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Research Article

A New Robust Application for Singularly Perturbed Volterra-Integro Differential Equations

Derya Arslan

Department of Mathematics, Faculty of Art and Science, Bitlis Eren University, 13200, Bitlis, Turkey
ayredlanu@gmail.com

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Abstract. In this study, it is aimed to approximately solve the singularly perturbed Volterra-integro differential equation by the Adomian decomposition method. The solution procedure is easy and fast. Firstly, the equation is written in operator form. Then the integral operator is applied to all sides of the equation. The series solution is obtained by applying some operations to the given equation and then converting it into a recurrence relation. Error values show that the solution results obtained for the two applied examples are very close to each other. The proposed method gives successful results with 21 and 23 iterations.

Keywords. Integro differential equation, Singularly perturbed equation, ADM (Adomian Decomposition Method), Error evaluation, Series solution

Mathematics Subject Classification (2020). 34K28, 45D05, 65L05, 65R20

1. Introduction

In this paper, it is considered the following singularly perturbed Volterra-integro-differential equation and its initial conditions:

$$\varepsilon y''(t)+a(t)y'(t)+\int_0^t H(t,s)y(s)ds=h(t), \quad 0<t<\ell, \tag{1.1}$$

$$y(0)=A, \tag{1.2}$$

$$y'(0) = \frac{B}{\varepsilon}, \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is a very small perturbation parameter; A and B are constants; $h(t)$, $a(t) > 0$ and $H(t,s)$ are assumed to be sufficiently smooth functions.

Integro-differential equations have many mathematical formulas in the natural sciences. These equations whose integral upper bound is accepted as variable are called Volterra integro-differential equations. Volterra integro-differential equations are seen in many application areas of science and engineering. For example, physics, chemistry, biology, fluid dynamics, atomic physics, diffusion, population and epidemic dynamics, glucose tolerance etc. (Burton [8], Jerri [20], Kauthen [21], Kythe and Puri [22], Lodge *et al*. [23], Ramos [31], and Salama and Bakr [35]). In most cases, most of these problems are solved by approximate methods, since it is difficult to obtain exact solutions by analytical methods. For this reason, there are different approaches preferred to solve Volterra integro-differential equations in the literature: Piecewise-quasilinearization method (Ramos [32]), exponential technique and implicit Runge-Kutta method (Ramos [31]), coupled method (Tao and Zhang [36]), finite difference method (Cimen [14], Mbroh *et al*. [28], Şevgin [34], and Yapman and Amiraliyev [37]) and differential transform method (Celik and Tabatabaei [11]), numerical integration method (Arslan [6]). Existence and uniqueness studies of the solutions of Volterra integro-differential equations are also included in the literature, see, Cakir and Arslan [8], Jerri [20], Kythe and Puri [22], Nefedov *et al*. [29], and Roos *et al*. [33]. When a positive ε parameter is multiplied by the highest order derivative, the equation is called a singularly perturbed equation. Singularly perturbed equations and its problems are involved in many fields such as neurobiology, mathematical biology, fluids and population dynamics, heat transport problems, nanofluids, viscoelasticity, simultaneous control systems etc. (Doolan *et al*. [15], Farrell *et al*. [17], and O'Malley [30]). The ε perturbation parameter creates unstability in the solution of the problem. For this reason, numerical methods are used to avoid unstable solutions. One of the most convenient and widely used methods is the finite difference method (Arslan [5], Cakir *et al*. [9], Mbroh *et al*. [28], Şevgin [34], and Yapman and Amiraliyev [37]). In this study, it is aimed to approximately solve singularly perturbed Volterra integro-differential equation with the Adomian decomposition method, which is at least as reliable as these methods. The solution procedure in the Adomian decomposition method is easy and fast. The series solution is obtained by applying some operations to the given equation and then converting it into a recurrence relation.

The motivation of this article is to present an accurate and reliable approach to the approximate solution of the Volterra integro-differential problem. Because our problem is a singularly perturbed with a very small parameter (Amiraliyev and Amirali [4], Doolan *et al*. [15], Farrell *et al*. [17], [30], and O'Malley [33]). This parameter creates the boundary layers in the problem. So, the behavior of the solution changes abruptly and rapidly. This state produces unlimited derivatives in solving singularly perturbed problems. In addition, the fact that the problem contains integral terms makes it more difficult to reach an analytical solution. Many classical analytical and numerical methods that have been and are being applied so far cannot solve this problem. For this reason, the Adomian decomposition method, which gives stable solutions for the ε parameter is used in the study.

