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Some Combinatorial Identities of q-Harmonic and q-Hyperharmonic Numbers

Research Article

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Abstract. In this paper, by means of q-difference operator we derive q-analogue for several well known results for harmonic numbers. Also we give some identities concerning q-hyperharmonic numbers.

Keywords. Harmonic numbers; hyperharmonic numbers; q-harmonic numbers; q-hyperharmonic numbers, q-difference operator

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1. Introduction

The classical harmonic numbers are defined by for $n \in \mathbb{N}$

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

and $H_0 = 0$. They have interesting applications in many areas, such as, number theory, combinatorics and analysis of algorithms. These numbers have been studied extensively.

In [1], for $n \geq 1$ some properties are following:

$$\sum_{k=1}^{n-1} H_k = n\left(H_n - 1\right), \tag{1.1}$$

$$\sum_{k=1}^{n-1} kH_k = \frac{n^{\underline{2}}}{2}\left(H_n - \frac{1}{2}\right), \tag{1.2}$$

$$\sum_{k=m}^{n-1} \binom{k}{m} H_k = \binom{n}{m+1}\left(H_n - \frac{1}{m+1}\right), \tag{1.3}$$

where $n^{\underline{2}} = n(n-1)$ and m is nonnegative integer.

In [2], Spivey proved some combinatorial identities related to harmonic numbers via finite differences. In [3], Chu studied finite and infinite series involving the classical harmonic numbers. The harmonic numbers and their generalizations have been studied in several papers; among other references, see [4–6, 17–21]. For example in [4], Choi presented further identities about certain interesting finite series associated with binomial coefficients, harmonic numbers and generalized harmonic numbers. In [5], Conway and Guy defined hyperharmonic number of order r, $H_n^{(r)}$, for $n,r \geq 1$ by the following recurrence relations:

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)},$$

where $H_n^{(0)} = \frac{1}{n}$ and if $n \leq 0$ or $r < 0$, $H_n^{(r)} = 0$.

In [6], Benjamin et al. gave some combinatorics properties of hyperharmonic numbers by taking repeated partial sums of the harmonic numbers. In [21], Bahşi and Solak defined a special matrix whose entries are hyperharmonic numbers and gave some properties of this matrix via hyperharmonic numbers. Recently, in [22], Tuglu et al. defined harmonic and hyperharmonic Fibonacci numbers. Moreover they gave some combinatorial identities for the harmonic Fibonacci numbers by using the difference operator. Now we briefly summarize basic properties of q-calculus. Let n be a positive integer and $q \in (0,1)$. The q-integer $[n]_q$ and q-factorial $[n]_q!$ are respectively defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1},$$

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q[n-1]_q \ldots [1]_q & \text{if } n = 1,2,\ldots. \end{cases}$$

The q-binomial coefficients $\binom{n}{k}_q$ is given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}\,; \quad 1 \leq k \leq n$$

with $\binom{n}{0}_q = 1$ and $\binom{n}{k}_q = 0$ for $n < k$ [7]. The well known q-analogue of Pascal's rule are following:

$$\binom{n}{j}_q = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q$$

$$\binom{n}{j}_q = \binom{n-1}{j}_q + q^{n-j} \binom{n-1}{j-1}_q.$$

For more details and properties related to q-calculus we refer to [7, 8].

q-analogues of H_n are given by Dilcher, for $n \geq 0$

$$H_n(q) = \sum_{j=1}^{n} \frac{1}{[j]_q}$$

and

$$\widetilde{H}_n(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q}$$

where $H_0(q) = \widetilde{H}_0(q) = 0$ (see [9]).

Many authors investigated q-harmonic numbers in various aspects [10–14, 23]. In [10], Mansour et al. examined q-analogs of certain previously established combinatorial identities. In [11], Mansour derived q-analogue of harmonic number sums by using partial fraction approach. In [12], Mezö studied three different q-analogue of classical harmonic numbers. In [13], Chu and Yan derived several interesting identities concerning q-harmonic numbers by using the q-finite differences and derivative operator.

