Communications in Mathematics and Applications
Vol. 16, No. 2, pp. 443–450, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i2.3133

http://www.rgnpublications.com

Research Article

Generating Functions of a New Class of Semi-Orthogonal Polynomials $X_n(x;a,\alpha)$ Using Lie Group Theory

R. R. Jagtap*[1], P. G. Andhare[1] and S. B. Gaikwad[2]

[1] *Department of Mathematics, R. B. Narayanrao Borawake College (affiliated to Savitribai Phule Pune University), Shrirampur, Ahilyanagar, Maharasthra, India*
[2] *Department of Mathematics, New Arts, Commerce and Science College (affiliated to Savitribai Phule Pune University), Ahilyanagar, Maharasthra, India*
***Corresponding author:** rrjmaths@gmail.com

Received: March 2, 2025 **Revised:** May 14, 2025 **Accepted:** May 25, 2025

Abstract. In this paper, by applying the group theoretic method introduced by Weisner, we determined new generating relations of a new class of semi-orthogonal polynomials $X_n(x;a,\alpha)$. By giving proper analytical reasoning to the index m of the semi-orthogonal polynomial, we derived three linear partial differential operators with the help of the ascending and descending differential recurrence relation of the polynomial. These linear partial differential operators generate a Lie group.

Keywords. X_n polynomials, Generating functions, Weisner method

Mathematics Subject Classification (2020). 33C45, 33C50

1. Introduction

Louis Weisner [14,15] has developed a group theoretic method to obtain the generating functions for a large class of functions under certain conditions. He also showed the group theoretic crucialness in the study of Hypergeometric functions, Hermite functions and Bessel functions and their generating functions. For a given set, the necessary and sufficient condition is that it must have descending and ascending recurrence relations.

Andhare and Choudhary [1] obtained new generating relations for a class of polynomials $Y_m(a,x)$ by using Lie group theory. By using the Weisner Method, Bhagavan and Tadikonda [3] obtained three new generating relations for the Chebyshev polynomials. Srinivasulu and Bhagavan [12] obtained new generating relations for the two variable Hypergeometric polynomial $R_n(\beta;\alpha;x,y)$ by Weisner method. Many researcher used this method to obtain the generating functions in the theory of special functions for several semi-orthogonal polynomials (see, Chongdar [4], Elkhazendar *et al*. [5], Grosswald [6], Manocha [7], McBride [8], Miller Jr. [9], Pathan *et al*. [10], Srinivasulu and Bhagavan [12], Srivastava and Manocha [13], and Weisner [14, 15]).

In the study of pure and other branches of applied mathematics, mathematical physics and approximation theory orthogonal polynomials (Rainville [11]) and semi-orthogonal polynomials play vital role. It has applications in various branches of engineering and science.

Bajpai [2] investigated a new class of semi-orthogonal polynomials $X_n(x;a,y)$. The semi-orthogonal polynomials $X_n(x;a,y)$ are defined as,

$$X_n(x;a,y) = {}_2F_0\left(-n,a;-;-\frac{x}{y}\right), \tag{1.1}$$

where $n = 0,1,2,\ldots$.

These X_n polynomials have relation with Bessel polynomials (Bajpai [2], and Chongdar [4]), Hermite polynomials and Laguerre polynomials.

Replacing y by α and n by m in (1.1), we get

$$X_m(x;a,\alpha) = {}_2F_0\left(-m,a;-;-\frac{x}{\alpha}\right) = \sum_{k=0}^{\infty} \frac{(-m)_k(a)_k\left(-\frac{x}{\alpha}\right)^k}{k!}. \tag{1.2}$$

For the semi-orthogonal polynomial $X_n(x;a,\alpha)$, we get following pure recurrence relation, where $m = 0,1,2,\ldots$,

$$X_{m+1}(x;a,\alpha) = \frac{1}{\alpha}\left[(\alpha + mx + ax)X_m(x;a,\alpha) - mxX_{m-1}(x;a,\alpha)\right], \tag{1.3}$$

and also it satisfies the following differential recurrence relations,

$$\frac{d}{dx}X_m(x;a,\alpha) = \frac{m}{x}X_m(x;a,\alpha) - \frac{m}{x}X_{m-1}(x;a,\alpha), \tag{1.4}$$

$$\frac{d}{dx}X_m(x;a,\alpha) = \frac{\alpha}{x^2}X_{m+1}(x;a,\alpha) - \frac{\alpha}{x^2}(\alpha + ax)X_m(x;a,\alpha). \tag{1.5}$$

