpha v(n+1) = \bar{f}(n) + \sum_{s=0}^{n-1} \bar{F}(n,s,v(s)) + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} \bar{H}(n,s,\sigma,v(\sigma))\right) \tag{6.4}$$

with condition (1.2), where $\bar{f}, \bar{F}, \bar{H}$ are defined as f, F, H.

The following theorem present the closeness of solutions.

Theorem 9. *Suppose that equations* (4.8)-(4.9) *hold. Furthermore, assume that there exist constants* $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, *for which*

$$|f(j) - \bar{f}(j)| \le \epsilon_1,$$

$$\left(\sum_{s=0}^{n-1} |F(j,s,w(s)) - \bar{F}(j,s,w(s))|\right) \le \epsilon_2,$$

$$\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} |H(j,s,\sigma,w(\sigma)) - \bar{H}(j,s,\sigma,w(\sigma))|\right) \le \epsilon_3.$$

If $u(n)$ *and* $v(n)$ *are respectively solutions of* (6.4) *and* (1.1) *with* (1.2), *then*

$$|v(n) - u(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)(\epsilon_1 + \epsilon_2 + \epsilon_3) \prod_{k=j+1}^{n-1} [1 + B(n-1,\alpha;k)(L_1 + L_2)], \tag{6.5}$$

for $n \in \mathbb{N}_0$.

Proof. Let $u(n)$ and $v(n)$ be the solutions of (1.1)-(1.2) and (6.4) with (1.2), respectively. Then by hypotheses, we have

$$\begin{aligned}|v(n)-u(n)| &\le |u_0-u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j)-\bar{f}(j)| + \sum_{s=0}^{j-1}|F(j,s,v(s))-\bar{F}(j,s,u(s))| \\ &\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,v(\sigma))-\bar{H}(j,s,\sigma,u(\sigma))|\right)\Bigg] \\ &\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j)-\bar{f}(j)| + \sum_{s=0}^{j-1}|F(j,s,v(s))-F(j,s,u(s))| \\ &\quad + \sum_{s=0}^{j-1}|F(j,s,u(s))-\bar{F}(j,s,u(s))| + \sum_{s=0}^{j-1}\Bigg(\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,v(\sigma))-H(j,s,\sigma,u(\sigma))| \\ &\quad + \sum_{\sigma=0}^{s-1}|H(j,s,\sigma,u(\sigma))-\bar{H}(j,s,\sigma,u(\sigma))|\Bigg)\Bigg] \\ &\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)[|f(j)-\bar{f}(j)| + L_1|v(j)-u(j)| + L_2|v(j)-u(j)| \\ &\quad + \sum_{s=0}^{j-1}|F(j,s,u(s))-\bar{F}(j,s,u(s))| + \sum_{j=0}^{j-1}\sum_{\sigma=0}^{s-1}|H(j,s,\sigma,u(\sigma))-\bar{H}(j,s,\sigma,u(\sigma))|\Bigg] \\ &\le \sum_{j=0}^{n-1} B(n-1,\alpha,;j)[(L_1+L_2)|v(j)-u(j)| + (\epsilon_1+\epsilon_2+\epsilon_3)]. \end{aligned} \tag{6.6}$$

The subsequent equation (6.7) is the result of applying Theorem 5 to the above inequality,

$$|v(n)-u(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)(\epsilon_1+\epsilon_2+\epsilon_3)\prod_{k=j+1}^{n-1}[1+B(n-1,\alpha;k)(L_1+L_2)]. \tag{6.7}$$

The solutions to problems (1.1) and (6.4) with condition (1.2) are close to one another, as can be inferred from (6.7), if $\bar{f}$, $\bar{F}$, and $\bar{H}$ are, respectively, close to f, F, and H. □

6.3 Dependence on Parameters

We next consider the following Volterra difference equations

$$\nabla^\alpha u(n+1) = f(n,\mu_1) + \sum_{s=0}^{n-1} F(n,s,u(s),\mu_1) + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} H(n,s,\sigma,u(\sigma),\mu_1)\right) \tag{6.8}$$

and

$$\nabla^\alpha v(n+1) = f(n,\mu_2) + \sum_{s=0}^{n-1} F(n,s,v(s),\mu_2) + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} H(n,s,\sigma,v(\sigma),\mu_2)\right) \tag{6.9}$$

with condition (1.2), where $u(n) \in D(\mathbb{N}_0,\mathbb{R})$ is a function $f \in D(\mathbb{N}_0\times\mathbb{R},\mathbb{R})$, $F: E_1\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $H: E_2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and μ_1, μ_2 are real constant.

