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Research Article

Newly Defined Conformable Double Laplace Transform and Partial Differential Equations with Conformable Derivative

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Abstract. This study provides a comprehensive investigation of the newly defined conformable double Laplace transform and the conformable derivatives. Several distinct properties of the conformable double Laplace transform are explored. The effectiveness and reliability of the method are demonstrated by obtaining analytical solutions to initial and boundary value problems associated with well-known fractional partial differential equations involving conformable derivatives.

Keywords. Conformable derivative, Conformable integral, Conformable double Laplace transform, Conformable partial differential equations, Klein-Gordan equation, Kdv equation

Mathematics Subject Classification (2020). 44A10

1. Introduction

Fractional calculus was born in the letters of L'Hospital to Leibniz in 1965 (Podlubny [20]). Recently, fractional calculus has played an important role in various fields such as engineering, physics, biology, chemistry, medicine, etc. In the last and present centuries, fractional calculus has attracted many researchers (Miller and Ross [16], Oldham and Spanier [17]).

Many of the real-world problems arising in mathematical physics and engineering are described by fractional differential equations. There are several definitions of fractional derivatives, such as the Grunwald-Letnikov, Riemann-Liouville, Caputo, Riesz, and Hadamard etc. Out of these various definitions, the most popular ones are Riemann-Liouville and Caputo. But these derivatives have some limitations, as they do not obey the chain rule and product rule of differentiation. Also, the appearance of integral forms in their definitions makes our calculations hard. Due to this, some difficulties arise in the applications of the fractional derivative in science and engineering. To address these issues, researchers seek straightforward and efficient definitions that uphold the fundamental properties of classical derivatives. In 2014, Khali *et al*. [13] defined the new definitions of derivative called the *conformable derivative* (CD). These derivatives seem to be the natural extension of classical derivatives and hence satisfy the basic properties of classical derivatives. Since the conformable derivative is defined in the limit form, it is easy to handle. Now it has been accepted so quickly and has become the subject of interest.

For instance, Korkman [14] used the wave transform for the conformable KP-equation, modified Kawahara equation, Kdv equation, and Burgers equation. Ali *et al*. [2] combined the G'/G method and modified Kudryashov method to obtain an exact solution of conformable time fractional $(1+2)$ *dimensional extended Zakharov-Kuznestov equation* (2D-FZKE). Chen and Jiang [5] investigated the simplest equation method and applied it to obtain an exact solution of some time fractional CPDE. Eltayeb *et al*. [8] obtained exact and approximate solutions of singular coupled conformable pseudoparabolic and pseudohyperbolic equations. Hashemi [9] investigated the invariant subspace method to get an exact solution of the coupled system of CFDEs. Çenesiz and Kurt [4] obtained the exact solution of the conformable heat equation by using the conformable Fourier transform. Kadkhoda and Jafari [10] applied the new Sine-Gordon method to obtain an analytical solution of the Conformable space-time Duffing model and space-time Eckhaus equation. Yavuz [23] developed an analytical solution of CDE using the conformable fractional Adomian decomposition method and conformable homotopy perturbation method. Kaplan [11] applied the modified simple equation method and exponential rational function method for solving a nonlinear conformable time fractional equations. Özkan and Kurt [18] investigated the solution of some conformable fractional partial integro-differential equation. Ayata and Özkan [3] have discussed the conformable *Newell-Whitehead-Segel* (NWS) equation and found the solution by using the Laplace transform method. Khalil and Shaab [12] solved CDE by applying the Fourier series and separation of variables. Saad and Brahim [21] used the Fourier series method to obtain an analytical solution of the conformable fractional telegraph equation. Thabet *et al*. [22] discussed the differential transform method to obtain an analytical solution of a system of CPDEs.

The *conformable partial differential equations* (CPDE) arise in various fields of science and engineering. The solution of such equations is needed for the study of geometrical and physical interpretation. Due to the appearance of conformable derivatives, the solution of conformable differential equations seems to be more complicated with the existing method. Therefore, an efficient and reliable method for solving such differential equations is needed. As we know, the Laplace transform method plays a vital role in finding solutions to differential equations. Recently, Debnath [6], and Dhunde and Waghmare [7] have investigated the double Laplace transform and obtained the solution of partial differential equations arising in mathematical physics. For the differential equations involving conformable derivatives, Abdeljawad [1] in

2015 defined the concept of the fractional Laplace transform, which is known as the conformable Laplace transform. In 2018, Özkan and Kurt [19] extended the concept of conformable Laplace transform and defined the *conformable double Laplace transform* (CDLT). They implemented the method to solve some homogeneous and non-homogeneous partial differential equations.

The CDLTM provides an innovative analytical method for solving CFPDEs with multiple fractional orders. Utilizing the local and limit-based characteristics of the conformable derivative, this method effectively generalizes the classical double Laplace transform to the fractional domain, maintaining its fundamental operational properties. In contrast to traditional fractional transform methods, which typically rely on complex convolution theorems or special functions, the conformable double Laplace transform streamlines the solution process, reduces computational complexity, and closely emulates the behavior of integer-order systems. Consequently, it acts as a powerful and efficient tool for analyzing a wide variety of space-time fractional models in physics and engineering, particularly those involving memory effects or anomalous diffusion behavior.

2. Preliminaries

Definition 2.1 (Conformable Derivative (CD) [1])**.** The left conformable fractional derivative of the function $y:[a,\infty)\to\mathbb{R}$, starting from a of order α, $0<\alpha\le 1$ is defined as [1],

$$(T_a^\alpha)y(t)=\lim_{\epsilon\to 0}\frac{y(t+\epsilon(t-a)^{1-\alpha})-y(t)}{\epsilon},$$

For $a=0$, we write Khalil's definition as,

$$(T^\alpha)y(t)=\lim_{\epsilon\to 0}\frac{y(t+\epsilon t^{1-\alpha})-y(t)}{\epsilon},\quad t\ge 0,\ 0<\alpha\le 1.$$

Similarly the right conformable fractional derivative terminating at b of order α is defined as,

$$({}_bT^\alpha)y(t)=-\lim_{\epsilon\to 0}\frac{y(t+\epsilon(b-t)^{1-\alpha})-y(t)}{\epsilon},\quad 0<\alpha\le 1.$$