From (1.4) and (1.5), the following differential equation can be determined which is the type of linear and ordinary

$$\left[x^2\frac{d^2}{dx^2} + [\alpha + (a-m+1)x]\frac{d}{dx} - ma\right]X_m(x;a,\alpha) = 0. \tag{1.6}$$

If we use the operator notation,

$$X\left(x,\frac{d}{dx},-ma\right) = x^2\frac{d^2}{dx^2} + [\alpha + (a-m+1)x]\frac{d}{dx} - ma, \tag{1.7}$$

by using (1.7), the equation (1.6) can be rewritten as

$$X\left(x,\frac{d}{dx},-ma\right)X_m(x;a,\alpha) = 0. \tag{1.8}$$

2. Linear Differential Operators

For the semi-orthogonal polynomial $X_m(x;a,\alpha)$, we define the first order partial differential operators A, B and C which are linear, such that

$$A[X_m(x;a,\alpha)y^{-ma}] = a_m X_m(x;a,\alpha)y^{-ma}, \tag{2.1}$$

$$B[X_m(x;a,\alpha)y^{-ma}] = b_m X_{m-1}(x;a,\alpha)y^{-ma+a}, \tag{2.2}$$

$$C[X_m(x;a,\alpha)y^{-ma}] = c_m X_{m+1}(x;a,\alpha)y^{-ma-a} \tag{2.3}$$

and

$$E[X_m(x;a,\alpha)y^{-ma}] = -X_m(x;a,\alpha)y^{-ma}, \tag{2.4}$$

where, a_m, b_m and c_m are functions of m and which do not dependent on x and y, but not needed independent of the parameters a and α. We want to determine the first order linear differential operators A, B and C by a method given by Srivastava and Manocha [13].

Proof of Equation (2.1). Let $A = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)$, where each R_i $(i=1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$\begin{aligned}
A[X_m(x;a,\alpha)y^{-ma}] &= a_m X_m(x;a,\alpha)y^{-ma}\\
&= \left\{R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\}\\
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha) + R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha)\\
&\quad + R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= R_1(x,y)y^{-ma}\left\{\frac{m}{x}X_m(x;a,\alpha) - \frac{m}{x}X_{m-1}(x;a,\alpha)\right\}\\
&\quad - \frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha) + R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= \frac{-m}{x}R_1(x,y)y^{-ma}X_{m-1}(x;a,\alpha)\\
&\quad + \left\{\frac{m}{x}R_1(x,y) - \frac{ma}{y}R_2(x,y) + R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}$$

Now equating the coefficients of $X_m(x;a,\alpha)$ and $X_{m-1}(x;a,\alpha)$ on both sides, we get

$$R_1(x,y) = 0, \quad -\frac{ma}{y}R_2(x,y) + R_3(x,y) = a_m.$$

Choosing,

$$R_2(x,y) = -\frac{y}{a}, \quad R_3(x,y) = 0.$$

We get

$$A = -\frac{y}{a}\frac{\partial}{\partial y}.$$

□

Proof of Equation (2.2). Let $B = R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)$, where, each R_i $(i=1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$\begin{aligned}
B[X_m(x;a,\alpha)y^{-ma}] &= b_m X_{m-1}(x;a,\alpha)y^{-ma+a}\\
&= \left\{R_1(x,y)\frac{\partial}{\partial x} + R_2(x,y)\frac{\partial}{\partial y} + R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\}
\end{aligned}$$

$$\begin{aligned}
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha)+R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha)\\
&\quad +R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= R_1(x,y)y^{-ma}\left\{\frac{m}{x}X_m(x;a,\alpha)-\frac{m}{x}X_{m-1}(x;a,\alpha)\right\}\\
&\quad -\frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha)+R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= \frac{-m}{xy^a}R_1(x,y)y^{-ma+a}X_{m-1}(x;a,\alpha)\\
&\quad +\left\{\frac{m}{x}R_1(x,y)-\frac{ma}{y}R_2(x,y)+R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}$$

