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Research Article

Performance Assessment of Production Inventory System With Catastrophes Including Negative Customers and Machine Breakdowns

A. Abi Shunmuga Kanni*[‡] and M. Amirthakodi

PG and Research Department of Mathematics, Kamaraj College (affiliated to Manonmaniam Sundaranar University, Abishekapatti), Thoothukudi, Tamilnadu, India
*__Corresponding author:__ aabisk1997@gmail.com

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Abstract. This study looks at a production inventory system that experiences breakdowns in machines, negative customers and catastrophes. If a customer arrives and raises their level in the waiting hall with a probability of r, they are considered ordinary; if they drop their level without inventory with a probability of $1-r=\bar{r}$, they are known as negative customers. Upon completion of service, a customer exits the system, causing the inventory level to decrease by one. Catastrophic events force the inventory level to drop to zero. The production policy is (s,S), where S is the fixed maximum inventory level. A machine could break down during production, in which case it will be fixed at random. The steady-state joint probability distribution of the inventory level, the number of customers in orbit, and the machine status are derived using the matrix-geometric method. Several performance measures are calculated, and the results are used to develop a cost function. Finally, numerical results are presented to demonstrate the system's behavior.

Keywords. Production inventory system, Service facility, Negative customers, (s,S) Policy, Machine breakdown, Catastrophes

Mathematics Subject Classification (2020). 60J27

[‡]Research Scholar, Reg. No: 20212102092006.

1. Introduction

This study examines the impact of machine breakdowns, negative customer effects and catastrophic events on production inventory systems. As discussed by Berman and Sapna [2], and Buzacott and Shanthikumar [3], stochastic models are used to model systems that are subject to random variations or uncertainties. The production process refers to the conversion of raw materials into finished goods. The production rate can fluctuate due to breakdowns, where production may be halted, leading to inventory shortages or delays. Gelenbe [4] introduces an innovative approach to queueing network theory by incorporating both positive and negative customers. Anitha *et al*. [1] explored a production inventory system affected by machine breakdowns, using exponential distributions for production time, interfailure time, and repair time. The production inventory system was further examined by Karthick *et al*. [5], who introduced two distinct customer types alongside machine breakdowns. Sivakumar and Arivarignan [10] builds on the concept of queueing theory and inventory management, specifically, in systems dealing with perishable goods and the concept of negative customers. A catastrophe in the production inventory system typically refers to a significant disruption or crisis in managing the inventory of materials or products within a production environment. According to Melikov *et al*. [6, 7], and Ozkar *et al*. [9], when a catastrophe strikes, all inventory items are destroyed, including those that were already allocated to customers for release. Following such an event, customers who were disrupted and had their inventory destroyed are re-entered into the queue. This situation introduces the potential for delays in customer fulfillment, as the queue is likely to expand with the addition of returning customers.

This paper analyses the production inventory system combining machine breakdowns, catastrophes and negative customers through a matrix geometric method.

Notations

- $[A]_{ij}$: Element or sub-matrix at ith row, jth column of the matrix A.
- $\mathbf{e}$: A column vector of the appropriate dimension filled with ones.
- $\bar{r} = 1 - r$, $0 \le r \le 1$.
- E_m^n : The set of all natural numbers from m to n, inclusive of m and n.

2. Model Description

Consider a production inventory system with a maximum stock capacity of S units. Customers arrive according to a Poisson process with rate ω and upon arrival, each customer either increases their level in the waiting hall with probability r or decreases their level with probability $\bar{r}$. When a customer lowers by one with a unit item, the service rate denoted by τ occurs and the service facility uses an exponential distribution. Catastrophes may occur in the system, and it is assumed that they follow an exponential distribution with a parameter of ζ. Once the inventory falls below a certain threshold, s $(< S)$, the machine is activated to produce the item. The production of a unit item is assumed to follow an exponential distribution with a parameter κ. When the inventory level reaches S, manufacturing is halted. The machine may break down during production, in which case the breakdown time follows an exponential distribution with a parameter ϵ. The breakdown is repaired after a random amount of time, and the repair duration is assumed to follow an exponential distribution with a parameter η.

3. Mathematical Analysis

Let $C_1(t)$ represent the number of consumers in the waiting hall at time t, $C_2(t)$ represent the amount of merchandise that is on hand at time t and $C_3(t)$ represent the machine's status at time t,

$$C_3(t) = \begin{cases} 0, & \text{the machine is idle,} \\ 1, & \text{the machine is switched on,} \\ 2, & \text{the machine is under repair.} \end{cases}$$

Using the input and output process assumptions, it is possible to demonstrate that the stochastic process $\{(C_1(t), C_2(t), C_3(t)), t \geq 0\}$ is a continuous time Markov process with state space,

$$\Omega = \begin{cases} (c_1, c_2, 0), & c_1 = 0, 1, 2, \dots; c_2 \in E^S_{s+1}; \\ (c_1, c_2, c_3), & c_1 = 0, 1, 2, \dots; c_2 \in E^{S-1}_1;\ c_3 = 1, 2. \end{cases}$$

The ordering of the above states space is denoted by, $(\ll 0 \gg, \ll 1 \gg, \ll 2 \gg, \dots)$, where

$$\langle\langle c_1 \rangle\rangle = (\langle c_1, 0\rangle, \langle c_1, 1\rangle, \dots, \langle c_1, S\rangle), \quad c_1 = 0, 1, 2, \dots$$

and

$$\langle c_1, c_2 \rangle = \begin{cases} ((c_1, c_2, 1), (c_1, c_2, 2)), & c_1 = 0, 1, 2, \dots; c_2 \in E^s_0; \\ ((c_1, c_2, 0), (c_1, c_2, 1), (c_1, c_2, 2)), & c_1 = 0, 1, 2, \dots; c_2 \in E^{S-1}_{s+1}; \\ (c_1, c_2, 0), & c_1 = 0, 1, 2, \dots; c_2 = S; \end{cases}$$

is the infinitesimal generating matrix Q (block matrix), which is a tri-diagonal matrix because the number of customers in the waiting hall can either go up or down by one or stay the same. Let

$$Q = \begin{array}{c} \\ \langle\langle 0 \rangle\rangle \\ \langle\langle 1 \rangle\rangle \\ \langle\langle 2 \rangle\rangle \\ \langle\langle 3 \rangle\rangle \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} \langle\langle 0 \rangle\rangle & \langle\langle 1 \rangle\rangle & \langle\langle 2 \rangle\rangle & \langle\langle 3 \rangle\rangle & \langle\langle 4 \rangle\rangle & \cdots \end{array} \\ \begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