Definition 2.2 (Conformable Partial Derivative (time) [22])**.** The conformable (time) fractional partial derivative of the function $y(x,t):\mathbb{R}\times(0,\infty)\to\mathbb{R}$, of order β, $0<\beta\le 1$ is defined as,

$$(T_t^\beta y(x,t))=\lim_{\epsilon\to 0}\frac{y(x,t+\epsilon t^{1-\beta})-y(x,t)}{\epsilon}.$$

Definition 2.3 (Conformable Partial Derivative (space) [22])**.** The conformable (space) fractional partial derivative of the function $y(x,t):\mathbb{R}\times(0,\infty)\to\mathbb{R}$, of order α, $0<\alpha\le 1$ is defined as,

$$(T_x^\alpha y(x,t))=\lim_{\epsilon\to 0}\frac{y(x+\epsilon x^{1-\alpha},t)-y(x,t)}{\epsilon}.$$

Definition 2.4 (Conformable Integral [13])**.** For the function $f:[a,\infty)\to\mathbb{R}$, the left conformable fractional integral of order α starting at a is defined as [13],

$$(I_a^\alpha)f(t)=\int_a^t f(x)d_\alpha x=\int_a^t f(x)\,(x-a)^{\alpha-1}\,dx.$$

For $a=0$, we write

$$(I^\alpha)f(t)=\int_0^t f(x)d_\alpha x=\int_0^t f(x)\,x^{\alpha-1}dx.$$

The right conformable fractional integral of order α terminating at b of f is defined as

$$({}_bI^{\alpha})f(t)=\int_t^b f(x)d_{\alpha}x=\int_t^b f(x)\,(b-x)^{\alpha-1}\,dx.$$

Properties of Conformable Derivative

Let $0<\alpha\le 1$ and y_1,y_2 be α-differentiable at $t>0$, then

(1) $T_{\alpha}(ay_1+by_2)=aT_{\alpha}(y_1)+bT_{\alpha}(y_2)$, for all constants a and b,

(2) $T_{\alpha}(y)=0$, for all constant functions $y(t)=c$,

(3) $T_{\alpha}(y_1y_2)=y_1T_{\alpha}(y_2)+y_2T_{\alpha}(y_1)$,

(4) $T_{\alpha}\left(\frac{y_1}{y_2}\right)=\left(\frac{y_2T_{\alpha}(y_1)-y_1T_{\alpha}(y_2)}{y_2^2}\right)$, $y_2\neq 0$,

(5) if y is differentiable then $T_{\alpha}(y)=t^{1-\alpha}y'(t)$.

Conformable Derivative of Standard Functions

(1) $T_{\alpha}t^k=kt^{k-\alpha}$, for all real number k,

(2) $T_{\alpha}(1)=0$,

(3) $T_{\alpha}e^{at}=at^{1-\alpha}e^{at}$, $T_{\alpha}\left(e^{\frac{at^{\alpha}}{\alpha}}\right)=ae^{\frac{at^{\alpha}}{\alpha}}$, for all $a\in\mathbb{R}$,

(4) $T_{\alpha}\sin(at)=at^{1-\alpha}\cos(at)$, $T_{\alpha}\sin\left(a\frac{t^{\alpha}}{\alpha}\right)=a\cos\left(a\frac{t^{\alpha}}{\alpha}\right)$, for all $a\in\mathbb{R}$,

(5) $T_{\alpha}\cos(at)=-at^{1-\alpha}\sin(at)$, $T_{\alpha}\cos\left(a\frac{t^{\alpha}}{\alpha}\right)=-a\sin\left(a\frac{t^{\alpha}}{\alpha}\right)$, for all $a\in\mathbb{R}$,

(6) $T_{\alpha}\left(\frac{t^{\alpha}}{\alpha}\right)=1$.

3. Conformable Double Laplace Transform (CDLT)

Definition 3.1 (Function of Exponential Order)**.** A function $y(t)$ is said to be of exponential order a if there exists a positive constant M such that,

$$|y(t)|\le Me^{at},\quad \text{for sufficiently large } t.$$

A function $y(x,t)$ of two variables is said to be of exponential order a and b if there exists a positive constant M such that

$$|y(x,t)|\le Me^{ax+bt},\quad \text{for sufficiently large } x \text{ and } t,$$

that is, it does not grow faster than the exponential function e^{ax+bt}.

Definition 3.2 (Conformable Single Laplace Transform (CSLT))**.** Let $f(t)$ be a function of exponential order then the Conformable Single Laplace Transform (CSLT) of $f(t)$ is defined as [1],

$$L_{\alpha}[f(t)]=\int_0^{\infty}e^{-p\frac{t^{\alpha}}{\alpha}}f(t)\,d_{\alpha}t=F(p),$$

where $Re(p)>0$, $0<\alpha\le 1$ and the integrals are fractional integrals in a conformable sense. The corresponding conformable inverse Laplace transform denoted as L_{α}^{-1} and is defined as,

$$L_{\alpha}^{-1}[F(p)]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-p\frac{t^{\alpha}}{\alpha}}F(p)\,d_{\alpha}p=f(t),$$

where $c\ge 0$, $0<\alpha\le 1$ and the integrals are fractional integrals in conformable sense.

Definition 3.3 (Conformable Double Laplace Transform(CDLT))**.** Let $y(x,t)$ be a piece wise continuous function defined on $[0,\infty)\times[0,\infty)$ and be of exponential order then the conformable double Laplace transform (CDLT) of $y(x,t)$ is defined as [19],

$$L_x^\alpha L_t^\beta[y(x,t)] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-q\frac{t^\beta}{\beta}} y(x,t)\, d_\alpha x\, d_\beta t,$$

where $Re(p), Re(q) > 0$, $0 < \alpha, \beta \le 1$ and the fractional integrals are in conformable sense.
The conformable single Laplace transform of the function of two variables $y(x,t)$ with respect to x is given by,

$$L_x^\alpha[y(x,t)] = Y(p,t) = \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} y(x,t)\, d_\alpha x.$$

The conformable single Laplace transform of the function of two variables $y(x,t)$ with respect to t is given by,

$$L_t^\beta[y(x,t)] = Y(x,q) = \int_0^\infty e^{-q\frac{t^\beta}{\beta}} y(x,t)\, d_\beta t.$$

Definition 3.4 (Inverse Conformable Double Laplace Transform)**.** If $L_x^\alpha L_t^\beta[y(x,t)] = Y(p,q)$ then, the inverse conformable double Laplace transform of the function $Y(p,q)$ is denoted as $(L_x^\alpha)^{-1}(L_t^\beta)^{-1}[Y(p,q)]$ and is defined as,

$$(L_x^\alpha)^{-1}(L_t^\beta)^{-1}[Y(p,q)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-p\frac{x^\alpha}{\alpha}}\, d_\alpha p\, \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{-q\frac{t^\beta}{\beta}} Y(p,q)\, d_\beta q = y(x,t),$$

where $c,d \ge 0$, $0 < \alpha, \beta \le 1$ and the fractional integrals are in conformable sense.

