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Dissipative Hyperbolic Geometric Flow on Modified Riemann Extensions

Research Article

H.G. Nagaraja* and Harish D.

Department of Mathematics, Bangalore University, Central College Campus, Bengaluru 560001, India
***Corresponding author:** hgnraj@yahoo.com

Abstract. We study the properties of Modified Riemann extensions evolving under dissipative hyperbolic geometric flow with examples.

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1. Introduction

Dissipative hyperbolic geometric flow was introduced by Dai-Kong-Liu. It is a new geometric tool which was motivated by the well developed theory of the dissipative hyperbolic equations. We define it on general class of pseudo Riemann metrics and study the necessary and sufficient conditions for a Riemann extension under dissipative geometric flow to stay as a Riemann extension. More precisely we study the conditions under which the solutions of Dissipative geometric flow on a Riemann extension are also Riemann extensions.

Patterson and Walker [6] defined Riemann extensions and showed how a Riemannian structure can be given to the $2n$ dimensional tangent bundle of an n-dimensional manifold with given non-Riemannian structure. This shows Riemann extension provides a solution of the general problem of embedding a manifold M carrying a given structure in a manifold M' carrying another structure, the embedding being carried out in such a way that the structure on M' induces in a natural way the given structure on M. The Riemann extension of Riemannian or non-Riemannian spaces can be constructed with the help of the Christoffel coefficients Γ^i_{jk} of corresponding Riemann space or with connection coefficients Π^i_{jk} in the case of the space of affine connection [4]. The theory of Riemann extensions has been extensively studied by Afifi [8]. Modified Riemann extensions introduced recently in [1] and their properties we list briefly in the next section.

2. Preliminaries

Let ∇ be a torsion-free affine connection of M. The modified Riemann extension of (M,∇) is the cotangent bundle T^*M equipped with a metric $\bar{g}$ whose local components given by

$$\bar{g}_{ij} = -2\omega_l \Gamma^l_{ij} + c_{ij}, \;\; \bar{g}_{ij^*} = \delta^j_i \quad \text{and} \quad \bar{g}_{i^*j^*} = 0.$$

The contravariant components are

$$\bar{g}^{ij} = 0, \;\; \bar{g}^{ij^*} = \delta^j_i, \;\; \bar{g}^{i^*j^*} = 0 \quad \text{and} \quad \bar{g}^{i^*j^*} = 2\omega_l \Gamma^l_{ij} - c_{ij}$$

for i, j ranging from 1 to n and i^*, j^* ranging from $n+1$ to $2n$, where ω_l are extended coordinates.

We note following results for the connection coefficients on extended space,

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij},$$

$$\bar{\Gamma}^{k^*}_{ij} = \omega_l R^l_{kji} + \frac{1}{2}(\nabla_i c_{jk} + \nabla_j c_{ik} - \nabla_h c_{ij}),$$

$$\bar{\Gamma}^k_{i^*j} = 0, \; \bar{\Gamma}^{k^*}_{i^*j} = -\Gamma^i_{jk}, \; \bar{\Gamma}^k_{i^*j^*} = 0, \quad \text{and}$$

$$\bar{\Gamma}^{k^*}_{i^*j^*} = 0.$$

The components of the Riemann curvature tensor of the extended space are given by

$$\bar{R}^i_{jkl} = R^i_{jkl},$$

$$\bar{R}^{i^*}_{jkl} = \frac{1}{2}[\nabla_j(\nabla_l c_{ki} - \nabla_i c_{kl}) - \nabla_k(\nabla_l c_{ji} - \nabla_i c_{jl}) - R^m_{jkl} c_{mi} - R^m_{jki} c_{lm}] + \omega_a(\nabla_j R^a_{ilk} - \nabla_k R^a_{ilj}),$$

$$\bar{R}^{i^*}_{j^*kl} = R^j_{ilk},$$

$$\bar{R}^{i^*}_{jk^*l} = -R^k_{ilj}, \quad \text{and}$$

$$\bar{R}^{i^*}_{jkl^*} = R^l_{kji}.$$

The others are zero. i^*, j^*, k^*, l^* ranges from $n+1$ to $2n$. We lower the index in the middle position, to get

$$R_{ijkl} = g_{mk} R^m_{ijl}. \tag{2.1}$$

Further, $\bar{R}_{ij} = R_{ij} + R_{ji}$, $\bar{R}_{i^*j} = 0$ and $\bar{R}_{i^*j^*} = 0$. Here bar is used for components of extended space. However in last section we do not use bar though they are components of modified Riemann extension as the calculations involve only the extended space.

3. Main Results and Discussion

Definition 3.1. Let M be a pseudo Riemannian Manifold with pseudo Riemannian metric g_{ij}. Then the dissipative hyperbolic geometric flow is defined by

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} + 2g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t} - \left(\lambda + 2g^{pq}\frac{\partial g_{pq}}{\partial t}\right)\frac{\partial g_{ij}}{\partial t} + \frac{1}{n-1}\left(\left(g^{pq}\frac{\partial g_{pq}}{\partial t}\right)^2 + \frac{\partial g^{pq}}{\partial t}\frac{\partial g_{pq}}{\partial t}\right) g_{ij} \tag{3.1}$$

where λ is a positive constant.