where

$$[A_0]_{c_2 c_2'} = \begin{cases} X_{00}, & c_2' = c_2,\ c_2 \in E^s_0; \\ X_{01}, & c_2' = c_2,\ c_2 \in E^{S-1}_{s+1}; \\ X_{02}, & c_2' = c_2,\ c_2 = S; \\ \mathbf{0}, & \text{otherwise,} \end{cases} \qquad X_{00} = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} r\omega & 0 \\ 0 & r\omega \end{pmatrix} \end{array}$$

$$X_{01} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} r\omega & 0 & 0 \\ 0 & r\omega & 0 \\ 0 & 0 & r\omega \end{pmatrix} \end{array}, \qquad X_{02} = 0 \begin{array}{c} 0 \\ \begin{pmatrix} r\omega \end{pmatrix} \end{array}$$

$$[A_2]_{c_2c_2'} = \begin{cases} W_{20}, & c_2' = c_2 - 1,\, c_2 \in E_1^s; \\ W_{21}, & c_2' = c_2 - 1,\, c_2 = s+1; \\ W_{22}, & c_2' = c_2 - 1,\, c_2 \in E_{s+2}^{S-1}; \\ W_{23}, & c_2' = c_2 - 1,\, c_2 = S; \\ X_{20}, & c_2' = c_2,\, c_2 \in E_0^s; \\ X_{21}, & c_2' = c_2,\, c_2 \in E_{s+1}^{S-1}; \\ X_{22}, & c_2' = c_2,\, c_2 = S; \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad W_{20} = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix} \end{array}, \qquad W_{21} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \tau & 0 \\ \tau & 0 \\ 0 & \tau \end{pmatrix} \end{array},$$

$$W_{22} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix} \end{array}, \qquad W_{23} = 0 \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \tau & 0 & 0 \end{pmatrix} \end{array}, \qquad X_{20} = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \bar{r}\omega & 0 \\ 0 & \bar{r}\omega \end{pmatrix} \end{array},$$

$$X_{21} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \bar{r}\omega & 0 & 0 \\ 0 & \bar{r}\omega & 0 \\ 0 & 0 & \bar{r}\omega \end{pmatrix} \end{array}, \qquad X_{22} = 0 \begin{array}{c} 0 \\ \begin{pmatrix} \bar{r}\omega \end{pmatrix} \end{array},$$

$$[B_0]_{c_2c_2'} = \begin{cases} Y_1, & c_2' = 0,\, c_2 \in E_1^s; \\ Y_2, & c_2' = 0,\, c_2 \in E_{s+1}^{S-1}; \\ Y_3, & c_2' = 0,\, c_2 = S; \\ X_{11}, & c_2' = c_2,\, c_2 = 0; \\ X_{22}, & c_2' = c_2,\, c_2 \in E_1^s; \\ X_{33}, & c_2' = c_2,\, c_2 \in E_{s+1}^{S-1}; \\ X_{44}, & c_2' = c_2,\, c_2 = S; \\ Z_{01}, & c_2' = c_2 + 1,\, c_2 \in E_0^{s-1}; \\ Z_{02}, & c_2' = c_2 + 1,\, c_2 = s; \\ Z_{03}, & c_2' = c_2 + 1,\, c_2 \in E_{s+1}^{S-2}; \\ Z_{04}, & c_2' = c_2 + 1,\, c_2 = S-1; \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad Y_1 = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} \end{array}, \qquad Y_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \zeta & 0 \\ \zeta & 0 \\ 0 & \zeta \end{pmatrix} \end{array},$$

$$Y_3 = 0 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \zeta & 0 \end{pmatrix} \end{array}, \qquad Z_{01} = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \qquad Z_{02} = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad Z_{03} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$Z_{04}=\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\!\!\begin{array}{c} 0\\ \left(\begin{array}{c} 0\\ \kappa\\ 0\end{array}\right)\end{array},\qquad X_{11}=\begin{array}{c} \\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{cc}\quad 1 & \qquad\qquad 2\end{array}\\ \left(\begin{array}{cc} -(r\omega+\kappa+\epsilon) & \epsilon\\ \eta & -(r\omega+\eta)\end{array}\right)\end{array},$$

$$X_{22}=\begin{array}{c} \\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{cc}\quad 1 & \qquad\qquad\qquad 2\end{array}\\ \left(\begin{array}{cc} -(r\omega+\kappa+\epsilon+\zeta) & \epsilon\\ \eta & -(r\omega+\eta+\zeta)\end{array}\right)\end{array},$$

$$X_{33}=\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{ccc}\quad 0 & \qquad\qquad\qquad 1 & \qquad\qquad\qquad 2\end{array}\\ \left(\begin{array}{ccc} -(r\omega+\zeta) & 0 & 0\\ 0 & -(r\omega+\kappa+\epsilon+\zeta) & \epsilon\\ 0 & \eta & -(r\omega+\eta+\zeta)\end{array}\right)\end{array},\qquad X_{44}=0\ \begin{array}{c}0\\ \left(-(r\omega+\zeta)\right)\end{array},$$

$$[A_1]_{c_2c_2'}=\begin{cases} Y_1, & c_2'=0,\ c_2\in E_1^s;\\ Y_2, & c_2'=0,\ c_2\in E_{s+1}^{S-1};\\ Y_3, & c_2'=0,\ c_2=S;\\ \widetilde{X}_{11}, & c_2'=c_2,\ c_2=0;\\ \widetilde{X}_{22}, & c_2'=c_2,\ c_2\in E_1^s;\\ \widetilde{X}_{33}, & c_2'=c_2,\ c_2\in E_{s+1}^{S-1};\\ \widetilde{X}_{44}, & c_2'=c_2,\ c_2=S;\\ Z_{01}, & c_2'=c_2+1,\ c_2\in E_0^{s-1};\\ Z_{02}, & c_2'=c_2+1,\ c_2=s;\\ Z_{03}, & c_2'=c_2+1,\ c_2\in E_{s+1}^{S-2};\\ Z_{04}, & c_2'=c_2+1,\ c_2=S-1;\\ \mathbf{0}, & \text{otherwise,}\end{cases}$$

$$\widetilde{X}_{11}=\begin{array}{c} \\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{cc}\quad 1 & \qquad\qquad 2\end{array}\\ \left(\begin{array}{cc} -(\omega+\kappa+\epsilon) & \epsilon\\ \eta & -(\omega+\eta)\end{array}\right)\end{array},\qquad \widetilde{X}_{44}=0\ \begin{array}{c}0\\ \left(-(\omega+\zeta+\tau)\right)\end{array},$$

$$\widetilde{X}_{22}=\begin{array}{c} \\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{cc}\quad 1 & \qquad\qquad\qquad 2\end{array}\\ \left(\begin{array}{cc} -(\omega+\kappa+\epsilon+\zeta+\tau) & \epsilon\\ \eta & -(\omega+\eta+\zeta+\tau)\end{array}\right)\end{array},$$

$$\widetilde{X}_{33}=\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\!\!\begin{array}{c}\begin{array}{ccc}\quad 0 & \qquad\qquad\qquad 1 & \qquad\qquad\qquad 2\end{array}\\ \left(\begin{array}{ccc} -(\omega+\zeta+\tau) & 0 & 0\\ 0 & -(\omega+\kappa+\epsilon+\zeta+\tau) & \epsilon\\ 0 & \eta & -(\omega+\eta+\zeta+\tau)\end{array}\right)\end{array}.$$