Conformable Double Laplace Transform of Some Standard Functions

Let $L_x^\alpha L_t^\beta[y(x,t)] = Y(p,q)$, then

(1) $L_x^\alpha L_t^\beta[1] = \frac{1}{pq}$,

(2) $L_x^\alpha L_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^m \left(\frac{t^\beta}{\beta}\right)^n\right] = \frac{\Gamma(m+1)\Gamma(n+1)}{p^{m+1}q^{n+1}}$,

(3) $L_x^\alpha L_t^\beta\left[e^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}\right] = \frac{1}{(p-a)(q-b)}$,

(4) $L_x^\alpha L_t^\beta\left[\sin\frac{ax^\alpha}{\alpha}\right] = \frac{1}{q}\frac{a}{p^2+a^2}$, $L_x^\alpha L_t^\beta\left[\cos\frac{at^\beta}{\beta}\right] = \frac{1}{p}\frac{q}{q^2+a^2}$,

(5) $L_x^\alpha L_t^\beta[f(x)g(t)] = L_x^\alpha[f(x)]L_t^\beta[g(t)]$,

(6) $L_x^\alpha L_t^\beta[f(x)] = \frac{1}{q} L_x^\alpha[f(x)]$,

(7) $L^\alpha L_t^\beta[g(t)] = \frac{1}{p} L_t^\beta[g(t)]$.

Properties of CDLT

In the conformable fractional framework, the *Conformable Double Laplace Transform* (CDLT) retains the essential operational characteristics of the classical double Laplace transform, such as linearity, shifting and convolution.The basic and essential properties of the CDLT are presented as follows.

Theorem 3.1 (Existence condition of CDLT)**.** *If a function $y(x,t)$ is piece wise continuous on $(0,\infty)$ and is of exponential order a and b, then the CDLT of $y(x,t)$ exists for all $Re(p) > a$, $Re(q) > b$.*

Proof. Let $y(x,t)$ be a piece wise continuous on $(0,\infty)$. Suppose that $y(x,t)$ is a function of exponential order a and b.
Therefore, there is a real number $M>0$ such that

$$|y(x,t)| \le M e^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}}.$$

From the definition of CDLT, we can write

$$\begin{aligned}
L_x^\alpha L_t^\beta[y(x,t)] &= \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-q\frac{t^\beta}{\beta}} y(x,t)\, d_\alpha x\, d_\beta t \\
&= \left|\int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}-q\frac{t^\beta}{\beta}} y(x,t)\, d_\alpha x\, d_\beta t\right| \\
&\le \int_0^\infty \int_0^\infty \left|e^{-p\frac{x^\alpha}{\alpha}-q\frac{t^\beta}{\beta}} y(x,t)\, d_\alpha x\, d_\beta t\right| \\
&= M\int_0^\infty e^{-(p-a)\frac{x^\alpha}{\alpha}}\, d_\alpha x \int_0^\infty e^{-(q-b)\frac{t^\beta}{\beta}}\, d_\beta t \\
&\le M\frac{1}{(p-a)(q-b)}.
\end{aligned}$$

Now for any $Re(p)>a$ and $Re(q)>b$, we have

$$\lim_{\substack{p\to\infty\\ q\to\infty}} |Y(p,q)| = 0 \quad \text{or} \quad \lim_{\substack{p\to\infty\\ q\to\infty}} Y(p,q) = 0.$$

Hence, the existence of CDLT of $y(x,t)$ is established. □

Theorem 3.2 (Linearity Property)**.** *If CDLT of $y_1(x,t)$ and $y_2(x,t)$ exists then, for all constants c and d,*

$$L_x^\alpha L_t^\beta[c y_1(x,t) + d y_2(x,t)] = c L_x^\alpha L_t^\beta y_1(x,t) + d L_x^\alpha L_t^\beta y_2(x,t).$$

Proof. The definition of CDLT allows us for a straightforward derivation of this result. □

Theorem 3.3 (First Shifting Property)**.** *If $L_x^\alpha L_t^\beta y(x,t) = Y(p,q)$ then,*

$$L_x^\alpha L_t^\beta\left[e^{a\frac{x^\alpha}{\alpha}+b\frac{t^\beta}{\beta}} y(x,t)\right] = Y(p-a, q-b).$$

Proof. See [19]. □

Theorem 3.4 (Change of Scale Property)**.** *If $L_x^\alpha L_t^\beta y(x,t) = Y(p,q)$ then,*

$$L_x^\alpha L_t^\beta[y(\eta x, \theta t)] = \frac{1}{\eta^\alpha \theta^\beta} Y\left(\frac{p}{\eta}, \frac{q}{\theta}\right).$$

Proof. See [19]. □

We now proceed to derive the CDLT for conformable partial derivatives. This derivation is of fundamental importance, as it provides the necessary framework for extending the classical double Laplace transform into the conformable fractional setting. By establishing the transform of conformable partial derivatives, we create a systematic approach that enables the effective formulation and solution of *conformable fractional partial differential equations* (CFPDEs). Thus, this derivation forms an essential basis for the application of the CDLT to diverse problems arising across physical and engineering domains.