Recently in [14], Mansour and Shattuck defined q-analogue of the hyperharmonic numbers as follows:

$$H_n^{(r)}(q) = \sum_{t=1}^{n} q^t H_n^{(r-1)}(q)$$

where $H_n^{(0)}(q) = \frac{1}{q[n]_q}$. For $r = 1$ it will be seen that

$$\widetilde{H}_n(q) = qH_n^{(1)}(q). \tag{1.4}$$

They considered a q-analogue of the hyperharmonic numbers and gave some combinatorics properties of these numbers.

The Carlitz-Gould q-difference operator is defined by

$$\Delta_q^1 f(x) = f(x+1) - f(x)$$

and

$$\Delta_q^{n+1} f(x) = \Delta_q^n f(x+1) - q^n \Delta_q^n f(x),$$

[15]. q-difference operator has very interesting property. For example in [16],

$$\Delta_q^1 [x]_q^{\underline{m}} = q^{x+1-m} [m]_q [x]_q^{\underline{m-1}}$$

where $[x]_q^{\underline{m}} = [x]_q [x-1]_q \dots [x-m+1]_q$. The inverse of q-difference operator is defined by

$$\Delta_q^{-1} f(k)\Big|_0^n = \sum_0^n f(x)\delta_q x = \sum_{k=0}^{n-1} f(k)$$

[15]. Since definition of q-difference operator, we can easily obtain

$$\sum q^x [x]_q^m \delta_q x = q^m \frac{[x]_q^{m+1}}{[m+1]_q}, \tag{1.5}$$

$$\sum_0^n u_q(x)\Delta_q v_q(x)\delta_q x = u_q(x)v_q(x)\Big|_0^n - \sum_0^n v_q(x+1)\Delta_q u_q(x)\delta_q x. \tag{1.6}$$

Motivated by such applications, we present q-analogues of certain previously established combinatorial identities by using q-difference operator. Moreover, we give some new combinatorial identities involving q-hyperharmonic numbers.

2. Main Results

Theorem 2.1. *Let k,m be non-negative integers and $0 \le m \le k$, then*

$$\sum_{k=0}^{n-1} q^k [k]_q^m \widetilde{H}_k(q) = \frac{[n]_q^{m+1}}{[m+1]_q}\left(q^m \widetilde{H}_n(q) - \frac{q^{2m+1}}{[m+1]_q}\right).$$

Proof. Let $u_q(k) = \widetilde{H}_k(q)$ and $\Delta_q v_q(k) = q^k [k]_q^m$ be in (1.6). Definition of q-difference operator and (1.5), $\Delta_q u_q(k) = \frac{q^{k+1}}{[k+1]_q}$ and $v_q(k) = q^m \frac{[k]_q^{m+1}}{[m+1]_q}$. Then by using the equation (1.6),

$$\begin{aligned}\sum_{k=0}^{n-1} q^k [k]_q^m \widetilde{H}_k(q) &= q^m \frac{[n]_q^{m+1}}{[m+1]_q}\widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\sum_0^n q^k [k]_q^m \delta_q k \\ &= q^m \frac{[n]_q^{m+1}}{[m+1]_q}\widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\left(q^m \frac{[k]_q^{m+1}}{[m+1]_q}\Bigg|_0^n\right) \\ &= \frac{[n]_q^{m+1}}{[m+1]_q}\left(q^m \widetilde{H}_n(q) - \frac{q^{2m+1}}{[m+1]_q}\right). \qquad \square\end{aligned}$$

Corollary 2.2. *For $n \ge 1$, we have*

$$\sum_{k=0}^{n-1} q^k \widetilde{H}_k(q) = [n]_q \left(\widetilde{H}_n(q) - q\right). \tag{2.1}$$

Proof. By putting $m = 0$ in Theorem 2.1, one has

$$\sum_{k=0}^{n-1} q^k \widetilde{H}_k(q) = [n]_q \left(\widetilde{H}_n(q) - q\right). \qquad \square$$

Remark 2.3. Setting $q = 1$ in (2.1), we obtain equation (1.1).

Corollary 2.4. *For $n \ge 1$, we have*

$$\sum_{k=0}^{n-1} q^k [k]_q \widetilde{H}_k(q) = \frac{[n]_q^2}{[2]_q}\left(q\widetilde{H}_n(q) - \frac{q^3}{[2]_q}\right). \tag{2.2}$$

Proof. By putting $m = 1$ in Theorem 2.1, one has

$$\sum_{k=0}^{n-1} q^k [k]_q \widetilde{H}_k(q) = \frac{[n]_q^2}{[2]_q}\left(q\widetilde{H}_n(q) - \frac{q^3}{[2]_q}\right). \qquad \square$$

Remark 2.5. Setting $q = 1$ in (2.2), we obtain equation (1.2).