3.1 Stability Analysis

Now, let us examine the generating matrix $D = A_0 + A_1 + A_2$ is provided by

$$[D]_{c_2 c_2'} = \begin{cases} Y_0, & c_2' = 0, c_2 = 1; \\ Y_1, & c_2' = 0, c_2 \in E_2^s; \\ Y_2, & c_2' = 0, c_2 \in E_{s+1}^{S-1}; \\ Y_3, & c_2' = 0, c_2 = S; \\ W_{20}, & c_2' = c_2 - 1, c_2 \in E_2^s; \\ W_{21}, & c_2' = c_2 - 1, c_2 = s+1; \\ W_{22}, & c_2' = c_2 - 1, c_2 \in E_{s+2}^{S-1}; \\ W_{23}, & c_2' = c_2 - 1, c_2 = S; \\ \widetilde{X}_0, & c_2' = c_2, c_2 = 0; \\ \widetilde{X}_1, & c_2' = c_2, c_2 \in E_1^s; \\ \widetilde{X}_2, & c_2' = c_2, c_2 \in E_{s+1}^{S-1}; \\ \widetilde{X}_3, & c_2' = c_2, c_2 = S; \\ Z_{01}, & c_2' = c_2 + 1, c_2 \in E_0^{s-1}; \\ Z_{02}, & c_2' = c_2 + 1, c_2 = s; \\ Z_{03}, & c_2' = c_2 + 1, c_2 \in E_{s+1}^{S-2}; \\ Z_{04}, & c_2' = c_2 + 1, c_2 = S-1; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $Y_0 = Y_1 + W_{20}$; $\widetilde{X}_0 = X_{00} + X_{20} + \widetilde{X}_{11}$; $\widetilde{X}_1 = X_{00} + X_{20} + \widetilde{X}_{22}$; $\widetilde{X}_2 = X_{01} + X_{21} + \widetilde{X}_{33}$; $\widetilde{X}_3 = X_{02} + X_{22} + \widetilde{X}_{44}$. This structure clearly shows that the generating matrix D is ergodic. There is a steady-state probability distribution.

Assuming that Π stands for D's steady state probability distribution, the equation

$$\Pi D = 0, \tag{3.1}$$

$$\Pi \mathbf{e} = 1, \tag{3.2}$$

where the vector $\Pi = (\pi^{\langle 0 \rangle}, \pi^{\langle 1 \rangle}, \pi^{\langle 2 \rangle}, \dots, \pi^{\langle S \rangle})$.

From the above two equations, we get the following set of equations

$$\begin{aligned} &\pi^{\langle j \rangle} \widetilde{X}_0 + \pi^{\langle j+1 \rangle} Z_{01} = \mathbf{0}, && j = 0, \\ &\pi^{\langle j-1 \rangle} Y_0 + \pi^{\langle j \rangle} \widetilde{X}_1 + \pi^{\langle j+1 \rangle} Z_{01} = \mathbf{0}, && j = 1, \\ &\pi^{\langle 0 \rangle} Y_1 + \pi^{\langle j-1 \rangle} W_{20} + \pi^{\langle j \rangle} \widetilde{X}_1 + \pi^{\langle j+1 \rangle} Z_{01} = \mathbf{0}, && j \in E_2^{s-1}, \\ &\pi^{\langle 0 \rangle} Y_1 + \pi^{\langle j-1 \rangle} W_{20} + \pi^{\langle j \rangle} \widetilde{X}_1 + \pi^{\langle j+1 \rangle} Z_{02} = \mathbf{0}, && j = s, \\ &\pi^{\langle 0 \rangle} Y_2 + \pi^{\langle j-1 \rangle} W_{21} + \pi^{\langle j \rangle} \widetilde{X}_2 + \pi^{\langle j+1 \rangle} Z_{03} = \mathbf{0}, && j = s+1, \\ &\pi^{\langle 0 \rangle} Y_2 + \pi^{\langle j-1 \rangle} W_{22} + \pi^{\langle j \rangle} \widetilde{X}_2 + \pi^{\langle j+1 \rangle} Z_{03} = \mathbf{0}, && j \in E_{s+2}^{S-2}, \\ &\pi^{\langle 0 \rangle} Y_2 + \pi^{\langle j-1 \rangle} W_{22} + \pi^{\langle j \rangle} \widetilde{X}_2 + \pi^{\langle j+1 \rangle} Z_{04} = \mathbf{0}, && j = S-1, \\ &\pi^{\langle 0 \rangle} Y_3 + \pi^{\langle j-1 \rangle} W_{23} + \pi^{\langle j \rangle} \widetilde{X}_3 = \mathbf{0}, && j = S. \end{aligned}$$

Lemma 3.1. *The stability condition of the system under study is given by*

$$r\omega < \bar{r}\omega + (1 - \pi^{\langle 0\rangle}\mathbf{e})\tau\,. \tag{3.3}$$

Proof. From the well known results of Neuts [8] on the positive recurrence of D, we have

$$\Pi A_0\mathbf{e} < \Pi A_2\mathbf{e} \tag{3.4}$$

and by exploiting the structure of the matrices A_0, A_2 and Π stated result follows. □

3.2 Steady State Analysis

The irreducible structure of the rate matrix Q and Lemma 3.1 make it evident that the continuous-time Markov process $\{(C_1(t), C_2(t), C_3(t)),\ t \geq 0\}$ with state space Ω is regular. For this reason, the limiting distribution Φ,

$$\Phi^{(c_1,c_2,c_3)} = \lim_{t\to\infty} Pr[C_1(t) = c_1, C_2(t) = c_2, C_3(t) = c_3 \mid C_1(0), C_2(0), C_3(0)]$$

exists and does not depend on the initial state, that is $\Phi = (\Phi^{(0)}, \Phi^{(1)}, \dots)$ satisfies

$$\Phi Q = 0, \quad \Phi\mathbf{e} = 1\,. \tag{3.5}$$

4. System Performance Measures

In this section, we derive several important system performance metrics.