Theorem 10. *Suppose that the functions f, F, H satisfying the conditions*

$$|f(n,\mu_1)-f(n,\mu_2)| \le L_1|\mu_1-\mu_2|,$$

$$\sum_{s=0}^{n-1}|F(n,s,v(s),\mu_1)-F(n,s,u(s),\mu_1)| \le L_2|v(n)-u(n)|,$$

$$\sum_{s=0}^{n-1} |F(n,s,u(s),\mu_1) - F(n,s,u(s),\mu_2)| \le L_3|\mu_1 - \mu_2|,$$

$$\sum_{s=0}^{n-1}\sum_{\sigma=0}^{s-1} |H(n,s,\sigma,v(\sigma),\mu_1) - H(n,s,\sigma,u(\sigma),\mu_1)| \le L_4|v(n) - u(n)|,$$

$$\sum_{s=0}^{n-1}\sum_{\sigma=0}^{s-1} |H(n,s,\sigma,v(\sigma),\mu_1) - H(n,s,\sigma,u(\sigma),\mu_2)| \le L_5|\mu_1 - \mu_2|,$$

where L_1, L_2, L_3, L_4, L_5 *are non negative constants. If* $u(n)$ *and* $v(n)$ *are respectively solutions of* (6.8) *and* (6.9) *with condition* (1.2)*, then*

$$|v(n) - u(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1 + L_3 + L_5)|\mu_1 - \mu_2| \prod_{k=j+1}^{n-1} [1 + B(n-1,\alpha;k)(L_2 + L_4)], \tag{6.10}$$

for $n \in \mathbb{N}_0$.

Proof. From the assumptions, it follows that

$$\begin{aligned}
&|v(n) - u(n)| \\
&\le |u_0 - u_0| + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j,\mu_1) - f(j,\mu_2)| + \sum_{s=0}^{j-1} |F(j,s,v(s),\mu_1) - F(j,s,u(s),\mu_2)| \\
&\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1} |H(j,s,\sigma,v(\sigma),\mu_1) - H(j,s,\sigma,v(\sigma),\mu_2)|\right)\Bigg] \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Bigg[|f(j,\mu_1) - f(j,\mu_2)| + \sum_{s=0}^{j-1} |F(j,s,v(s),\mu_1) - F(j,s,u(s),\mu_1)| \\
&\quad + \sum_{s=0}^{j-1} |F(j,s,u(s),\mu_1) - F(j,s,u(s),\mu_2)| \\
&\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1} |H(j,s,\sigma,v(\sigma),\mu_1) - H(j,s,\sigma,u(\sigma),\mu_1)|\right) \\
&\quad + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1} |H(j,s,\sigma,u(\sigma),\mu_1) - H(j,s,\sigma,u(\sigma),\mu_2)|\right)\Bigg] \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)[L_1|\mu_1 - \mu_2| + L_2|v(j) - u(j)| + L_3|\mu_1 - \mu_2| + L_4|v(j) - u(j)| + L_5|\mu_1 - \mu_2|] \\
&\le \sum_{j=0}^{n-1} B(n-1,\alpha;j)[(L_2 + L_4)|v(j) - u(j)| + (L_1 + L_3 + L_5)|\mu_1 - \mu_2|]. \qquad (6.11)
\end{aligned}$$

With the help of Theorem 5 and the inequality (6.11), we get

$$|v(n) - u(n)| \le \sum_{j=0}^{n-1} B(n-1,\alpha;j)(L_1 + L_3 + L_5)|\mu_1 - \mu_2| \prod_{k=j+1}^{n-1} [1 + B(n-1,\alpha;k)(L_2 + L_4)]. \tag{6.12}$$

This demonstrates how the parameters μ_1 and μ_2 affect the solution of equations (6.8) and (6.9) with condition (1.2). □

7. Example

We consider the following problem:

$$\nabla^{\frac{1}{2}} u(n+1) = n + \sum_{s=0}^{n-1} s u(s) + \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} 100\sin(\sigma)\right), \tag{7.1}$$

$$u(0) = 0. \tag{7.2}$$

Solution. From the equation (3.2), one can write the corresponding solution of the given problem (7.1)-(7.2) as

$$u(n) = \sum_{j=0}^{n-1} B\left(n-1, \frac{1}{2}; j\right)\left[j + \sum_{s=0}^{j-1} s u(s) + \sum_{s=0}^{j-1}\left(\sum_{\sigma=0}^{s-1} 100\sin(u(\sigma))\right)\right]. \tag{7.3}$$

Comparing this equation with the equation (1.1), we have

$$f(n) = n,\ F(n,s,u(s)) = s u(s),\ H(n,s,\sigma,u(\sigma)) = 100\sin(u(\sigma)).$$