Choosing,

$$R_1(x,y)=xy^a,\ \ R_2(x,y)=\frac{y^{a+1}}{a},\ \ R_3(x,y)=0.$$

We get

$$B=xy^a\frac{\partial}{\partial x}+\frac{y^{a+1}}{a}\frac{\partial}{\partial y}.$$

□

Proof of Equation (2.3). Let $C=R_1(x,y)\frac{\partial}{\partial x}+R_2(x,y)\frac{\partial}{\partial y}+R_3(x,y)$, where each R_i $(i=1,2,3)$ are functions of x and y, but not needed independent of the parameters a and α,

$$\begin{aligned}
C[X_m(x;a,\alpha)y^{-ma}] &= c_mX_{m+1}(x;a,\alpha)y^{-ma-a}\\
&= \left\{R_1(x,y)\frac{\partial}{\partial x}+R_2(x,y)\frac{\partial}{\partial y}+R_3(x,y)\right\}\{X_m(x;a,\alpha)y^{-ma}\},\\
&= R_1(x,y)y^{-ma}\frac{d}{dx}X_m(x;a,\alpha)+R_2(x,y)y^{-ma}\left(\frac{-ma}{y}\right)X_m(x;a,\alpha)\\
&\quad +R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= R_1(x,y)y^{-ma}\left\{\frac{\alpha}{x^2}X_{m+1}(x;a,\alpha)-\frac{\alpha}{x^2}(\alpha+ax)X_m(x;a,\alpha)\right\}\\
&\quad -\frac{ma}{y}R_2(x,y)y^{-ma}X_m(x;a,\alpha)+R_3(x,y)y^{-ma}X_m(x;a,\alpha)\\
&= \frac{\alpha y^a}{x^2}R_1(x,y)y^{-ma-a}X_{m+1}(x;a,\alpha)\\
&\quad +\left\{-\frac{(\alpha+ax)}{x^2}R_1(x,y)-\frac{ma}{y}R_2(x,y)+R_3(x,y)\right\}y^{-ma}X_m(x;a,\alpha).
\end{aligned}$$

Now equating the coefficients of $X_m(x;a,\alpha)$ and $X_{m+1}(x;a,\alpha)$ on both sides and choosing

$$R_1(x,y)=\frac{x^2}{\alpha y^a},\ \ R_2(x,y)=0,\ \ R_3(x,y)=\frac{\alpha+ax}{\alpha y^a},$$

we get

$$C=\frac{x^2}{\alpha y^a}\frac{\partial}{\partial x}+\frac{\alpha+ax}{\alpha y^a}.$$

Therefore, we get the first order linear differential operators

$$A=-\frac{y}{a}\frac{\partial}{\partial y}\ ;\quad B=xy^a\frac{\partial}{\partial x}+\frac{y^{a+1}}{a}\frac{\partial}{\partial y}\ ;\quad C=\frac{x^2}{\alpha y^a}\frac{\partial}{\partial x}+\frac{\alpha+ax}{\alpha y^a}\ ;\quad E=-1. \tag{2.5}$$

These linear differential operators satisfy the commutation relations

$$\left.\begin{aligned}&[A,B]=AB-BA=-B,\\&[A,C]=AC-CA=C,\\&[B,C]=BC-CB=-1=E\\&\text{and}\\&[A,E]=[B,E]=[C,E]=0.\end{aligned}\right\}\tag{2.6}$$

These commutator relations exhibits that the linear differential operators A, B, C, E generate a Lie group and the operator C commutes with the operators B and A.