Expected Inventory level

$$\Im_I = \sum_{c_1=0}^{\infty}\sum_{c_2=1}^{S} c_2\Phi^{(c_1,c_2)}\mathbf{e}$$

Expected number of customers in the waiting hall

$$\Im_O = \sum_{c_1=1}^{\infty} c_1\Phi^{(c_1)}\mathbf{e}$$

Expected value of arrivals of negative customers

$$\Im_N = \sum_{c_1=1}^{\infty} \bar{r}\omega\Phi^{(c_1)}\mathbf{e}$$

Expected Production startup rate

$$\Im_P = \sum_{c_1=0}^{\infty}\sum_{c_2=s+1}^{S} \zeta\Phi^{(c_1,c_2,0)} + \sum_{c_1=1}^{\infty} \tau\Phi^{(c_1,s+1,0)}$$

Expected Repair rate

$$\Im_R = \sum_{c_1=0}^{\infty}\sum_{c_2=0}^{S-1} \eta\Phi^{(c_1,c_2,2)}$$

Expected Catastrophe rate

$$\Im_{CT} = \sum_{c_1=0}^{\infty}\sum_{c_2=1}^{S} \zeta\Phi^{(c_1,c_2)}\mathbf{e}$$

5. Cost Analysis

The long-run total expected cost rate for this model is defined to be

$$TC(s,S) = c_h \Im_I + c_o \Im_O + c_n \Im_N + c_s \Im_P + c_r \Im_R + c_{ct} \Im_{CT},$$

where

c_h : The inventory carrying cost per unit item per unit time
c_o : Waiting cost of a customer in the waiting hall per unit time
c_n : Loss per unit time due to arrival of a negative customer
c_s : Production startup cost for per production initiation
c_r : Machine service cost per repair per unit time
c_{ct} : Catastrophe cost per unit time

6. Numerical Analysis

The function $TC(s,S)$ appears to be convex based on extensive numerical experimentation. Optimal values of the total cost rate, denoted as TC^*, s^* and S^*, are obtained through a simple numerical search process. A typical three-dimensional plot of the expected total cost function is presented in Figure 1.

The optimal cost value $TC^* = 1.2224$ is achieved at $(s^*, S^*) = (4, 18)$, for the fixed parameter values $\omega = 2.3$; $\tau = 3.5$; $r = 0.7$; $\zeta = 0.05$; $\kappa = 3.2$; $\epsilon = 0.9$; $\eta = 2.4$; $c_h = 0.01$; $c_o = 0.8$; $c_n = 0.027$; $c_s = 1.2$; $c_r = 0.152$; $c_{ct} = 0.65$.

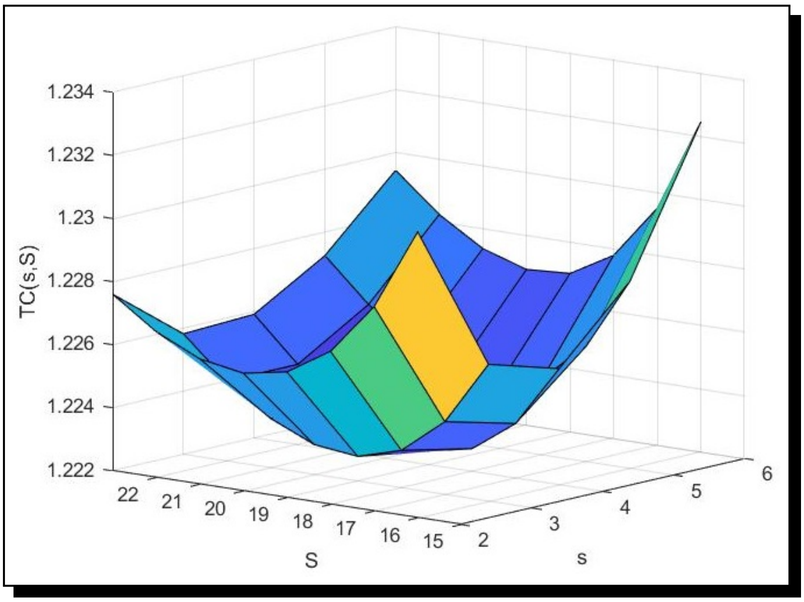

Figure 1. A typical three-dimensional plot illustrating the convexity of the total expected cost rate

The impact of variations in cost rates and system parameters on the optimal values is analyzed and summarized in the following tables. In each cell, the lower entry represents the optimal cost rate, while the upper entries provide the corresponding S^* and s^* values. Given the fixed parameter values $c_h = 0.01$, $c_o = 0.8$, $c_n = 0.027$, $c_s = 1.2$, $c_r = 0.152$ and $c_{ct} = 0.65$, the results presented in Tables 1, 2, and 3 demonstrate that the optimal cost increases with rising values of ω, ζ and ϵ, whereas it decreases as the parameters τ, κ and η increase. S^* increases monotonically with ω, τ and ϵ increases and decreases monotonically as ζ, κ and η increases. s^* increases monotonically with ω and ϵ increases and decreases monotonically as τ, ζ, κ and η increases.

Table 1. Impact of parameter on the optimal values for fixed $\omega = 2.25$

ω = 2.25		ϵ	0.8						0.9						1					
η			2.37		2.4		2.43		2.37		2.4		2.43		2.37		2.4		2.43	
τ	ζ	κ																		
3.4	0.04		18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4
		3.12	1.1768		1.1747		1.1727		1.2011		1.1985		1.1961		1.2277		1.2245		1.2214	
			18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
		3.20	1.1666		1.1646		1.1627		1.1.1889		1.1866		1.1843		1.2132		1.2103		1.2075	
			18	3	18	3	17	3	18	4	18	4	18	3	18	4	18	4	18	4
		3.28	1.1573		1.1555		1.1538		1.1783		1.1762		1.1742		1.2006		1.1980		1.1955	
	0.05		18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
		3.12	1.2075		1.2049		1.2025		1.2353		1.2322		1.2293		1.2654		1.2616		1.2581	
			18	3	18	3	18	3	18	4	18	3	18	3	18	4	18	4	18	4
		3.20	1.1951		1.1928		1.1906		1.2210		1.2182		1.2155		1.2486		1.2452		1.2419	
			17	3	17	3	17	3	18	3	18	3	18	3	18	4	18	4	18	4
		3.28	1.1843		1.1822		1.1801		1.2083		1.2057		1.2032		1.2339		1.2309		1.2279	
	0.06		18	3	18	3	18	3	18	3	18	3	18	3	18	4	18	4	18	4
		3.12	1.2371		1.2342		1.2314		1.2684		1.2648		1.2614		1.3020		1.2978		1.2937	
			18	3	18	3	17	3	18	3	18	3	18	3	18	3	18	3	18	3
		3.20	1.2231		1.2204		1.2179		1.2519		1.2487		1.2456		1.2830		1.2791		1.2753	
			17	3	17	3	17	3	18	3	17	3	17	3	18	3	18	3	18	3
		3.28	1.2107		1.2083		1.2060		1.2375		1.2345		1.2317		1.2661		1.2625		1.2591	
3.5	0.04		18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4
		3.12	1.1490		1.1469		1.1449		1.1734		1.1708		1.1683		1.2001		1.1968		1.1938	
			18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
		3.20	1.1388		1.1368		1.1349		1.1612		1.1588		1.1566		1.1855		1.1826		1.1798	
			18	3	18	3	17	3	18	4	18	4	18	3	18	4	18	4	18	4
		3.28	1.1295		1.1277		1.1260		1.1506		1.1484		1.1464		1.1728		1.1702		1.1678	
	0.05		18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
		3.12	1.1793		1.1768		1.1744		1.2072		1.2042		1.2012		1.2374		1.2336		1.2300	
			18	3	18	3	18	3	18	4	18	3	18	3	18	4	18	4	18	4
		3.20	1.1670		1.1647		1.1625		1.1929		1.1901		1.1874		1.2205		1.2171		1.2139	
			17	3	17	3	17	3	18	3	18	3	18	3	18	4	18	4	18	4
		3.28	1.1562		1.1540		1.1520		1.1802		1.1776		1.1751		1.2050		1.2028		1.1999	
	0.06		18	3	18	3	18	3	18	3	18	3	18	3	18	4	18	4	18	4
		3.12	1.2086		1.2057		1.2030		1.2400		1.2364		1.2330		1.2736		1.2694		1.2653	
			18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3
		3.20	1.1946		1.1920		1.1894		1.2234		1.2202		1.2171		1.2545		1.2506		1.2469	
			17	3	17	3	17	3	18	3	18	3	17	3	18	3	18	3	18	3
		3.28	1.1823		1.1799		1.1775		1.2090		1.2060		1.2032		1.2376		1.2341		1.2307	