Lemma 3.5. *If $L_x^\alpha L_t^\beta y(x,t) = Y(p,q)$, then CDLT of α-th and β-th order conformable partial derivative is,*

$$L_x^\alpha L_t^\beta [T_x^\alpha y(x,t)] = pL_x^\alpha L_t^\beta [y(x,t)] - L_t^\beta [y(0,t)] = pY(p,q) - Y(0,q), \tag{3.1}$$

$$L_x^\alpha L_t^\beta [T_t^\beta y(x,t)] = qL_x^\alpha L_t^\beta [y(x,t)] - L_x^\alpha [y(x,0)] = qY(p,q) - Y(p,0). \tag{3.2}$$

Proof. The proof relies on the integration by parts formula, which remains valid for conformable integrals.
The definition of CDLT allow us to write,

$$L_x^\alpha L_t^\beta [T_x^\alpha y(x,t)] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - q\frac{t^\beta}{\beta}} T_x^\alpha y(x,t)\, d_\alpha x\, d_\beta t \tag{3.3}$$

$$= \int_0^\infty e^{-q\frac{t^\beta}{\beta}} \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} T_x^\alpha y(x,t)\, d_\alpha x\, d_\beta t. \tag{3.4}$$

Let us consider the integral,

$$I = \int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} T_x^\alpha y(x,t)\, d_\alpha x.$$

To use integration by parts, let us choose $u = e^{-p\frac{x^\alpha}{\alpha}}$ and $v = T_x^\alpha y(x,t)$.
Therefore we obtain,

$$I = \left[e^{-p\frac{x^\alpha}{\alpha}} I^\alpha [T_x^\alpha y(x,t)]\right]_0^\infty - \int_0^\infty (-p) e^{-p\frac{x^\alpha}{\alpha}} I^\alpha [T_x^\alpha y(x,t)]\, d_\alpha x$$

$$= -y(0,t) + p\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} y(x,t)\, d_\alpha x,$$

where I^α is the conformable integral and $I^\alpha(T_x^\alpha y(x,t)) = y(x,t)$.
With this equation (3.4) becomes,

$$\begin{aligned} L_x^\alpha L_t^\beta [T_x^\alpha y(x,t)] &= \int_0^\infty e^{-q\frac{t^\beta}{\beta}} [-y(0,t) + p\int_0^\infty e^{-p\frac{x^\alpha}{\alpha}} y(x,t)\, d_\alpha x\,] d_\beta t \\ &= p\int_0^\infty \int_0^\infty e^{-q\frac{t^\beta}{\beta} - p\frac{x^\alpha}{\alpha}} y(x,t)\, d_\alpha x\, d_\beta t - \int_0^\infty e^{-q\frac{t^\beta}{\beta}} y(0,t) d_\beta t \\ &= pL_x^\alpha L_t^\beta [y(x,t)] - L_t^\beta [y(0,t)] \\ &= pY(p,q) - Y(0,q). \end{aligned}$$

Similarly, we can prove the statement (3.2). □

Lemma 3.6. *The CDLT of mixed conformable partial derivative is given by,*

$$L_x^\alpha L_t^\beta [T_x^\alpha T_t^\beta y(x,t)] = pqY(p,q) - pY(p,0) - qY(0,q) + Y(0,0).$$

Proof. The definition of CDLT and Lemma 3.5 allows us to write,

$$\begin{aligned} L_x^\alpha L_t^\beta [T_x^\alpha (T_t^\beta y(x,t))] &= p\, L_x^\alpha L_t^\beta [T_t^\beta y(x,t)] - L_t^\beta [T_t^\beta y(0,t)] \\ &= p[qY(p,q) - Y(p,0)] - [qY(0,q) - Y(0,0)] \\ &= pqY(p,q) - pY(p,0) - qY(0,q) + Y(0,0). \end{aligned}$$

□

Theorem 3.7. *Let $L_x^\alpha L_t^\beta y(x,t) = Y(p,q)$. If $T_x^{i\alpha} y(x,t)$ and $T_t^{j\beta} y(x,t)$ exists for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, where $T_x^{i\alpha}$ and $T_t^{j\beta}$ denotes i-times αth order and j-times βth order conformable* *partial derivative with respect to x and t respectively then,*

$$L_x^\alpha L_t^\beta[T_x^{m\alpha} y(x,t)] = p^m Y(p,q) - p^{m-1}Y(0,q) - \sum_{i=1}^{m-1} p^{m-1-i} L_t^\beta[T_x^{i\alpha} y(0,t)], \tag{3.5}$$

$$L_x^\alpha L_t^\beta[T_t^{n\beta} y(x,t)] = q^n Y(p,q) - q^{n-1}Y(p,0) - \sum_{j=1}^{n-1} q^{n-1-j} L_x^\alpha[T_t^{j\beta} y(x,0)]. \tag{3.6}$$

Proof. To prove (3.5), we use principle of mathematical induction.
The case $m = 1$ follows from Lemma 3.5.
Now assume the result (3.5) for $m = k$.
Therefore, we must have

$$L_x^\alpha L_t^\beta[T_x^{k\alpha} y(x,t)] = p^k Y(p,q) - p^{k-1}Y(0,q) - \sum_{i=1}^{k-1} p^{k-1-i} L_t^\beta[T_x^{i\alpha} y(0,t)]. \tag{3.7}$$

To prove the result (3.5) for $m = k+1$, let us consider,

$$L_x^\alpha L_t^\beta[T_x^{(k+1)\alpha} y(x,t)] = L_x^\alpha L_t^\beta[T_x^\alpha[y(x,t)][T_x^{k\alpha}[y(x,t)]].$$

Utilizing Lemma 3.5 in conjunction with assumption (3.7), we arrive at,

$$\begin{aligned} L_x^\alpha L_t^\beta[T_x^{(k+1)\alpha} y(x,t)] &= pL_x^\alpha L_t^\beta[T_x^{k\alpha} y(x,t)] - L_t^\beta[T_x^{k\alpha} y(0,t)] \\ &= p\left[p^k Y(p,q) - p^{k-1}Y(0,q) - \sum_{i=1}^{k-1} p^{k-1-i} L_t^\beta[T_x^{i\alpha} y(0,t)]\right] - L_t^\beta[T_x^{k\alpha} y(0,t)] \\ &= p^{k+1} Y(p,q) - p^k Y(0,q) - \sum_{i=1}^{k} p^{k-i} L_t^\beta[T_x^{i\alpha} y(0,t)]. \end{aligned}$$

Hence, the result (3.5) is true for $m = k+1$.
Thus, by the principle of mathematical induction, the assertion (3.5) is valid for all positive integers m.
Using a similar approach, we can demonstrate (3.6). □

The subsequent theorems provide a formal relation between the CDLT and the classical Laplace transform.