This study will proceed as follows: In Section 2, how the Adomian decomposition method works will be given. Then, an application will be made on the sample problem of the proposed method. The approximate results obtained will be compared with the actual results and presented with tables and graphs in the same section.

2. Adomian Decomposition Method and Its Application

In the 1980s, George Adomian developed the Adomian decomposition method, named after him, for the solution of nonlinear functional equations (Yapman and Amiraliyev [37]). The Adomian decomposition method gives analytical solution results in terms of infinite power series obtained very simply by an important formula applied to a wide class of linear and nonlinear equations (Adomian *et al*. [1–3]). The difficulty, however, is in proving the convergence of the series of functions. The studies of the Adomian decomposition method continued with many other authors, see, Cakir and Arslan [10], Malaikah [24], Maturi [25, 26], and Maturi and Malaikah [27]. Also, Cherruault and Adomian [13], and Cherruault [12] studied to convergence rate of the Adomian decomposition method. El-Kalla [16] offered a different view regarding the error analysis of the ADM. The Adomian Decomposition method is defined as follows (Adomian and Rach [3], Adomian [1, 2], and O'Malley [30]):

$$Fy(t) = h(t),$$
$$Ly + Ry + Ny = h(t),$$
$$Ly = h(t) - Ry - Ny,$$

where L is invertible (L^{-1}) and differential operator. For example, if L is a third-order operator, then L^{-1} is a three-fold integration. If the differential equation is n-order

$$L(\cdot) = \frac{d^n(\cdot)}{dt^n}$$

and

$$L^{-1}(\cdot) = \int_0^t \cdots \int_0^t (\cdot)dt \cdots dt, \quad (n\text{-times}).$$

R is the reminder term and linear operator. Ny is nonlinear term, it is defined as

$$Ny = \sum_{n=0}^{\infty} A_n, \tag{2.1}$$

where A_n are Adomian polynomials of $y_0, y_1, y_2, \dots, y_n$. They are calculated the following formula:

$$An = \frac{1}{n!}\frac{d^n}{d\gamma^n}\left[\sum \gamma^j y_j\right]_{\gamma=0}, \quad j = 0, 1, 2, 3, \dots.$$

If it is applied the L^{-1} operator, which is the inverse of the L, to equation (1.1), it has:

$$L^{-1}y = L^{-1}h - L^{-1}Ry.$$

It is obtained series solution of given the differential equation.

$$y = y(0) + L^{-1}h - L^{-1}Ry - L^{-1}Ny, \tag{2.2}$$

According to definition of Adomian decomposition method, eq. (2.2) solution is represented as infinite sum of series.

$$y=\sum_{n=0}^{\infty} y_n = y_0+y_1+y_2+\dots. \tag{2.3}$$

If the equations (2.1) and (2.3) are written in equation (2.2), the following expression is easily obtained:

$$\sum_{n=0}^{\infty} y_n = y(0)+L^{-1}h-L^{-1}R\sum_{n=0}^{\infty} y_n - L^{-1}\sum_{n=0}^{\infty} A_n,$$

Each term of infinite sum of series (2.3) is given by the following recurrence relation:

$$y_0 = y(0)+L^{-1}h,$$

$$y_{n+1} = -L^{-1}Ry_n - L^{-1}Ny_n, \quad n=0,1,2,\dots,$$

where y_1 is obtained by using y_0.

Let us examine the following example to demonstrate the correctness of the theory:

$$\varepsilon y''(t)+2y'(t)+\int_0^t y(s)ds = \frac{t}{2}-\frac{\varepsilon}{4}(1-e^{\frac{-t}{\varepsilon}}), \quad 0<t<1, \tag{2.4}$$

$$y(0)=0, \tag{2.5}$$

$$y'(0)=\frac{1}{\varepsilon}, \tag{2.6}$$

$$Ly=\frac{t}{2}-\frac{\varepsilon}{4}(1-e^{\frac{-t}{\varepsilon}})+2y'(t)+\int_0^t y(s)ds, \tag{2.7}$$

where

$$L(\cdot)=\frac{d^2(\cdot)}{dt^2}.$$

Thus,

$$L^{-1}(\cdot)=\int_0^t\int_0^t(\cdot)dt\,dt.$$

By applying L^{-1} on the both sides of equation (2.4) and using (2.5)-(2.6), it is obtained as

$$y=\frac{t}{\varepsilon}-\frac{1}{\varepsilon}\left[\int_0^t\int_0^t\left(\frac{t}{2}-\frac{\varepsilon}{4}(1-e^{\frac{-t}{\varepsilon}})\right)dt\,dt-\int_0^t\int_0^t 2y'(t)dt\,dt-\int_0^t\int_0^t\left(\int_0^t y(s)ds\right)dt\,dt\right].$$