Table *Contd.*

$\omega = 2.25$	ϵ		0.8			0.9			1		
η			2.37	2.4	2.43	2.37	2.4	2.43	2.37	2.4	2.43
τ	ζ	κ									
3.6	0.04	3.12	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.1231	1.1211	1.1191	1.1476	1.1450	1.1425	1.1743	1.1711	1.1680
		3.20	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.1129	1.1109	1.1090	1.1353	1.1330	1.1307	1.1597	1.1568	1.1540
		3.28	18 3	18 3	18 3	18 4	18 4	18 3	18 4	18 4	18 4
			1.1035	1.1018	1.1001	1.1247	1.1226	1.1205	1.1470	1.1444	1.1419
	0.05	3.12	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.1532	1.1506	1.1482	1.1811	1.1780	1.1751	1.2113	1.2076	1.2040
		3.20	18 3	18 3	18 3	18 4	18 3	18 3	18 4	18 4	18 4
			1.1408	1.1385	1.1363	1.1668	1.1640	1.1612	1.1944	1.1910	1.1878
		3.28	18 3	17 3	17 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.1299	1.1278	1.1258	1.1540	1.1514	1.1489	1.1797	1.1766	1.1737
	0.06	3.12	18 3	18 3	18 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.1821	1.1792	1.1765	1.2135	1.2099	1.2065	1.2472	1.2429	1.2389
		3.20	18 3	18 3	18 3	18 3	18 3	18 3	18 3	18 3	18 3
			1.1681	1.1654	1.1629	1.1969	1.1937	1.1906	1.2281	1.2241	1.2204
		3.28	17 3	17 3	17 3	18 3	18 3	18 3	18 3	18 3	18 3
			1.1558	1.1534	1.1510	1.1825	1.1795	1.1767	1.2111	1.2076	1.2042

Table 2. Impact of parameter on the optimal values for fixed $\omega = 2.3$

$\omega = 2.3$	ϵ		0.8			0.9			1		
η			2.37	2.4	2.43	2.37	2.4	2.43	2.37	2.4	2.43
τ	ζ	κ									
3.4	0.04	3.12	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	18 4
			1.2080	1.2058	1.2036	1.2341	1.2313	1.2286	1.2626	1.2591	1.2558
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.1969	1.1948	1.1929	1.2207	1.2182	1.2157	1.2467	1.2436	1.2406
		3.28	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.1872	1.1852	1.1834	1.2092	1.2069	1.2047	1.2330	1.2301	1.2274
	0.05	3.12	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2404	1.2377	1.2351	1.2700	1.2667	1.2635	1.3023	1.2982	1.2943
		3.20	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2272	1.2247	1.2223	1.2544	1.2514	1.2486	1.2839	1.2803	1.2767
		3.28	18 3	18 3	18 3	18 4	18 3	18 3	18 4	18 4	18 4
			1.2154	1.2132	1.2110	1.2409	1.2382	1.2355	1.2680	1.2646	1.2615
	0.06	3.12	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2717	1.2685	1.2655	1.3050	1.3012	1.2976	1.3409	1.3362	1.3318
		3.20	18 3	18 3	18 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.2564	1.2536	1.2509	1.2872	1.2837	1.2803	1.3201	1.3160	1.3120
		3.28	18 3	18 3	18 3	18 3	18 3	18 3	18 3	18 3	18 3
			1.2431	1.2405	1.2380	1.2715	1.2683	1.2652	1.3021	1.2982	1.2945