Theorem 3.2. *Dissipative hyperbolic geometric flow on Riemann extensions is given by*

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} - \lambda \frac{\partial g_{ij}}{\partial t}. \tag{3.2}$$

Proof. In equation (3.1) the expression $g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t}$, $\frac{\partial g_{ip}}{\partial t}$ and $\frac{\partial g_{jq}}{\partial t}$ to be non zero, i,p,j,q must be lesser than n.

But then, $g^{pq} = 0$ and hence the term is equal to zero. On similar lines we can simplify the expression to get the result. □

It may be noted that when $\lambda = 0$, the equation (3.1) reduces to hyperbolic geometric flow[4]. Though the Ricci tensor in equation (3.1) can be arbitrary chosen, to obtain a solution which is a modified Riemann extension, the Ricci tensor has to satisfy the following condition.

Theorem 3.3. *A Riemann extension under dissipative geometric flow to remain as Riemann extension, the Ricci tensor must satisfy the equation,*

$$\frac{\partial^3 R_{\mu\nu}}{\partial t^3} + 3\lambda \frac{\partial^2 R_{\mu\nu}}{\partial t^2} + 2\lambda^2 \frac{\partial R_{\mu\nu}}{\partial t} = 0. \tag{3.3}$$

Proof. From elementary calculations it can be shown that the Ricci tensor for a modified Riemann extension is given by

$$R_{\mu\nu} = \frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(g^{\alpha m}\frac{\partial g_{\mu\nu}}{\partial x^m}\right) - \frac{1}{2}\frac{\partial}{\partial x^\nu}\left(g^{\alpha m}\frac{\partial g_{\mu\alpha}}{\partial x^m}\right) - \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\mu\nu}}{\partial x^m}\frac{\partial g_{\alpha\beta}}{\partial x^n} + \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\mu\beta}}{\partial x^m}\frac{\partial g_{\nu\alpha}}{\partial x^n}. \tag{3.4}$$

Differentiating partially with respect to 't' and simplifying we get

$$\frac{\partial R_{\mu\nu}}{\partial t} = -\frac{1}{2}g^{\alpha m}\frac{\partial^3 g_{\mu\alpha}}{\partial t \partial x^\nu \partial x^m} + \frac{1}{2}g^{\alpha m}\frac{\partial^3 g_{\mu\nu}}{\partial t \partial x^\alpha \partial x^m} - \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\mu\nu}}{\partial x^m}\frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^n} - \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\alpha\beta}}{\partial x^n}\frac{\partial^2 g_{\mu\nu}}{\partial t \partial x^m} + \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\mu\beta}}{\partial x^m}\frac{\partial^2 g_{\nu\alpha}}{\partial t \partial x^n} + \frac{1}{4}g^{\alpha m}g^{\beta n}\frac{\partial g_{\nu\alpha}}{\partial x^n}\frac{\partial^2 g_{\nu\beta}}{\partial t \partial x^m}, \tag{3.5}$$

$$\frac{\partial^2 R_{\mu\nu}}{\partial t^2} = \frac{\lambda}{2}g^{\alpha m}\left[\frac{\partial^3 g_{\mu\alpha}}{\partial x^\nu \partial x^m \partial t} - \frac{\partial^3 g_{\mu\nu}}{\partial x^\alpha \partial x^m \partial t}\right] - \frac{1}{2}g^{\alpha m}g^{\beta n}\frac{\partial^2 g_{\mu\nu}}{\partial t \partial x^m}\frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^n} + \frac{1}{2}g^{\alpha m}g^{\beta n}\frac{\partial^2 g_{\mu\beta}}{\partial t \partial x^m}\frac{\partial^2 g_{\nu\alpha}}{\partial t \partial x^n} + \frac{\lambda}{4}g^{\alpha m}g^{\beta n}\frac{\partial}{\partial t}\left(\frac{\partial g_{\mu\nu}}{\partial x^m}\frac{\partial g_{\alpha\beta}}{\partial x^n}\right) - \frac{\lambda}{4}g^{\alpha m}g^{\beta n}\frac{\partial}{\partial t}\left(\frac{\partial g_{\mu\beta}}{\partial x^m}\frac{\partial g_{\nu\alpha}}{\partial x^n}\right). \tag{3.6}$$

and

$$\begin{aligned}\frac{\partial^3 R_{\mu\nu}}{\partial t^3} = &-\frac{\lambda^2}{2} g^{\alpha m}\frac{\partial^3 g_{\mu\alpha}}{\partial x^\nu \partial x^m \partial t} + \frac{\lambda^2}{2} g^{\alpha m}\frac{\partial^3 g_{\mu\nu}}{\partial x^\alpha \partial x^m \partial t} + \frac{3\lambda}{2} g^{\alpha m} g^{\beta n}\frac{\partial^2 g_{\mu\nu}}{\partial t \partial x^m}\frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^n}\\ &-\frac{\lambda^2}{4} g^{\alpha m} g^{\beta n}\frac{\partial}{\partial t}\left(\frac{\partial g_{\alpha\beta}}{\partial x^n}\frac{\partial g_{\mu\nu}}{\partial x^m}\right) - \frac{3\lambda}{2} g^{\alpha m} g^{\beta n}\frac{\partial^2 g_{\mu\beta}}{\partial t \partial x^m}\frac{\partial^2 g_{\nu\alpha}}{\partial t \partial x^n}\\ &+\frac{\lambda^2}{4} g^{\alpha m} g^{\beta n}\frac{\partial}{\partial t}\left(\frac{\partial g_{\nu\alpha}}{\partial x^n}\frac{\partial g_{\mu\beta}}{\partial x^m}\right).\end{aligned} \tag{3.7}$$