The extended form of group generated by each of operators B and C can be expressed as,

$$e^{b'B}f(x,y)=f\left(\frac{xy^{-a}}{y^{-a}-b'},(y^{-a}-b')^{\frac{-1}{a}}\right),\tag{2.7}$$

$$e^{c'C}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{\alpha x}{\alpha-c'xy^{-a}},y\right)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{x}{1-\frac{c'xy^{-a}}{\alpha}},y\right),\tag{2.8}$$

$$e^{c'C}e^{b'B}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{\alpha xy^{-a}}{(y^{-a}-b')(\alpha-c'xy^{-a})},(y^{-a}-b')^{\frac{-1}{a}}\right),$$

$$e^{c'C}e^{b'B}f(x,y)=e^{c'y^{-a}}\left(1-\frac{c'xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-b')\left(1-\frac{c'xy^{-a}}{\alpha}\right)},(y^{-a}-b')^{\frac{-1}{a}}\right).\tag{2.9}$$

□

3. Generating Function Relations

By assigning different values to b' and c', the generating relations are determined for the following three cases:

Case 1. $b'=1$, $c'=0$.

Case 2. $b'=0$, $c'=1$.

Case 3. $b'c'\neq 0$.

Case 1. Putting $b'=1$ in equation (2.7),

$$e^{B}f(x,y)=f\left(\frac{xy^{-a}}{y^{-a}-1},(y^{-a}-1)^{\frac{-1}{a}}\right),$$

$$\exp B\{y^{-ma}X_m(x;a,\alpha)\}=(1-t)^m X_m\left(\frac{x}{1-t};a,\alpha\right),$$

$$\sum_{p=0}^{\infty}\frac{(-m)_p X_{m-p}(x;a,\alpha)t^p}{p!}=(1-t)^m X_m\left(\frac{x}{1-t};a,\alpha\right),\tag{3.1}$$

where $t=y^{-a}$.

Case 2. Putting $b'=0$, $c'=1$ in equation (2.8),

$$e^{C}f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)f\left(\frac{x}{1-\frac{xy^{-a}}{\alpha}},y\right),$$

$$\exp C\{y^{-ma}X_m(x;a,\alpha)\}=\exp(y^{-a})\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{x}{1-\frac{xy^{-a}}{\alpha}},y\right),$$

$$\sum_{p=0}^{\infty}\frac{X_{m+p}(x;a,\alpha)z^p}{p!}=\exp(z)\left(1-\frac{xz}{\alpha}\right)^{-a}X_m\left(\frac{x}{1-\frac{xz}{\alpha}};a,\alpha\right),$$

$$\exp(z)\left(1-\frac{xz}{\alpha}\right)^{-a}X_m\left(\frac{x}{1-\frac{xz}{\alpha}};a,\alpha\right)=\sum_{p=0}^{\infty}\frac{X_{m+p}(x;a,\alpha)z^p}{p!}, \tag{3.2}$$

where $z=y^{-a}$.

Case 3. From equation (2.9) for $b'c'\neq 0$, putting $b'=w$ and $c'=1$, we get

$$e^Ce^{wB}f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)},(y^{-a}-w)^{\frac{-1}{a}}\right),$$

$$\exp(C)\exp(wB)f(x,y)=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}f\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)},(y^{-a}-w)^{\frac{-1}{a}}\right),$$

$$\exp(C)\exp(wB)\{y^{-ma}X_m(x;a,\alpha)\}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}((y^{-a}-w)^{\frac{-1}{a}})^{-ma}\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right),$$

$$\exp(C)\exp(wB)\{y^{-ma}X_m(x;a,\alpha)\}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}(y^{-a}-w)^m\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right),$$

$$\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^p(y^{-a})^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}=e^{y^{-a}}\left(1-\frac{xy^{-a}}{\alpha}\right)^{-a}(y^{-a}-w)^m\cdot X_m\left(\frac{xy^{-a}}{(y^{-a}-w)\left(1-\frac{xy^{-a}}{\alpha}\right)};a,\alpha\right).$$

Putting $y^{-a}=z$ in the above equation,

$$\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^pz^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}=e^z\left(1-\frac{xz}{\alpha}\right)^{-a}(z-w)^mX_m\left(\frac{xz}{(z-w)(1-\frac{xz}{\alpha})};a,\alpha\right)$$

or we can write,

$$e^z\left(1-\frac{xz}{\alpha}\right)^{-a}(z-w)^mX_m\left(\frac{xz}{(z-w)(1-\frac{xz}{\alpha})};a,\alpha\right)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-m)_pw^pz^{m+q-p}X_{m+q-p}(x;a,\alpha)}{p!q!}. \tag{3.3}$$

Equations (3.1), (3.2) and (3.3) are new generating relations for the new class of semi-orthogonal polynomials $X_m(x;a,\alpha)$.