Table *Contd.*

$\omega = 2.3$	ϵ		0.8						0.9						1					
η			2.37		2.4		2.43		2.37		2.4		2.43		2.37		2.4		2.43	
τ	ζ	κ																		
3.5	0.04	3.12	18	4	18	4	18	4	18	4	18	4	18	4	19	4	19	4	18	4
			1.1794		1.1771		1.1749		1.2056		1.2027		1.2000		1.2341		1.2306		1.2273	
		3.20	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4
			1.1682		1.1661		1.1642		1.1921		1.1896		1.1871		1.2182		1.2150		1.2120	
		3.28	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.1585		1.1565		1.1547		1.1805		1.1782		1.1760		1.2044		1.2015		1.1988	
	0.05	3.12	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4
			1.2113		1.2087		1.2061		1.2411		1.2377		1.2345		1.2734		1.2693		1.2654	
		3.20	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.1981		1.1956		1.1932		1.2254		1.2224		1.2196		1.2550		1.2513		1.2478	
		3.28	18	3	18	3	18	3	18	4	18	3	18	3	18	4	18	4	18	4
			1.1864		1.1841		1.1819		1.2119		1.2091		1.2064		1.2390		1.2356		1.2325	
	0.06	3.12	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.2423		1.2391		1.2362		1.2756		1.2718		1.2682		1.3115		1.3069		1.3025	
		3.20	18	3	18	3	18	3	18	3	18	3	18	3	18	4	18	4	18	4
			1.2270		1.2242		1.2215		1.2578		1.2543		1.2509		1.2908		1.2866		1.2826	
		3.28	18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3
			1.2137		1.2111		1.2086		1.2421		1.2389		1.2358		1.2727		1.2688		1.2651	
3.6	0.04	3.12	18	4	18	4	18	4	18	4	18	4	18	4	19	4	19	4	19	4
			1.1527		1.1504		1.1482		1.1790		1.1761		1.1734		1.2076		1.2041		1.2007	
		3.20	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4	18	4
			1.1415		1.1394		1.1375		1.1655		1.1629		1.1604		1.1916		1.1884		1.1854	
		3.28	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.1317		1.1298		1.1279		1.1538		1.1515		1.1493		1.1777		1.1749		1.1722	
	0.05	3.12	18	4	18	4	18	4	18	4	18	4	18	4	19	4	18	4	18	4
			1.1843		1.1817		1.1791		1.2141		1.2108		1.2076		1.2466		1.2425		1.2386	
		3.20	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.1711		1.1686		1.1662		1.1985		1.1954		1.1926		1.2281		1.2244		1.2208	
		3.28	18	3	18	3	18	3	18	4	18	3	18	3	18	4	18	4	18	4
			1.1593		1.1570		1.1549		1.1849		1.1821		1.1794		1.2120		1.2087		1.2055	
	0.06	3.12	18	3	18	3	18	3	18	4	18	4	18	4	18	4	18	4	18	4
			1.2149		1.2118		1.2088		1.2483		1.2445		1.2409		1.2843		1.2797		1.2752	
		3.20	18	3	18	3	18	3	18	3	18	3	18	3	18	4	18	4	18	4
			1.1996		1.1968		1.1941		1.2304		1.2269		1.2236		1.2635		1.2593		1.2553	
		3.28	18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3	18	3
			1.1863		1.1837		1.1813		1.2147		1.2115		1.2085		1.2453		1.2415		1.2378	

Table 3. Impact of parameter on the optimal values for fixed $\omega = 2.35$

ω = 2.35	ϵ		0.8			0.9			1		
η			2.37	2.4	2.43	2.37	2.4	2.43	2.37	2.4	2.43
τ	ζ	κ									
3.4	0.04	3.12	18 4	18 4	18 4	19 4	19 4	18 4	19 5	19 5	19 5
			1.2401	1.2377	1.2353	1.2681	1.2650	1.2621	1.2983	1.2946	1.2911
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	19 4
			1.2279	1.2257	1.2236	1.2535	1.2507	1.2480	1.2813	1.2779	1.2746
		3.28	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2174	1.2154	1.2135	1.2408	1.2383	1.2359	1.2663	1.2632	1.2602
	0.05	3.12	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	19 4
			1.2741	1.2712	1.2684	1.3059	1.3022	1.2988	1.3404	1.3360	1.3318
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	18 4	18 4
			1.2598	1.2572	1.2547	1.2888	1.2856	1.2825	1.3205	1.3165	1.3126
		3.28	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2473	1.2449	1.2425	1.2741	1.2711	1.2683	1.3030	1.2994	1.2960
	0.06	3.12	18 4	18 4	18 3	18 4	18 4	18 4	19 4	19 4	18 4
			1.3072	1.3039	1.3007	1.3426	1.3385	1.3345	1.3810	1.3760	1.3712
		3.20	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2907	1.2877	1.2847	1.3233	1.3196	1.3161	1.3585	1.3539	1.3496
		3.28	18 3	18 3	18 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.2763	1.2735	1.2708	1.3065	1.3031	1.2997	1.3388	1.3347	1.3308
3.5	0.04	3.12	18 4	18 4	18 4	19 4	19 4	19 4	19 5	19 5	19 5
			1.2106	1.2081	1.2057	1.2386	1.2355	1.2325	1.2689	1.2652	1.2616
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	19 4
			1.1983	1.1961	1.1940	1.2239	1.2211	1.2185	1.2518	1.2484	1.2451
		3.28	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.1878	1.1858	1.1839	1.2112	1.2087	1.2063	1.2368	1.2337	1.2307
	0.05	3.12	18 4	18 4	18 4	19 4	18 4	18 4	19 4	19 4	19 4
			1.2441	1.2412	1.2385	1.2759	1.2723	1.2689	1.3106	1.3062	1.3019
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	18 4	18 4
			1.2298	1.2272	1.2247	1.2589	1.2556	1.2525	1.2906	1.2866	1.2827
		3.28	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2173	1.2149	1.2125	1.2441	1.2412	1.2384	1.2731	1.2695	1.2660
	0.06	3.12	18 4	18 3	18 3	18 4	18 4	18 4	19 4	19 4	19 4
			1.2769	1.2736	1.2703	1.3123	1.3082	1.3042	1.3507	1.3457	1.3409
		3.20	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2604	1.2573	1.2543	1.2930	1.2893	1.2857	1.3282	1.3236	1.3193
		3.28	18 3	18 3	18 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.2459	1.2431	1.2404	1.2761	1.2727	1.2694	1.3085	1.3044	1.3004

Table *Contd.*

$\omega = 2.35$	ϵ		0.8			0.9			1		
η			2.37	2.4	2.43	2.37	2.4	2.43	2.37	2.4	2.43
τ	ζ	κ									
3.6	0.04	3.12	18 4	18 4	18 4	19 4	19 4	19 4	19 5	19 5	19 5
			1.1831	1.1806	1.1782	1.2111	1.2080	1.2051	1.2415	1.2378	1.2343
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	19 4
			1.1708	1.1686	1.1664	1.1965	1.1937	1.1910	1.2244	1.2210	1.2177
		3.28	18 4	18 4	18 4	18 4	18 4	18 4	19 4	18 4	18 4
			1.1602	1.1582	1.1563	1.1837	1.1812	1.1788	1.2093	1.2062	1.2032
	0.05	3.12	18 4	18 4	18 4	19 4	19 4	18 4	19 4	19 4	19 4
			1.2163	1.2134	1.2106	1.2482	1.2446	1.2411	1.2829	1.2784	1.2742
		3.20	18 4	18 4	18 4	18 4	18 4	18 4	19 4	19 4	18 4
			1.2020	1.1993	1.1968	1.2311	1.2278	1.2247	1.2628	1.2588	1.2550
		3.28	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.1894	1.1869	1.1846	1.2163	1.2133	1.2105	1.2453	1.2417	1.2382
	0.06	3.12	18 4	18 3	18 3	18 4	18 4	18 4	19 4	19 4	19 4
			1.2487	1.2453	1.2421	1.2841	1.2800	1.2761	1.3226	1.3176	1.3128
		3.20	18 3	18 3	18 3	18 4	18 4	18 4	18 4	18 4	18 4
			1.2321	1.2291	1.2261	1.2648	1.2611	1.2575	1.3000	1.2955	1.2911
		3.28	18 3	18 3	18 3	18 3	18 3	18 3	18 4	18 4	18 4
			1.2176	1.2149	1.2122	1.2479	1.2444	1.2412	1.2803	1.2762	1.2722

From Tables 4, 5, and 6, under fixed parameters $\omega = 2.3$; $\tau = 3.5$; $r = 0.7$; $\zeta = 0.05$; $\kappa = 3.2$; $\epsilon = 0.9$; $\eta = 2.4$ it is observed that the optimal cost value increases with rising values of c_h, c_s, c_o, c_n, c_r and c_{ct}. S^* monotonically increases when c_s, c_o, c_n and c_{ct} increase and decreases monotonically as c_h and c_r increase. s^* increases monotonically with increases in c_o and c_n and monotonically decreases when c_h, c_s, c_r and c_{ct} increase.