Clearly,

$$\frac{\partial^3 R_{\mu\nu}}{\partial t^3} - \lambda^2 \frac{\partial R_{\mu\nu}}{\partial t} = \frac{3\lambda}{2} g^{\alpha m} g^{\beta n}\left(\frac{\partial^2 g_{\mu\nu}}{\partial t \partial x^m}\frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^n} - \frac{\partial^2 g_{\mu\beta}}{\partial t \partial x^m}\frac{\partial^2 g_{\nu\alpha}}{\partial t \partial x^n}\right) \tag{3.8}$$

and

$$\frac{\partial^2 R_{\mu\nu}}{\partial t^2} + \lambda \frac{\partial R_{\mu\nu}}{\partial t} = \frac{1}{2} g^{\alpha m} g^{\beta n}\left[\frac{\partial^2 g_{\mu\beta}}{\partial t \partial x^m}\frac{\partial^2 g_{\nu\alpha}}{\partial t \partial x^n} - \frac{\partial^2 g_{\mu\nu}}{\partial t \partial x^m}\frac{\partial^2 g_{\alpha\beta}}{\partial t \partial x^n}\right]. \tag{3.9}$$

From equations (3.8) and (3.9) we get

$$\frac{\dfrac{\partial^3 R_{\mu\nu}}{\partial t^3} - \lambda^2 \dfrac{\partial R_{\mu\nu}}{\partial t}}{\dfrac{\partial^2 R_{\mu\nu}}{\partial t^2} + \lambda \dfrac{\partial R_{\mu\nu}}{\partial t}} = -3\lambda\ . \tag{3.10}$$

Hence

$$\frac{\partial^3 R_{\mu\nu}}{\partial t^3} + 3\lambda \frac{\partial^2 R_{\mu\nu}}{\partial t^2} + 2\lambda^2 \frac{\partial R_{\mu\nu}}{\partial t} = 0\ . \tag{3.11}$$

□

Solving equation (3.11) we get

$$R_{\mu\nu} = A_{\mu\nu} e^{-2\lambda t} + B_{\mu\nu} e^{-\lambda t} + C_{\mu\nu} \tag{3.12}$$

where $A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu}$ are arbitrary tensor components for $\mu,\nu \le n$ and zero otherwise. Thus we have obtained the general form, the Ricci tensor must have for a modified Riemann extension under dissipative hyperbolic geometric flow to stay as modified Riemann extension.

Using this we now obtain the solution to equation (3.2).

Theorem 3.4. *A modified Riemann extension under dissipative hyperbolic geometric flow staying as modified Riemann extension, the metric is given by,*

$$\begin{aligned} g_{\mu\nu} = &\frac{2A_{\mu\nu}}{\lambda^2}\left(e^{-\lambda t} + \lambda t e^{-\lambda t} - 1\right) + \frac{2B_{\mu\nu}}{\lambda^2}\left(e^{-\lambda t} - \frac{1}{2}e^{-2\lambda t} - \frac{1}{2}\right)\\ &-(e^{-\lambda t} - 1)\left(\frac{2C_{\mu\nu}}{\lambda} + \frac{1}{\lambda}\frac{\partial}{\partial t} g_{\mu\nu}(0)\right) + g_{\mu\nu}(0). \end{aligned} \tag{3.13}$$

Proof. Substituting equation (3.12) in (3.2) we get,

$$\frac{\partial^2}{\partial t^2} g_{\mu\nu} + \lambda \frac{\partial g_{\mu\nu}}{\partial t} + 2A_{\mu\nu} e^{-\lambda t} + 2B_{\mu\nu} e^{-2\lambda t} + 2C_{\mu\nu} = 0. \tag{3.14}$$

Integrating we get,

$$\frac{\partial g_{\mu\nu}}{\partial t} + \lambda g_{\mu\nu} = 2\frac{A_{\mu\nu} e^{-\lambda t}}{\lambda} + \frac{B_{\mu\nu} t e^{-2\lambda t}}{\lambda} - 2C_{\mu\nu} t - F_{\mu\nu} \tag{3.15}$$

which is a first order differential equation solving and substituting the initial conditions we get the required result. □

4. Conclusion

Thus we have obtained the general solution for dissipative hyperbolic geometric flow on modified Riemann extension. It must be noted that for dissipative hyperbolic geometric flow on pseudo Riemann metric, the existence and uniqueness of solution must be proved first which is a difficult problem. But restricting the flow to modified Riemann extensions, we have solved the problem.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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