Thus, the following expression is found:

$$y_0=\frac{t}{\varepsilon}-\frac{1}{\varepsilon}\left[\int_0^t\int_0^t\left(\frac{t}{2}-\frac{\varepsilon}{4}(1-e^{\frac{-t}{\varepsilon}})\right]dt\,dt\right],$$

$$y_1=\frac{1}{\varepsilon}\left[-\int_0^t\int_0^t 2y_0'(t)dt\,dt-\int_0^t\int_0^t\left(\int_0^t y_0(s)ds\right)\right],$$

$$y_2=\frac{1}{\varepsilon}\left[-\int_0^t\int_0^t 2y_1'(t)dt\,dt-\int_0^t\int_0^t\left(\int_0^t y_1(s)ds\right)\right],$$

$$y_{n+1}=\frac{1}{\varepsilon}\left[-\int_0^t\int_0^t 2y_n'(t)dt\,dt-\int_0^t\int_0^t\left(\int_0^t y_n(s)ds\right)\right].$$

Consequently, the first few components are as follows for $n=0,1,2,\dots$ and $\varepsilon=0.9$:

$$y_0=-0.6249987500+1.125000875t-0.1250000000t^2+\dots,$$

$$y_1 = -0.07031234375 + 0.1406248285t - 1.140626969t^2 + \dots,$$
$$y_2 = -0.07910136914 + 0.7910136914t - 0.1582030547t^2 + \dots.$$

Hence, the solution in a series form is given by

$$y_{app}(t) = 3.203064231 \cdot \exp(-2.000002000 \cdot t) + 7.406135872 \cdot t - 4.023000367 \cdot 10^{-15} \cdot t^{22} + 3.498264688 \cdot 10^{-16} \cdot t^{23} + 1.764974059 \cdot 10^{-24} \cdot t^{29} - \dots.$$

According to Table 1, the exact and the approximate solution values were found for $\varepsilon = 0.9$ and $t = 0,\dots,1$ using the Adomian decomposition method. Error values showed that these values were very close to each other. It was seen that the proposed method gave successful results with 23 iterations.

Table 1. The computed Exact solutions, Approximate solutions and Errors for $\varepsilon = 0.9$

	Exact solution	ADM solution	NIM error	ADM error
$t = 0.0$	0.0000000000	0.000000000	0.00000000000	0.000000000
$t = 0.1$	0.0906347054	0.090634705	0.63918408e-2	0.000000004
$t = 0.2$	0.1648401110	0.164840112	0.106908703e-1	0.000000001
$t = 0.3$	0.2255943466	0.225594347	0.131706503e-1	0.000000004
$t = 0.4$	0.2753356976	0.275335698	0.141025156e-1	0.000000004
$t = 0.5$	0.3160604634	0.316060466	0.137438382e-1	0.000000026
$t = 0.6$	0.3494030748	0.349403076	0.123311100e-1	0.000000012
$t = 0.7$	0.3767016906	0.376701692	0.100764130e-1	0.000000014
$t = 0.8$	0.3990519025	0.399051906	0.71662090e-2	0.000000035
$t = 0.9$	0.4173507046	0.417350707	0.37616717e-2	0.000000024
$t = 1.0$	0.4323324937	0.432332494	0.00000000000	0.000000007

