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Research Article

New Application of Libera Integral Operator

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Abstract. In this paper, we introduce the new subclass $A_j(n,\alpha,\beta)$ of $A(n)$ using $Lf(z)$. Some interesting properties of functions $f(z)\in A(n)$ are studied with illustrative examples.

Keywords. Analytic function, Libera integral operator, α-spirallike functions of order β, Subordination, Convex

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1. Introduction

Let $A(n)$ denote the class of analytic functions $f(z)$ of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k, \quad n\in\mathbb{N}=\{2,3,4,\dots\} \tag{1.1}$$

in the open unit disk $U=\{z\in C:|z|<1\}$. For $f(z)\in A(n)$, Libera [2] defined the Libera integral operator $Lf(z)$ given by

$$Lf(z)=\frac{2}{z}\int_0^z f(t)dt. \tag{1.2}$$

Using the above Libera integral operator $Lf(z)$, we introduce the following iterative forms:

$$L_0f(z)=f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k, \tag{1.3}$$

$$L_1f(z)=Lf(z)=\frac{2}{z}\int_0^z f(t)dt=z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)a_kz^k, \tag{1.4}$$

$$L_2f(z)=L(Lf(z))=z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^2 a_k z^k \tag{1.5}$$

and

$$L_jf(z)=L(L_{j-1}f(z))=z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j a_k z^k \tag{1.6}$$

for $j=1,2,3,\ldots$.

With the above integral operator $L_jf(z)$, we say that $f(z)\in A_j(n,\alpha,\beta)$ if $f(z)\in A(n)$ satisfies

$$Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}\right)>\beta\cos,\alpha \quad z\in U, \tag{1.7}$$

for some real α $\left(|\alpha|<\frac{\pi}{2}\right)$ and β $(0\le\beta<1)$.

Noting that

$$\frac{L_{j-1}f(z)}{L_jf(z)}=\frac{1}{2}\left(1+\frac{z\left(L_jf(z)\right)'}{L_jf(z)}\right), \quad j\in\mathbb{N} \tag{1.8}$$

we know that

$$Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}\right)=Re\left\{\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_jf(z))'}{L_jf(z)}\right)\right\}>\beta\cos\alpha, \quad z\in U, \tag{1.9}$$

for $f(z)\in A_j(n,\alpha,\beta)$. Therefore, if we consider a function $f(z)\in A(n)$ given by

$$L_jf(z)=z(1-z^n)^t \tag{1.10}$$

with

$$t=-\frac{4}{n}(1-\beta)e^{-i\alpha}\cos\alpha, \tag{1.11}$$

then we have that

$$\begin{aligned} Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}\right)&=Re\left\{\frac{e^{i\alpha}}{2}\left(2-tn\frac{z^n}{1-z^n}\right)\right\} \\ &=\cos\alpha+2(1-\beta)\cos\alpha\, Re\left(\frac{z^n}{1-z^n}\right) \\ &>\cos\alpha-(1-\beta)\cos\alpha \\ &=\beta\cos\alpha, \quad z\in U. \end{aligned} \tag{1.12}$$

Thus, we say that $f(z)\in A_j(n,\alpha,\beta)$ for $f(z)\in A(n)$ given by (1.10).

Remark 1.1. Recently, Guney and Owa[1] considered the subclass $S(n,\alpha,\beta)$ and $C(n,\alpha,\beta)$ of $A(n)$ consisting of $f(z)$ which satisfies

$$Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right)>\beta\cos\alpha, \quad z\in U \tag{1.13}$$

and

$$Re\left(e^{i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>\beta\cos\alpha, \quad z\in U \tag{1.14}$$

respectively, for some real α $\left(|\alpha|<\frac{\pi}{2}\right)$ and β $(0\le\beta<1)$.
A function $f(z)$ in the class $S(n,\alpha,\beta)$ is said to be α-spirallike of order β in U.

[1] H. O. Güney and S. Owa, Some properties for subordinations of α-spirallike functions of order β, *Applied Mathematics E-Notes* (accepted for publication).

2. Conditions for the Class $A_j(n,\alpha,\beta)$

Theorem 2.1. *If $f(z)\in A(n)$ satisfies*

$$\left|\frac{z(L_jf(z))'}{L_jf(z)}-1\right|<2(1-\beta)\cos\alpha, \quad z\in U, \tag{2.1}$$

for some real α $\left(|\alpha|<\frac{\pi}{2}\right)$ and β $(0\le\beta<1)$, then the subclass $f(z)\in A_j(n,\alpha,\beta)$.