Table 4. Effects of costs on optimal values for fixed $c_h = 0.008$

$c_h = 0.008$	c_r		0.142			0.152			0.162		
c_{ct}			0.62	0.65	0.68	0.62	0.65	0.68	0.62	0.65	0.68
c_s	c_o	c_n									
1.15	0.79	0.022	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.1846	1.1853	1.1860	1.1895	1.1902	1.1909	1.1944	1.1951	1.1958
		0.027	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.1861	1.1868	1.1875	1.1910	1.1917	1.1924	1.1958	1.1966	1.1973
		0.032	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.1875	1.1883	1.1890	1.1924	1.1932	1.1939	1.1973	1.1981	1.1988
	0.8	0.022	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.1971	1.1979	1.1986	1.2020	1.2028	1.2035	1.2069	1.2076	1.2084
		0.027	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.1986	1.1993	1.2001	1.2035	1.2042	1.2050	1.2084	1.2091	1.2098
		0.032	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4	19 4
			1.2001	1.2008	1.2016	1.2050	1.2057	1.2064	1.2099	1.2106	1.2113

Table *Contd.*

			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.022	1.2097		1.2104		1.2111		1.2146		1.2153		1.2160		1.2195		1.2202		1.2209	
			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
	0.81	0.027	1.2112		1.2119		1.2126		1.2161		1.2168		1.2175		1.2210		1.2217		1.2224	
			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.032	1.2127		1.2134		1.2141		1.2176		1.2183		1.2190		1.2224		1.2232		1.2239	
			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.022	1.1860		1.1867		1.1875		1.1909		1.1916		1.1924		1.1958		1.1965		1.1973	
			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
	0.79	0.027	1.1875		1.1882		1.1890		1.1924		1.1931		1.1938		1.1973		1.1980		1.1987	
			19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.032	1.1890		1.1897		1.1904		1.1939		1.1946		1.1953		1.1988		1.1995		1.2002	
			19	4	19	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.022	1.1986		1.1993		1.2000		1.2035		1.2042		1.2049		1.2084		1.2091		1.2098	
			19	4	20	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
1.2	0.8	0.027	1.2001		1.2008		1.2015		1.2050		1.2057		1.2064		1.2099		1.2106		1.2113	
			19	4	20	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.032	1.2016		1.2023		1.2030		1.2064		1.2072		1.2079		1.2113		1.2121		1.2128	
			20	4	20	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.022	1.2111		1.2119		1.2126		1.2160		1.2168		1.2175		1.2209		1.2217		1.2224	
			20	4	20	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
	0.81	0.027	1.2126		1.2133		1.2141		1.2175		1.2182		1.2190		1.2224		1.2231		1.2239	
			20	4	20	4	20	4	19	4	19	4	19	4	19	4	19	4	19	4
		0.032	1.2141		1.2148		1.2155		1.2190		1.2197		1.2204		1.2239		1.2246		1.2253	
			20	4	20	4	20	4	20	4	20	4	20	4	19	4	19	4	19	4
		0.022	1.1874		1.1881		1.1888		1.1923		1.1930		1.1938		1.1973		1.1980		1.1987	
			20	4	20	4	20	4	20	4	20	4	20	4	19	4	20	4	20	4
	0.79	0.027	1.1889		1.1896		1.1903		1.1938		1.1945		1.1952		1.1987		1.1995		1.2002	
			20	4	20	4	20	4	20	4	20	4	20	4	19	4	20	4	20	4
		0.032	1.1903		1.1911		1.1918		1.1953		1.1960		1.1967		1.2002		1.2009		1.2017	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
		0.022	1.1999		1.2006		1.2014		1.2049		1.2056		1.2063		1.2098		1.2105		1.2112	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
1.25	0.8	0.027	1.2014		1.2021		1.2028		1.2064		1.2071		1.2078		1.2113		1.2120		1.2127	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
		0.032	1.2029		1.2036		1.2043		1.2078		1.2086		1.2093		1.2128		1.2135		1.2142	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
		0.022	1.2125		1.2132		1.2139		1.2174		1.2181		1.2189		1.2224		1.2231		1.2238	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
	0.81	0.027	1.2140		1.2147		1.2154		1.2189		1.2196		1.2203		1.2238		1.2246		1.2253	
			20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4	20	4
		0.032	1.2154		1.2162		1.2169		1.2204		1.2211		1.2218		1.2253		1.2260		1.2268	

Table 5. Effects of costs on optimal values for fixed $c_h = 0.01$

$c_h = 0.01$	c_r		0.142			0.152			0.162		
c_{ct}			0.62	0.65	0.68	0.62	0.65	0.68	0.62	0.65	0.68
c_s	c_o	c_n									
1.15	0.79	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2012	1.2020	1.2027	1.2061	1.2068	1.2075	1.2109	1.2116	1.2124
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2027	1.2034	1.2042	1.2076	1.2083	1.2090	1.2124	1.2131	1.2139
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2042	1.2049	1.2057	1.2091	1.2098	1.2105	1.2139	1.2146	1.2153
	0.8	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2138	1.2145	1.2153	1.2187	1.2194	1.2201	1.2235	1.2242	1.2249
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2153	1.2160	1.2167	1.2201	1.2209	1.2216	1.2250	1.2257	1.2264
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2168	1.2175	1.2182	1.2216	1.2223	1.2231	1.2265	1.2272	1.2279
	0.81	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2264	1.2271	1.2278	1.2312	1.2320	1.2327	1.2361	1.2368	1.2375
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2279	1.2286	1.2293	1.2327	1.2334	1.2342	1.2376	1.2383	1.2390
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2294	1.2301	1.2308	1.2342	1.2349	1.2356	1.2390	1.2398	1.2405
1.2	0.79	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2028	1.2035	1.2043	1.2077	1.2084	1.2091	1.2125	1.2132	1.2139
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2043	1.2050	1.2057	1.2091	1.2099	1.2106	1.2140	1.2147	1.2154
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2058	1.2065	1.2072	1.2106	1.2113	1.2121	1.2155	1.2162	1.2169
	0.8	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2154	1.2161	1.2168	1.2202	1.2209	1.2217	1.2251	1.2258	1.2265
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2169	1.2176	1.2183	1.2217	1.2224	1.2232	1.2265	1.2273	1.2280
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2184	1.2191	1.2198	1.2232	1.2239	1.2246	1.2280	1.2288	1.2295
	0.81	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2280	1.2287	1.2294	1.2328	1.2335	1.2342	1.2376	1.2384	1.2391
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2294	1.2302	1.2309	1.2343	1.2350	1.2357	1.2391	1.2398	1.2406
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2309	1.2316	1.2324	1.2358	1.2365	1.2372	1.2406	1.2413	1.2421