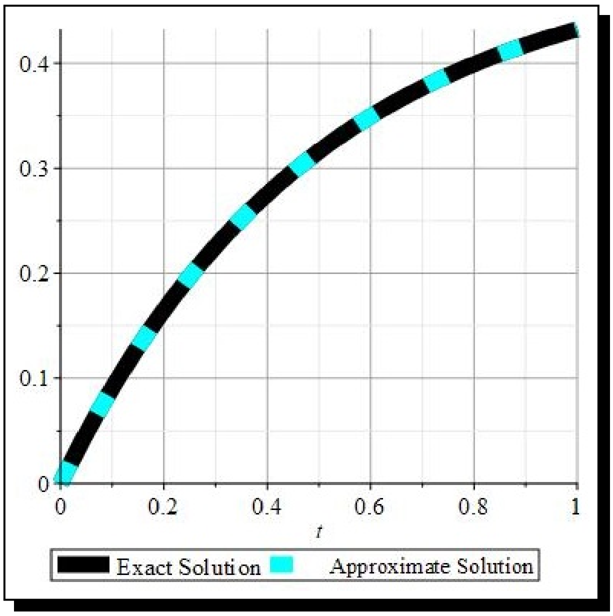

Figure 1. The comparison of Exact solutions and Approximate solutions for $\varepsilon = 0.9$

If it is taken $\varepsilon = 0.2$, the approximate solution is as follows:

$$y_0 = -0.002500000000 - 5.025000000t - 0.1250000000t^2 + \dots,$$
$$y_1 = -0.002512500000 + 0.02512500000t - 25.12562500t^2 + \dots,$$
$$y_2 = -0.002525062500 + 0.02525062500t - 0.1262531250t^2 + \dots.$$

Hence, the solution in a series form is given by

$$y_{app}(t) = 5.442432534 \cdot t - 8.073458743 \cdot 10^{-22} \cdot t^{34} - 260.1198297 \cdot t^{16} + 153.0116646 \cdot t^{17} - \dots.$$

According to Table 2, the exact and approximate solution results were obtained for $\varepsilon = 0.2$ and $t = 0, \dots, 0.5$ by Adomian decomposition method. The error results were found to be quite small. It was revealed that the method used in the study showed successful results with 21 iterations.

Table 2. The computed Exact solutions, Approximate solutions and Errors for $\varepsilon = 0.2$

	Exact solution	ADM solution	NIM error	ADM error
$t = 0.0$	0.0000000000	0.0000000000	0.000000000	0.000000000
$t = 0.1$	0.3160602794	0.3160602794	17.90945516	0.000000000
$t = 0.2$	0.4323323584	0.4323323585	19.65123356	0.000000003
$t = 0.3$	0.4751064658	0.4751064925	16.12887487	0.000000002
$t = 0.4$	0.4908421806	0.4908466215	7.854359720	0.000000444
$t = 0.5$	0.4966310265	0.4968676835	4.947451849	0.000236657

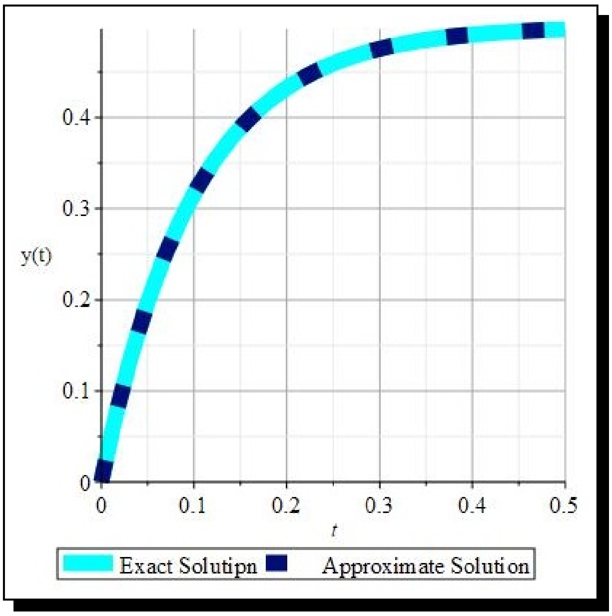

Figure 2. The comparison of Exact solutions and Approximate solutions for $\varepsilon = 0.2$

Conclusion

The boundary value problem of singularly perturbed Volterra integro-differential equation is solved by Adomian decomposition method. Since there are boundary layers at $t = 0$ and $t = 1$

points of the problem, the solution has changed abruptly and rapidly in the neighborhoods of these points, and therefore the solution curve leans towards the axes. It is seen that the errors are minimum for different values of $\varepsilon = 2^{-1}$, 2^{-5}, 2^{-10} and $t = 0, 0.1, 0.2, \dots, 1$ on the tables. Approximate, exact solution and error values are compared in Figures 1 and 2. According to these results, the method is stable, reliable and useful. In order to contribute to the literature, it can be said that the Adomian decomposition method can be applied to the fuzzy and fractional types of integral equations.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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