Theorem 3.8. $L_x^\alpha L_t^\beta y(x,t) = L_x L_t\big[y\big((\alpha x)^{\frac{1}{\alpha}}\big), \big(\beta t)^{\frac{1}{\beta}}\big)\big]$, *where* $L_x L_t$ *denotes double Laplace transform operator, which is defined as,*

$$L_x L_t[y(x,t)] = \int_0^\infty \int_0^\infty e^{-px-qt} y(x,t)\, dx\, dt.$$

Proof. We have

$$L_x^\alpha L_t^\beta[y(x,t)] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} y(x,t)\, d_\alpha x\, d_\beta t \tag{3.8}$$

$$= \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - s\frac{t^\beta}{\beta}} y(x,t) x^{\alpha-1} t^{\beta-1}\, dx\, dt. \tag{3.9}$$

Setting $\frac{x^\alpha}{\alpha} = u$ and $\frac{t^\beta}{\beta} = v$, we get $x^{\alpha-1}dx = du$ and $t^{\beta-1}dt = dv$.
Therefore, inequities become,

$$L_x^\alpha L_t^\beta[y(x,t)] = \int_0^\infty \int_0^\infty e^{-pu-qv} y((\alpha u)^{\frac{1}{\alpha}}, (\beta v)^{\frac{1}{\beta}})\, du\, dv \tag{3.10}$$

$$= \int_0^\infty \int_0^\infty e^{-px-qt} y((\alpha x)^{\frac{1}{\alpha}}, (\beta t)^{\frac{1}{\beta}})\, dx\, dt \tag{3.11}$$

$$= L_x L_t[y((\alpha x)^{\frac{1}{\alpha}}, (\beta t)^{\frac{1}{\beta}})]. \tag{3.12}$$

□

Theorem 3.9. *$L_x^\alpha L_t^\beta[y(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta})] = L_x L_t[y(x,t)]$, where $L_x L_t$ denotes double Laplace transform operator, which is defined as,*

$$L_x L_t[y(x,t)] = \int_0^\infty \int_0^\infty e^{-px-qt} y(x,t)\, dx\, dt.$$

Proof. We have

$$L_x^\alpha L_t^\beta\left[y\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - q\frac{t^\beta}{\beta}} y\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) d_\alpha x\, d_\beta t \tag{3.13}$$

$$= \int_0^\infty \int_0^\infty e^{-p\frac{x^\alpha}{\alpha} - q\frac{t^\beta}{\beta}} y\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) x^{\alpha-1} t^{\beta-1}\, dx\, dt. \tag{3.14}$$

Setting $\frac{x^\alpha}{\alpha} = u$ and $\frac{t^\beta}{\beta} = v$, we get $x^{\alpha-1}dx = du$ and $t^{\beta-1}dt = dv$.
Therefore, eq. (3.15) becomes

$$L_x^\alpha L_t^\beta\left[y\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right)\right] = \int_0^\infty \int_0^\infty e^{-pu-qv} y(u,v)\, du\, dv \tag{3.15}$$

$$= \int_0^\infty \int_0^\infty e^{-px-qt} y((x,t)\, dx\, dt \tag{3.16}$$

$$= L_x L_t[y(x,t)]. \tag{3.17}$$

□

4. Methodology

This section is dedicated to exploring the application of the CDLTM in solving *conformable fractional partial differential equations* (CFPDEs). This approach furnishes a structured and effective analytical tool for solving CFPDEs with multiple fractional orders. Through illustrative examples and theoretical results, we highlight the effectiveness of the CDLTM as a powerful tool for deriving analytical and simplified solutions to such equations.

To illustrate the method, we consider a linear space and time fractional CFPDE of the following form,

$$\sum_{i=0}^m a_i T_x^{i\alpha} y(x,t) + \sum_{j=0}^n b_j T_t^{j\beta} y(x,t) = f(x,t) \tag{4.1}$$

with initial and boundary condition,

$$T_t^{i\beta} y(x,0) = g_i(x), \quad i = 0,1,2,\dots,m-1, \tag{4.2}$$

$$T_x^{j\alpha} y(0,t) = h_j(t), \quad j = 0,1,2,\dots,n-1, \tag{4.3}$$

where $T_x^{i\alpha}$ and $T_t^{j\beta}$ denotes i-times αth order conformable fractional partial derivative with respect to x and $T_t^{j\beta}$ denotes j-times βth order conformable partial derivative with respect to t of a function $y(x,t)$.

Taking the CDLT on both sides of eq. (4.1) yields,

$$\sum_{i=0}^{m} a_i \left[p^i Y(p,q) - \sum_{r=0}^{i} p^{m-1-r} L_t^{\beta}(T_x^{r\alpha} y(0,t)) \right] + \sum_{j=0}^{n} b_j \left[q^j Y(p,q) - \sum_{s=0}^{j} q^{n-1-s} L_x^{\alpha}(T_t^{s\beta} y(x,0)) \right] = F(p,q). \tag{4.4}$$

Applying CSLT to initial and boundary conditions (4.2) and (4.3), we obtain

$$L_x^{\alpha}(T_t^{j\beta} y(x,0)) = L_x^{\alpha}(g_i(x)) = G_i(p), \tag{4.5}$$

$$L_t^{\beta}(T_x^{i\alpha} y(0,t)) = L_t^{\beta}(h_j(t)) = H_j(q). \tag{4.6}$$

Using eq. (4.5) and (4.6) in eq. (4.4), we obtain

$$\sum_{i=0}^{m} a_i \left[p^i Y(p,q) - \sum_{r=0}^{i} p^{m-1-r} H_r(q) \right] + \sum_{j=0}^{n} b_j \left[q^j Y(p,q) - \sum_{s=0}^{j} q^{n-1-s} G_s(p) \right] = F(p,q). \tag{4.7}$$

Simplifying, we get

$$Y(p,q) \left[\sum_{i=0}^{m} a_i p^i + \sum_{j=0}^{n} b_j q^j \right] - \sum_{r=0}^{i} p^{m-1-r} H_r(q) - \sum_{s=0}^{j} q^{n-1-s} G_s(p) = F(p,q), \tag{4.8}$$

$$Y(p,q) = \frac{F(p,q) + \sum_{r=0}^{i} p^{m-1-r} H_r(q) - \sum_{s=0}^{j} q^{n-1-s} G_s(p)}{\sum_{i=0}^{m} a_i p^i + \sum_{j=0}^{n} b_j q^j}. \tag{4.9}$$

Taking conformable inverse Laplace transform on both sides of equation (4.9),

$$\begin{aligned} y(x,t) &= (L_x^{\alpha} L_t^{\beta})^{-1} [Y(p,q)] \\ &= (L_x^{\alpha} L_t^{\beta})^{-1} \left[\frac{F(p,q) + \sum_{r=0}^{i} p^{m-1-r} H_r(q) - \sum_{s=0}^{j} q^{n-1-s} G_s(p)}{\sum_{i=0}^{m} a_i p^i + \sum_{j=0}^{n} b_j q^j} \right] \end{aligned} \tag{4.10}$$

which is the solution of CFPDE (4.1).