Mansour and Shattuck proved following theorem by using q-partial fraction methods ([14]). Now we give another proof for this theorem by using the q-difference operator.

Theorem 2.6. *If* $0 \le m \le k$, *then*

$$\sum_{k=m}^{n-1} q^{k-m}\binom{k}{m}_q \widetilde{H}_k(q) = \binom{n}{m+1}_q \left(\widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\right). \tag{2.3}$$

Proof. Let $u_q(k) = \widetilde{H}_k(q)$ and $\Delta_q v_q(k) = q^{k-m}\binom{k}{m}_q$ be in (1.6). From the definition of q-difference operator. $\Delta_q u_q(k) = \frac{q^{k+1}}{[k+1]_q}$ and $v_q(k) = \binom{k}{m+1}_q$. Then by using the equation (1.6), we have

$$\begin{aligned}
\sum_{k=m}^{n-1} q^{k-m}\binom{k}{m}_q \widetilde{H}_k(q) &= \binom{n}{m+1}_q \widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\sum_{k=m}^{n-1} q^{k-m}\binom{k}{m}_q \\
&= \binom{n}{m+1}_q \widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\binom{n}{m+1}_q \\
&= \binom{n}{m+1}_q \left(\widetilde{H}_n(q) - \frac{q^{m+1}}{[m+1]_q}\right). \qquad \square
\end{aligned}$$

Remark 2.7. Setting $q = 1$ in (2.3), we obtain equation (1.3).

Definition of q-hyperharmonic numbers, for $r = 2$ it will be seen that

$$H_n^{(2)}(q) = \sum_{t=1}^{n} q^t \sum_{j=1}^{t} \frac{q^{j-1}}{[j]_q} \tag{2.4}$$

The following theorems are related to $H_n^{(2)}(q)$. These theorems contains some combinatorial identities, which are easier to calculate than according to equation (2.4).

Theorem 2.8. *For* $0 \le m \le k$, *we have*

$$\sum_{k=0}^{n-1} q^k [k]_q^m H_k^{(2)}(q) = \frac{q^m}{[m+1]_q}\left([n]_q^{m+1} H_n^{(2)}(q) - \frac{q^m [n+1]_q^{m+2}}{[m+2]_q}\left(\widetilde{H}_{n+1}(q) - \frac{q^{m+2}}{[m+2]_q}\right)\right).$$

Proof. Let $u(k) = H_n^{(2)}(q)$ and $\Delta_q v(k) = q^k [k]_q^m$ be in (1.6). Then we obtain $\Delta_q u(k) = q^k \widetilde{H}_{k+1}(q)$ and $v(k) = q^m \frac{[k]_q^{m+1}}{[m+1]_q}$. By using the equation (1.6), we obtain

$$\sum_{k=0}^{n-1} q^k [k]_q^m H_k^{(2)}(q) = q^m \frac{[n]_q^{m+1}}{[m+1]_q} H_n^{(2)}(q) - \frac{q^{m-1}}{[m+1]_q}\sum_0^n q^{k+1}[k+1]_q^{m+1}\widetilde{H}_{k+1}(q)\delta_q k$$

$$= q^m \frac{[n]_q^{m+1}}{[m+1]_q} H_n^{(2)}(q) - \frac{q^{m-1}}{[m+1]_q}\left(q^{m+1}\frac{[n+1]_q^{m+2}}{[m+2]_q}\widetilde{H}_{n+1}(q) - \frac{q^{2m+3}[n+1]_q^{m+2}}{[m+2]_q^2}\right)$$

$$= \frac{q^m}{[m+1]_q}\left([n]_q^{m+1} H_n^{(2)}(q) - \frac{q^m[n+1]_q^{m+2}}{[m+2]_q}\left(\widetilde{H}_{n+1}(q) - \frac{q^{m+2}}{[m+2]_q}\right)\right). \qquad \square$$