Table *Contd.*

$c_h = 0.01$		c_r	0.142			0.152			0.162		
c_{ct}			0.62	0.65	0.68	0.62	0.65	0.68	0.62	0.65	0.68
c_s	c_o	c_n									
1.25	0.79	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2044	1.2051	1.2058	1.2092	1.2099	1.2107	1.2141	1.2148	1.2155
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2059	1.2066	1.2073	1.2107	1.2114	1.2122	1.2155	1.2163	1.2170
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2073	1.2081	1.2088	1.2122	1.2129	1.2136	1.2170	1.2178	1.2185
	0.8	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2170	1.2177	1.2184	1.2218	1.2225	1.2232	1.2266	1.2274	1.2281
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2184	1.2192	1.2199	1.2233	1.2240	1.2247	1.2281	1.2288	1.2296
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2199	1.2206	1.2214	1.2248	1.2255	1.2262	1.2296	1.2303	1.2311
	0.81	0.022	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2295	1.2302	1.2310	1.2344	1.2351	1.2358	1.2392	1.2399	1.2407
		0.027	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2310	1.2317	1.2325	1.2358	1.2366	1.2373	1.2407	1.2414	1.2421
		0.032	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4	18 4
			1.2325	1.2332	1.2339	1.2373	1.2381	1.2388	1.2422	1.2429	1.2436

Table 6. Effects of costs on optimal values for fixed $c_h = 0.012$

$c_h = 0.012$		c_r	0.142			0.152			0.162		
c_{ct}			0.62	0.65	0.68	0.62	0.65	0.68	0.62	0.65	0.68
c_s	c_o	c_n									
1.15	0.79	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2170	1.2178	1.2185	1.2218	1.2226	1.2233	1.2266	1.2274	1.2281
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2185	1.2193	1.2200	1.2233	1.2241	1.2248	1.2281	1.2288	1.2296
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2200	1.2207	1.2215	1.2248	1.2255	1.2263	1.2296	1.2303	1.2311
	0.8	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2296	1.2304	1.2311	1.2344	1.2352	1.2359	1.2392	1.2399	1.2407
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2311	1.2318	1.2326	1.2359	1.2366	1.2374	1.2407	1.2414	1.2422
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2326	1.2333	1.2341	1.2374	1.2381	1.2389	1.2422	1.2429	1.2436
	0.81	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2422	1.2429	1.2437	1.2470	1.2477	1.2485	1.2518	1.2525	1.2533
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2437	1.2444	1.2452	1.2485	1.2492	1.2500	1.2533	1.2540	1.2547
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2452	1.2459	1.2466	1.2500	1.2507	1.2514	1.2548	1.2555	1.2562

Table *Contd.*

$c_h = 0.012$		c_r	0.142			0.152			0.162		
c_{ct}			0.62	0.65	0.68	0.62	0.65	0.68	0.62	0.65	0.68
c_s	c_o	c_n									
1.2	0.79	0.022	17 4	17 4	17 3	17 4	17 3	17 3	17 4	17 3	17 3
			1.2188	1.2195	1.2202	1.2236	1.2243	1.2250	1.2283	1.2290	1.2297
		0.027	17 4	17 4	17 3	17 4	17 4	17 3	17 4	17 3	17 3
			1.2202	1.2210	1.2217	1.2250	1.2258	1.2265	1.2298	1.2305	1.2312
		0.032	17 4	17 4	17 3	17 4	17 4	17 3	17 4	17 3	17 3
			1.2217	1.2225	1.2232	1.2265	1.2273	1.2279	1.2313	1.2320	1.2327
	0.8	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2313	1.2321	1.2328	1.2361	1.2369	1.2376	1.2409	1.2417	1.2424
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2328	1.2336	1.2343	1.2376	1.2384	1.2391	1.2424	1.2431	1.2439
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2343	1.2350	1.2358	1.2391	1.2398	1.2406	1.2439	1.2446	1.2454
	0.81	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2439	1.2447	1.2454	1.2487	1.2495	1.2502	1.2535	1.2542	1.2550
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2454	1.2461	1.2469	1.2502	1.2509	1.2517	1.2550	1.2557	1.2565
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2469	1.2476	1.2484	1.2517	1.2524	1.2532	1.2565	1.2572	1.2579
1.25	0.79	0.022	17 3	17 3	17 3	17 3	17 3	17 3	17 3	17 3	17 3
			1.2205	1.2212	1.2219	1.2252	1.2259	1.2266	1.2300	1.2307	1.2314
		0.027	17 4	17 3	17 3	17 3	17 3	17 3	17 3	17 3	17 3
			1.2220	1.2227	1.2233	1.2267	1.2274	1.2281	1.2315	1.2322	1.2329
		0.032	17 4	17 3	17 3	17 3	17 3	17 3	17 3	17 3	17 3
			1.2234	1.2241	1.2248	1.2282	1.2289	1.2296	1.2330	1.2337	1.2344
	0.8	0.022	17 4	17 4	17 3	17 4	17 3	17 3	17 4	17 3	17 3
			1.2331	1.2338	1.2345	1.2378	1.2386	1.2393	1.2426	1.2433	1.2440
		0.027	17 4	17 4	17 3	17 4	17 4	17 3	17 4	17 3	17 3
			1.2345	1.2353	1.2360	1.2393	1.2401	1.2408	1.2441	1.2448	1.2455
		0.032	17 4	17 4	17 3	17 4	17 4	17 3	17 4	17 3	17 3
			1.2360	1.2368	1.2375	1.2408	1.2415	1.2422	1.2456	1.2463	1.2470
	0.81	0.022	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2456	1.2464	1.2471	1.2504	1.2512	1.2519	1.2552	1.2560	1.2567
		0.027	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 3
			1.2471	1.2479	1.2486	1.2519	1.2526	1.2534	1.2567	1.2574	1.2582
		0.032	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4	17 4
			1.2486	1.2493	1.2501	1.2534	1.2541	1.2549	1.2582	1.2589	1.2597

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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