We now proceed to implement the CDLTM on a well-known partial differential equation that incorporates both space- and time-fractional conformable derivatives. This application not only illustrates the practical utility of the CDLTM but also demonstrates its effectiveness in simplifying the solution process for complex fractional models.

Example 4.1. Consider a linear, space-time fractional Klein-Gordan equation,

$$T_t^{2\beta} y(x,t) - y(x,t) = T_x^{2\alpha} y(x,t), \quad x,t \geq 0,\ 0 < \alpha,\ \beta \leq 1 \tag{4.11}$$

with initial and boundary conditions,

$$y(x,0) = 1 + \sin\left(\frac{x^{\alpha}}{\alpha}\right), \quad T_t^{\beta} y(x,0) = 0, \tag{4.12}$$

$$y(0,t) = \cosh\left(\frac{t^{\beta}}{\beta}\right), \quad T_x^{\alpha} y(0,t) = 1. \tag{4.13}$$

Applying CDLT on both sides of eq. (4.11) and CSLT on both sides of eq. (4.12) and (4.13), we obtain

$$q^2Y(p,q)-qY(p,0)-L_t^{\beta}\big(T_x^{\alpha}y(0,t)\big)-Y(p,q)=p^2Y(p,q)-pY(0,q)-L_x^{\alpha}\big(T_t^{\beta}y(x,0)\big),$$

$$Y(p,q)=\frac{1}{q(p^2+1)}+\frac{q}{p(q^2-1)}. \tag{4.14}$$

Taking inverse CDLT on both sides of eq. (4.14), we obtain

$$(L_x^{\alpha}L_t^{\beta})^{-1}[Y(p,q)]=(L_x^{\alpha}L_t^{\beta})^{-1}\left[\frac{1}{q(p^2+1)}+\frac{q}{p(q^2-1)}\right],$$

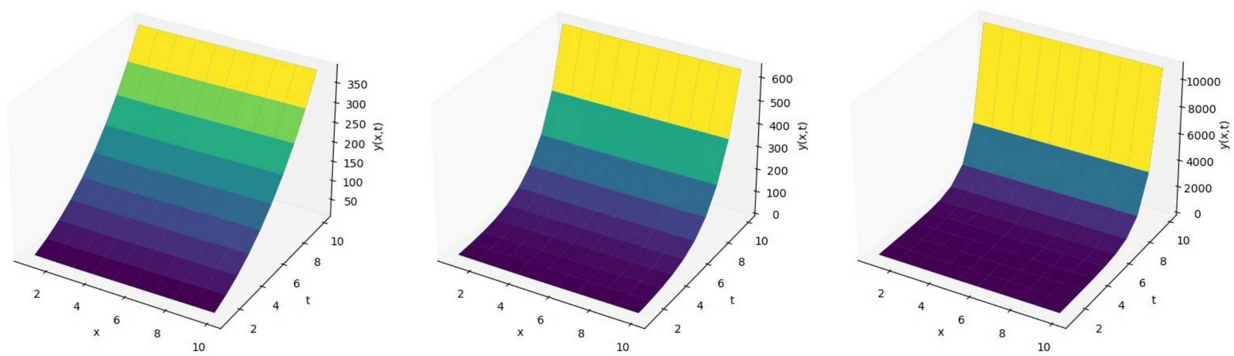
$$y(x,t)=(L_x^{\alpha}L_t^{\beta})^{-1}\left[\frac{1}{q}\,\frac{1}{p^2+1}\right]+\left(L_x^{\alpha}L_t^{\beta}\right)^{-1}\left[\frac{1}{p}\,\frac{q}{q^2-1}\right],$$

$$y(x,t)=\sin\left(\frac{x^{\alpha}}{\alpha}\right)+\cosh\left(\frac{t^{\beta}}{\beta}\right)$$

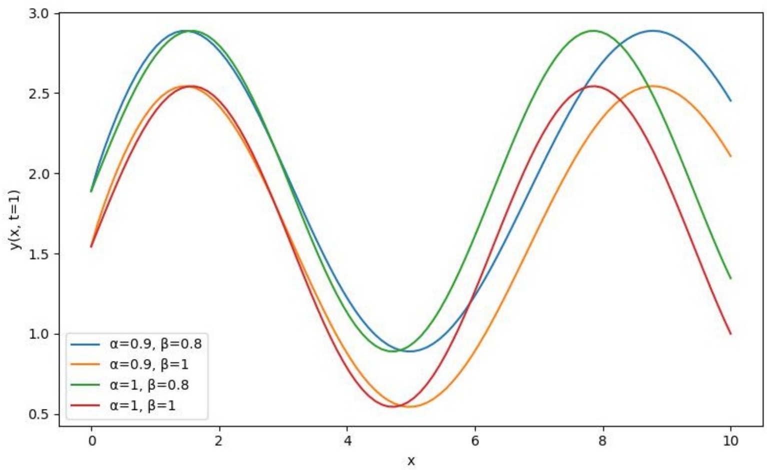
which represents the analytical solution of eq. (4.11).

In case of integer order, that is,for $\alpha=1$ and $\beta=1$ the exact solution is, $y(x,t)=\sin x+\cosh t$, which is same as solution obtained in [7].

We now plot 3D surface of solution $y(x,t)$ for various values of α and β using python.