Corollary 2.9. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n-1} q^k H_k^{(2)}(q) = [n]_q H_n^{(2)}(q) - \frac{[n+1]_q^2}{[2]_q}\left(\widetilde{H}_{n+1}(q) - \frac{q^2}{[2]_q}\right).$$

Proof. By putting $m = 0$ in Theorem 2.8, we obtain

$$\sum_{k=0}^{n-1} q^k H_k^{(2)}(q) = [n]_q H_n^{(2)}(q) - \frac{[n+1]_q^2}{[2]_q}\left(\widetilde{H}_{n+1}(q) - \frac{q^2}{[2]_q}\right). \qquad \square$$

Corollary 2.10. *For $n \geq 1$, we have*

$$\sum_{k=0}^{n-1} q^k H_k^{(2)}(q) = [n]_q H_n^{(2)}(q) - \frac{q[n+1]_q^2}{[2]_q}\left(H_{n+1}^{(1)}(q) - \frac{q}{[2]_q}\right).$$

Proof. The proof is trivial from the (1.4). $\square$

Corollary 2.11. *For $n \geq 1$, we have*

$$H_{n-1}^{(3)}(q) = [n]_q H_n^{(2)}(q) - \frac{q[n+1]_q^2}{[2]_q}\left(H_{n+1}^{(1)}(q) - \frac{q}{[2]_q}\right). \tag{2.5}$$

Proof. The proof is trivial from the definition of q-hyperharmonic numbers. $\square$

Setting $q = 1$ in (2.5), we obtain

$$H_{n-1}^{(3)} = nH_n^{(2)} - \frac{(n+1)^2}{2}\left(H_{n+1} - \frac{1}{2}\right).$$

A q-analogue of hyperharmonic numbers can be expressed with the following conjecture.

Conjecture 2.12. *For $n \geq 1$,*

$$H_{n-1}^{(r+1)}(q) = (-1)^r \frac{q^{\binom{r}{2}_q + r - 1}}{[r]_q}\binom{n+r-1}{r}_q + \sum_{j=0}^{r-1}(-1)^j q^{\binom{j+1}{2}}\binom{n+j}{j+1}_q H_{n+j}^{(r-j)}(q).$$

Theorem 2.13. *For $n \geq 1$, we have*

$$\sum_{k=m}^{n-1} q^{k-m}\binom{k}{m}_q H_k^{(2)}(q) = \binom{n}{m+1}_q H_n^{(2)}(q) - q^{m+1}\binom{n+1}{m+2}_q\left(H_{n+1}^{(1)}(q) - \frac{q^{m+1}}{[m+2]_q}\right). \tag{2.6}$$

Proof. Let $u_q(k) = H_n^{(2)}(q)$ and $\Delta_q v_q(k) = q^k [k]_q^{\underline{m}}$ be in (1.6). Then we have $\Delta_q u_q(k) = q^k \widetilde{H}_{k+1}(q)$ and $v_q(k) = \binom{k}{m+1}_q$. From (1.6), one has

$$\begin{aligned}
\sum_{k=m}^{n-1} q^{k-m} \binom{k}{m}_q H_k^{(2)}(q) &= \binom{k}{m+1}_q H_k^{(2)}(q) \Big|_m^n - \sum_m^n q^k \binom{k+1}{m+1}_q \widetilde{H}_{k+1}(q) \delta_q k \\
&= \binom{n}{m+1}_q H_n^{(2)}(q) - q^m \sum_m^n q^{k-m} \binom{k+1}{m+1}_q \widetilde{H}_{k+1}(q) \\
&= \binom{n}{m+1}_q H_n^{(2)}(q) - q^m \left(\binom{n+1}{m+2}_q \widetilde{H}_{n+1}(q) - \frac{q^{m+2}}{[m+2]_q} \binom{n+1}{m+2}_q \right) \\
&= \binom{n}{m+1}_q H_n^{(2)}(q) - q^{m+1} \binom{n+1}{m+2}_q \left(H_{n+1}^{(1)}(q) - \frac{q^{m+1}}{[m+2]_q} \right). \qquad \square
\end{aligned}$$

Remark 2.14. Setting $q = 1$ in (2.6), one has

$$\sum_{k=m}^{n-1} \binom{k}{m} H_k^{(2)} = \binom{n}{m+1} H_n^{(2)} - \binom{n+1}{m+2} \left(H_{n+1} - \frac{1}{m+2} \right),$$

where H_{n+1} is $(n+1)^{\text{th}}$ ordinary harmonic numbers.