Then, $F(n,s,u(s))$ and $H(n,s,\sigma,u(\sigma))$ satisfy the conditions

$$\sum_{s=0}^{n-1} |F(n,s,u(s)) - F(n,s,v(s))| = \sum_{s=0}^{n-1} |s u(s) - s v(s)| \le L_1 |u(n) - v(n)|, \tag{7.4}$$

$$\begin{aligned}\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} |H(n,s,\sigma,u(\sigma)) - H(n,s,\sigma,v(\sigma))|\right) &= \sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} |100\sin(u(\sigma)) - 100\sin(v(\sigma))|\right)\\ &= 100\sum_{s=0}^{n-1}\left(\sum_{\sigma=0}^{s-1} |\sin(u(\sigma)) - \sin(v(\sigma))|\right)\\ &= 100\sum_{\sigma=0}^{n-2}(n-1-\sigma)|\sin(u(\sigma)) - \sin(v(\sigma))|\\ &\le 100(n-1)|u(n) - v(n)|\\ &\le L_2|u(n) - v(n)|,\end{aligned} \tag{7.5}$$

where $L_1 = \max\limits_{0\le s\le n-1}\{s\} = n-1$ and $L_2 = 100(n-1)$ and $\sum\limits_{s=0}^{n-1}\left(\sum\limits_{\sigma=0}^{s-1}\right) u(\sigma) = \sum\limits_{\sigma=0}^{n-2}(n-1-\sigma)u(\sigma)$ was used to simplify the result.

Hence, in view of Theorem 6, we observe that

$$\begin{aligned}|v(n) - w(n)| &< \epsilon \exp\left[(L_1 + L_2)\binom{n+\alpha-1}{n-1}\right]\\ &< \epsilon \exp\left[(n-1+100(n-1))\binom{n+\alpha-1}{n-1}\right]\\ &< \epsilon \exp\left[101(n-1)\binom{n+\alpha-1}{n-1}\right].\end{aligned} \tag{7.6}$$

Since the arbitrary nature of ϵ, inequality (7.6) conclude that $u(n) \to 0$ as $n \to \infty$ and hence, we have $v(n) = w(n)$. This proves the uniqueness of the solutions.

In particular, for $n = 10$ and $\alpha = \frac{1}{2}$, the inequality (7.6) becomes

$$|v(n) - w(n)| < \epsilon \exp\left[101(10-1)\binom{10+\frac{1}{2}-1}{10-1}\right]$$

$$< \epsilon \exp\left[909\binom{9+\frac{1}{2}}{9}\right]$$

$$< \epsilon \exp\left[909\binom{\frac{19}{2}}{9}\right]. \tag{7.7}$$

Finally, referring the definition as in (2.1), one can have

$$|v(n)-w(n)| < \epsilon \exp\left[909\frac{\Gamma\left(\frac{19}{2}+1\right)}{\Gamma\left(\frac{19}{2}-9+1\right)\Gamma(9+1)}\right]$$

$$= \epsilon \exp\left[909\times\frac{19}{2}\frac{\Gamma\left(\frac{19}{2}\right)}{\Gamma\left(\frac{1}{2}+1\right)\Gamma(10)}\right]$$

$$= \epsilon \exp\left[909\times\frac{19}{2}\frac{\Gamma\left(\frac{19}{2}\right)}{\frac{1}{2}\sqrt{\pi}10!}\right]$$

$$= \epsilon \exp\left[909\times 19\frac{\frac{34459425\sqrt{\pi}}{512}}{\sqrt{\pi}10!}\right]$$

$$= \epsilon \exp\left[\frac{17271}{512}\times\frac{34459425}{10!}\right]$$

$$\cong \epsilon \exp(320.33). \tag{7.8}$$

Or equivalently, one can seen that

$$0 \le |v(n)-w(n)|\exp(-320.33) < \epsilon, \tag{7.9}$$

for every ϵ and n. Therefore, looking at the definition as in (2.1) and $n\to\infty$, we conclude that $v(n)=w(n)$. This proves the our required.

8. Conclusions

In the first part, we used a recurrence relation to the values of the unknown function at earlier arguments and the definition of the fractional sum operator to establish the existence of solution. Next, we derive the finite fractional difference inequality which is kind enough to study the uniqueness of the solution to the initial value problem (1.1)-(1.2). Further, the finite fractional difference inequalities help to discuss various properties of solutions like boundedness, continuous dependence on the initial data, the closeness of solutions, and dependence on parameters and functions involved therein. At last, we provided a suitable example to illustrate all of the findings.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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