4. Conclusion

Three new generating relations are obtained for the new class of semi-orthogonal polynomials $X_m(x;a,\alpha)$ by using the Weisner's method. This method is a very powerful technique to obtain generating relations from the differential recurrence relations of the ascending and the descending type of the semi-orthogonal polynomial $X_m(x;a,\alpha)$.

Acknowledgement

The authors are grateful to the editor and referee for their valuable and helpful comments and suggestions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P. G. Andhare and M. S. Choudhary, Generating functions for a class of polynomials $Y_m(a,x)$ using Lie group theory, *Ganita Sandesh* **18**(2) (2004), 149 – 156.

[2] S. D. Bajpai, Generating functions and semi-orthogonal properties of a new class of polynomials, *Rendiconti di Matematica, Serie VII* **13** (1993), 365 – 372, URL: https://www1.mat.uniroma1.it/ricerca/rendiconti/ARCHIVIO/1993(2)/365-372.pdf

[3] V. S. Bhagavan and S. Tadikonda, Chebyshev polynomials of generating functions by Weisner method, *AIP Conference Proceedings* **2512** (2024), 020078, DOI: 10.1063/5.0111907.

[4] A. K. Chongdar, On the bilateral generating functions of simple Bessel polynomial, *Proceedings of the Indian Academy of Sciences* (*Mathematical Sciences*) **98**(1) (1988), 59 – 62, URL: https://www.ias.ac.in/article/fulltext/pmsc/098/01/0059-0062.

[5] M. B. Elkhazendar, Group theoretical study of certain generating functions for modified Jacobi polynomials, *Journal of Informatics and Mathematical Sciences* **4**(1) (2012), 117 – 122, DOI: 10.26713/jims.v4i1.73.

[6] E. Grosswald, *Bessel Polynomials*, Lecture Notes in Mathematics series, Vol. 698, Springer-Verlag, Berlin — Heidelberg — New York, xvi + 188 pages (1978), URL: https://link.springer.com/book/10.1007/BFb0063135.

[7] H. L. Manocha, Lie theoretic generating functions, *Publications de l'Institut Mathématique* **20**(34) (1976), 179 – 184, URL: http://elib.mi.sanu.ac.rs/files/journals/publ/40/21.pdf.

[8] E. B. McBride, *Obtaining Generating Functions*, 1st edition, Springer, Berlin – Heidelberg, viii + 104 pages (1971).

[9] W. Miller Jr., *Lie Theory and Special Functions*, Academic Press, Inc., New York – London, 338 pages (1968).

[10] M. A. Pathan, A. N. Goyal and M. J. S. Shahwan, Lie-theoretic generating functions of multivariable generalized Bessel functions, *Reports on Mathematical Physics* **39**(2) (1997), 249 – 254, DOI: 10.1016/S0034-4877(97)88003-2.

[11] E. D. Rainville, *Special Functions*, MacMillan, New York and reprinted by Chelsea Publishing Company, Bronx, New York, 365 pages (1971).

[12] T. Srinivasulu and V. S. Bhagavan, Generating functions for Hypergeometric polynomials of two variables $R_n(\beta;\alpha;x,y)$ using Weisner group-theoretic method, *International Journal of Civil Engineering and Technology* **9**(3) (2018), 106 – 119.

[13] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 569 pages (1984).

[14] L. Weisner, Group-theoretic origin of certain generating functions, *Pacific Journal of Mathematics* **5**(6) (1955), 1033 – 1039, DOI: 10.2140/pjm.1955.5.1033.

[15] L. Weisner, Generating functions for Hermite functions, *Canadian Journal of Mathematics* **11** (1959), 141 – 147, DOI: 10.4153/CJM-1959-018-4.