3. Conclusion

In this paper, we studied some combinatorial identities of q-harmonic and q-hyperharmonic numbers by using the q-difference operator. On the other hand in [24], p,q-binomial coefficients is defined by

$$\binom{n}{k}_{pq} = \frac{[n]_{pq}!}{[n-k]_{pq}! - [k]_{pq}!},$$

where $[n]_{pq} = \frac{p^n - q^n}{p-q}$ and $[n]_{pq}! = [n]_{pq}[n-1]_{pq} \dots [1]_{pq}$. It would be interesting study some combinatorial properties of p,q-harmonic and p,q-hyperharmonic numbers of order r by using the p,q-difference operator and research their properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R. Graham, D. Knuth and K. Patashnik, *Concrete Mathematics*, Addison Wesley (1989).

[2] M.Z. Spivey, Combinatorial sums and finite difference, *Discrete Math.* **307** (2007), 3130–3146.

[3] W. Chu, Summation formulae involving harmonic numbers, *Filomat* **26** (2012), 143–152.

[4] J. Choi, Finite summation formulas involving binomial coefficients, harmonic numbers and generalized harmonic numbers, *J. Inequal. Appl.* **49** (2013), p. 11.

[5] J.H. Comway and R.K. Guy, *The Book of Numbers*, Springer-Verlag, New York (1996).

[6] A.T. Benjamin, D. Gaebler and R.A. Gaebler, Combinatorial approach to hyperharmonic numbers, *Integers* **3** (2003), 1–9.

[7] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York (2001).

[8] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopaedia of Mathematics and Its Applications, Vol. 71, Cambridge University Press, Cambridge (1999).

[9] K. Dilcher, Determinant expressions for q-harmonic congruences and degenerate Bernoulli numbers, *Electron. J. Combin.* **15** (2008), #R63.

[10] T. Mansour, M. Stattuck and C. Song, q-Analogs of identities involving harmonic numbers and binomial coefficients, *Appl. Appl. Math.* **7** (2012), 22–36.

[11] T. Mansour, Identities on harmonic and q-harmonic number sums, *Afr. Mat.* **23** (2012), 135–143.

[12] I. Mezö, Some possible q-generalizations of harmonic numbers, *Arxiv:1106.5029*.

[13] W. Chu and Q. Yan, Combinatorial identities on q-harmonic numbers, *J. Combin. Math. Combin. Comput.* **77** (2011), 173–185.

[14] T. Mansour and M. Shattuck, A q-analog of the hyperharmonic numbers, *Afr. Math.* **25** (2014), 147–160.

[15] T. Ernst, q-Stirling numbers, and umbral approach, *Adv. Dyn. Syst. Appl.* **3** (2008), 251–282.

[16] M.S. Kim and J.W. Son, A note on q-difference operators, *Commun. Korean Math Soc.* **17** (2002), 423–430.

[17] A. Sofo, Harmonic numbers and double binomial coefficients, *Integral Transforms Spec Funct.* **20** (2009), 847–857.

[18] A. Sofo and H.M. Srivastava, Identities for the harmonic numbers and binomial coefficients, *Ramanujan J.* **25** (2011), 93–113.

[19] J. Choi and H.M. Srivastava, Some summation formulas involving harmonic numbers and generalized harmonic numbers, *Math. Computer Modelling* **54** (2011), 2220–2234.

[20] G. Dattoli and H.M. Srivastava, A note on harmonic numbers, umbral calculus and generating functions, *Appl. Math. Lett.* **21** (2008), 686–693.

[21] M. Bahşi and S. Solak, An application of hyperharmonic numbers in matrices, *Hacet. J. Math. Stat.* **42** (2013), 387–393.

[22] N. Tuglu, C. Kızılateş and S. Kesim, On the harmonic and hyperharmonic Fibonacci numbers, *Adv. Difference Equ.* **2015**, p. 297, doi:10.1186/s13662-015-0635-z (2015).

[23] X. Wang and Y. Qu, Some generalized q-harmonic number identities, *Integral Transforms Spec. Funct.* **26** (2015), 971–984.

[24] R. Corcino, On p,q-binomial coefficients, *Integers* **8** (2008), #A29.