(a) $\alpha=0.3,\beta=0.3$ **(b)** $\alpha=0.5,\beta=0.7$ **(c)** $\alpha=\beta=1$



(d) Comparison of integer vs fractional order solution

Figure 1. First three plots are the solution $y(x,t)$ of eq. (4.11) for different values of α, β and plot (d) is the comparison of integer order versus fractional order solution

In Figure 1, plots (a) and (b) corresponds the solution for different combinations of α and β. Plot (c) depicts the solution in the integer-order case. Finally plot (d) provides two-dimensional solution $y(x,t)$ at fixed $t=1$, $x\geq 0$, for the parameter pairs $\alpha=0.9,1$ and $\beta=0.8,1$.This comparison highlights the behavior of the solution $y(x,t)$ in both the fractional-order and integer-order cases.

Example 4.2. Consider the space and time fractional third order Kdv equation,

$$T_t^{\beta}y(x,t)+T_x^{3\alpha}y(x,t)=-T_x^{\alpha}y(x,t), \quad x,t\geq 0,\ 0<\alpha,\ \beta\leq 1 \tag{4.15}$$

with initial and boundary conditions,

$$y(x,0)=e^{-\frac{x^{\alpha}}{\alpha}}, \qquad y(0,t)=e^{\frac{2t^{\beta}}{\beta}}, \tag{4.16}$$

$$T_x^{2\alpha}y(0,t)=-e^{\frac{2t^{\beta}}{\beta}}, \quad T_x^{\alpha}y(0,t)=e^{\frac{2t^{\beta}}{\beta}}. \tag{4.17}$$

Applying CDLT on both sides of eqn. (4.15) and CSLT on both sides of eqn. (4.16) and (4.17), we obtain

$$qY(p,q)-Y(p,0)+[p^3Y(p,q)-p^2Y(0,q)-pL_t^{\beta}(T_x^{\alpha}y(0,t))-L_t^{\beta}(T_x^{2\alpha}y(0,t))]$$
$$=-[pY(p,q)-Y(0,q)],$$
$$Y(p,q)=\left(\frac{1}{p+1}\right)\left(\frac{1}{q-2}\right). \tag{4.18}$$

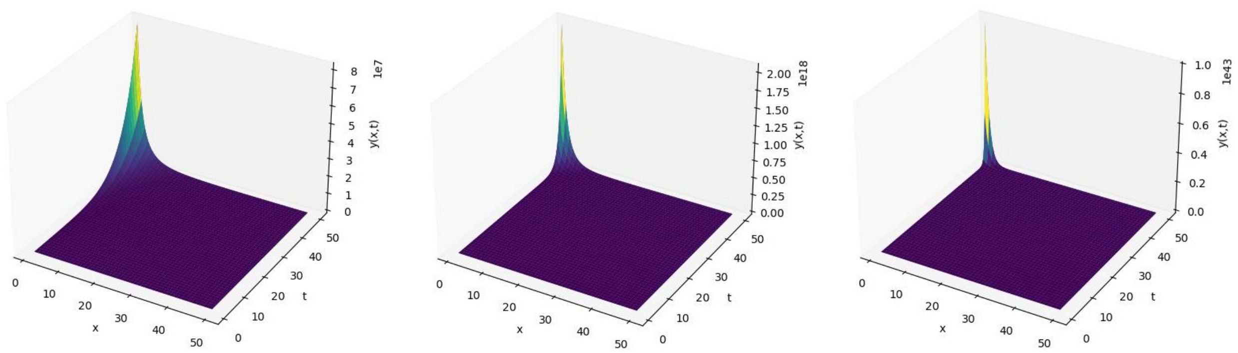
Taking inverse CDLT on both sides of eq. (4.18), we obtain,

$$(L_x^{\alpha}L_t^{\beta})^{-1}[Y(p,q)]=(L_x^{\alpha}L_t^{\beta})^{-1}\left[\frac{1}{p+1}\right](L_x^{\alpha}L_t^{\beta})^{-1}\left[\frac{1}{q-2}\right],$$
$$y(x,t)=e^{-\frac{x^{\alpha}}{\alpha}}\ e^{\frac{2t^{\beta}}{\beta}}$$

which is the analytical solution of eqn. (4.15).

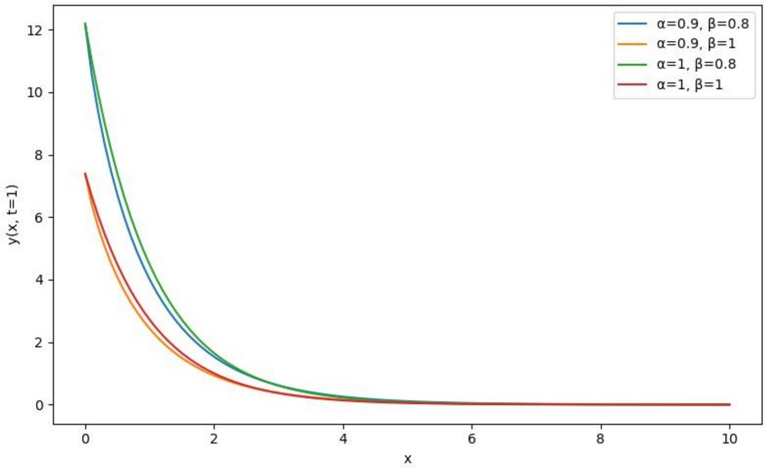
When both orders are integers, the solution unfolds as $y(x,t)=e^{-x+2t}$, perfectly mirroring the result found in [15].

In Figure 2, plots (a) and (b) display the solutions corresponding to different combinations of α and β. Plot (c) present the solution in integer-order case. The fourth plot (d) is the 2D-representation of the solution $y(x,t)$ at fix $t=1$, $x\geq 0$ and for pair of values $\alpha=0.9,1$ and $\beta=0.8,1$. This comparison highlights the differences between the integer-order and fractional-order solutions $y(x,t)$.



(a) $\alpha=\beta=0.3$ **(b)** $\alpha=0.5,\ \beta=0.7$ **(c)** $\alpha=0.5,\ \beta=0.7$

Figure 2 (*continued*)



(d) Comparison of integer vs fractional order solution

Figure 2. First three plots are the solution $y(x,t)$ of eq. (4.15) for different values of α, β and plot (d) is the comparison of integer order versus fractional order solution

Example 4.3. Consider a non-homogeneous space and time fractional Klein-Gordan equation,

$$T_t^{2\beta}y(x,t)-T_x^{2\alpha}y(x,t)-2y(x,t)=-2\sin\left(\frac{x^\alpha}{\alpha}\right)\sin\left(\frac{t^\beta}{\beta}\right),\quad x,t\geq 0,\ 0<\alpha,\ \beta\leq 1 \tag{4.19}$$

with initial and boundary conditions,

$$y(x,0)=0,\quad T_t^{\beta}y(x,0)=\sin\left(\frac{x^\alpha}{\alpha}\right), \tag{4.20}$$

$$y(0,t)=0,\quad T_x^{\alpha}y(0,t)=\sin\left(\frac{t^\beta}{\beta}\right), \tag{4.21}$$

Applying CDLT on both sides of eq. (4.19) and CSLT on both sides of eq. (4.20) and (4.21), we obtain

$$q^2Y(p,q)-qY(p,0)-\left(\frac{1}{1+p^2}\right)-2Y(p,q)=p^2Y(p,q)-pY(0,q)-\left(\frac{1}{1+q^2}\right)-2\left(\frac{1}{1+p^2}\right)\left(\frac{1}{1+q^2}\right)$$

$$q^2Y(p,q)-q.0-\frac{1}{1+p^2}-2Y(p,q)=p^2Y(p,q)-p.0-\frac{1}{1+q^2}-2\left(\frac{1}{1+p^2}\right)\left(\frac{1}{1+q^2}\right)$$

$$Y(p,q)=\left(\frac{1}{1+p^2}\right)\left(\frac{1}{1+q^2}\right). \tag{4.22}$$

Taking inverse CDLT on both sides of eq. (4.22), we obtain

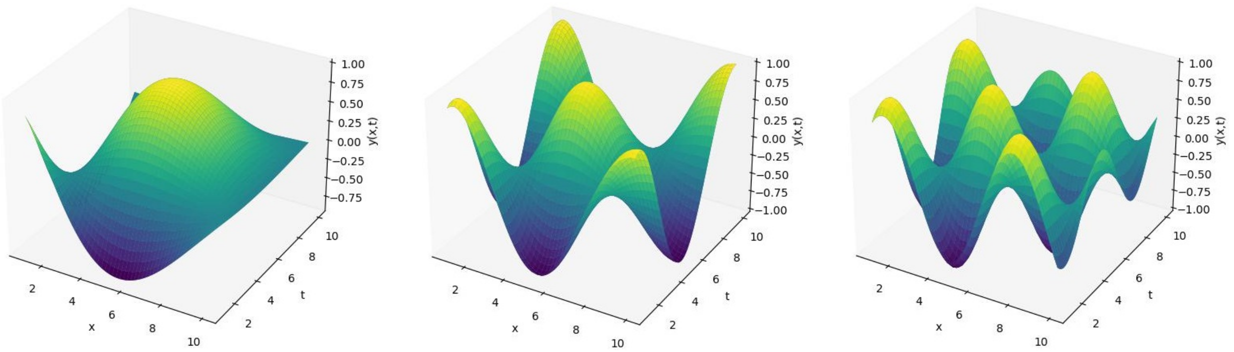
$$(L_x^\alpha L_t^\beta)^{-1}[Y(p,q)]=(L_x^\alpha L_t^\beta)^{-1}\left[\frac{1}{1+p^2}\right](L_x^\alpha L_t^\beta)^{-1}\left[\frac{1}{1+q^2}\right]$$

$$y(x,t)=\sin\left(\frac{x^\alpha}{\alpha}\right)\sin\left(\frac{t^\beta}{\beta}\right)$$

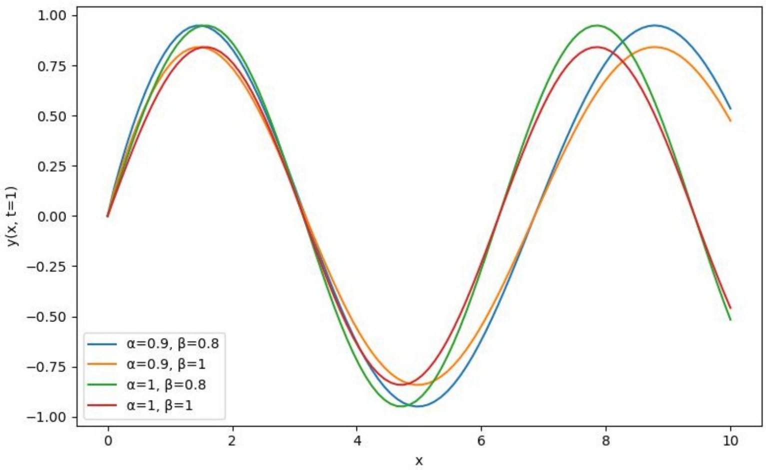
which is the analytical solution of eq. (4.19).

The solution of eq. (4.19) in case of integer order case is, $y(x,t)=\sin x\ \sin t$, which aligns with the solution obtained in [15].

Following are the plots of the solution $y(x,t)$ for various values of α and β.
In Figure 3, the first two plots (a) and (b) illustrate the solution for different combinations of α and β.



(a) $\alpha = 0.5, \beta = 0.7$ **(b)** $\alpha = \beta = 0.8$ **(c)** Solution for $\alpha = \beta = 1$



(d) Comparison of integer vs fractional order solution

Figure 3. First three plots are the solution $y(x,t)$ of eq. (4.19) for different values of α, β and plot (d) is the comparison of integer order versus fractional order solution

Plot (c) shows the solution in the case of an integer-order case. Fourth plot (d) is two-dimensional plot of solution $y(x,t)$ at fixed $t = 1$, $x \geq 0$ and for pair of values $\alpha = 0.9, 1$ and $\beta = 0.8, 1$, which represents a comparative study of the solution $y(x,t)$ for both the integer-order and the fractional-order cases.

5. Conclusion

In our present study, we demonstrates the connection between the classical double Laplace transform and the *conformable double Laplace transform* (CDLT). We derives the CDLT of conformable fractional partial derivatives and applies to solve several conformable fractional partial differential equations. The solutions are then graphed using Python to compare them with both fractional and integer-order cases. The study concludes that the CDLT method is a powerful and efficient technique for analyzing a abroad spectrum of space-time fractional models. To address nonlinear systems in mathematical physics and engineering, we suggest extending the CDLTM through the incorporation of Adomian decomposition and homotopy perturbation techniques.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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