Proof. Let us consider a function $w(z)$ given by

$$\frac{z(L_jf(z))'}{L_jf(z)}=1+2(1-\beta)\cos\alpha w(z), \tag{2.2}$$

for $f(z)$ which satisfies (2.1). Then $w(z)$ is analytic in U, $w(0)=0$, and $|w(z)|<1$ $(z\in U)$. For such $w(z)$, we know that

$$\frac{L_{j-1}f(z)}{L_jf(z)}=\frac{1}{2}\left(1+\frac{z(L_jf(z))'}{L_jf(z)}\right)=1+(1-\beta)\cos\alpha w(z) \tag{2.3}$$

and

$$\begin{aligned} Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}\right)&=\cos\alpha+(1-\beta)\cos\alpha Re(e^{i\alpha}w(z))\\ &\ge\cos\alpha-(1-\beta)\cos\alpha|e^{i\alpha}w(z)|\\ &>\beta\cos\alpha, \quad z\in U. \end{aligned} \tag{2.4}$$

This gives us that $f(z)\in A_j(n,\alpha,\beta)$. □

Next, we consider the following theorem.

Theorem 2.2. *If $f(z)\in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty}\left[\frac{2}{k+1}\right]^j\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k|\le1, \tag{2.5}$$

for some real α $\left(|\alpha|<\frac{\pi}{2}\right)$ and β $(0\le\beta<1)$, then the subclass $f(z)\in A_j(n,\alpha,\beta)$.

Proof. For $f(z)\in A(n)$, we know that

$$\left|\frac{z(L_jf(z))'}{L_jf(z)}-1\right|=\left|\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)a_kz^k}{1+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j a_kz^k}\right|<\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)|a_k|}{1-\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j|a_k|}. \tag{2.6}$$

Therefore, using (2.1), if $f(z)$ satisfies

$$\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)|a_k|}{1-\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j|a_k|}\le2(1-\beta)\cos\alpha, \tag{2.7}$$

then $f(z)\in A_j(n,\alpha,\beta)$. Further, we say that the inequality (2.7) is the same as (2.5).

We consider a function $f(z)\in A(n)$ given by

$$f(z)=z+\sum_{k=n+1}^{\infty}A_kz^k \tag{2.8}$$

with

$$A_k=\frac{(n+1)e^{i\theta}}{\left(\frac{2}{k+1}\right)^j\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)k(k+1)}, \quad 0\le\theta\le2\pi. \tag{2.9}$$

For such $f(z)$, we have

$$\sum_{k=n+1}^{\infty}\left[\frac{2}{k+1}\right]^{j}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|A_k| = \sum_{k=n+1}^{\infty}\frac{n+1}{k(k+1)}$$
$$= (n+1)\sum_{k=n+1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1. \tag{2.10}$$

Therefore, a function $f(z)$, given by (2.8), satisfies the equality in (2.5). □

3. Coefficient Problems for the Class $A_j(n,\alpha,\beta)$

We consider coefficient problems of $f(z) \in A(n)$ for the class $A_j(n,\alpha,\beta)$. To consider the above problems, we need the following lemma by Carathéodory [1].

Lemma 3.1. *If $p(z)$ given by*

$$p(z) = 1+\sum_{k=1}^{\infty} c_k z^k \tag{3.1}$$

is analytic in U and $Re\, p(z) > \alpha$ $(z \in U)$ for $0 \le \alpha < 1$, then

$$|c_k| \le 2(1-\alpha), \quad k = 1,2,3,\dots. \tag{3.2}$$

The equality in (3.2) *is given by*

$$p(z) = \frac{1+(1-2\alpha)z}{1-z}. \tag{3.3}$$

With Lemma 3.1, we have

Lemma 3.2. *If $p(z)$ given by*

$$p(z) = 1+\sum_{k=n}^{\infty} c_k z^k, \quad n = 1,2,3,\dots \tag{3.4}$$

is analytic in U and $Re\, p(z) > \alpha$, $z \in U$ for $0 \le \alpha < 1$, then

$$|c_k| \le 2(1-\alpha), \quad k = n, n+1, n+2, \dots. \tag{3.5}$$

The equality in (3.5) *is given by*

$$p(z) = \frac{1+(1-2\alpha)z^n}{1-z^n}. \tag{3.6}$$

Using Lemma 3.2, we show

Theorem 3.1. *If $f(z) \in A_j(n,\alpha,\beta)$, then*

$$|a_{n+1}| \le \frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\left(\frac{n+2}{2}\right)^{j} \tag{3.7}$$

and

$$|a_k| \le \frac{4|e^{i\alpha}-\beta\cos\alpha|}{k-1}\left(\frac{k+1}{2}\right)^{j}\prod_{l=n}^{k-2}\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{l}\right), \quad k = n+2, n+3, n+4, \dots. \tag{3.8}$$

Inequalities (3.7) *and* (3.8) *hold for function $f(z) \in A(n)$ such that*

$$L_j f(z) = \frac{z}{(1-z^{n-1})^t} \tag{3.9}$$

with

$$t = \frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1}. \tag{3.10}$$

Proof. We define a function $p(z)$ by

$$p(z) = \frac{e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)} - \beta\cos\alpha}{e^{i\alpha} - \beta\cos\alpha} \tag{3.11}$$

for $f(z) \in A_j(n,\alpha,\beta)$. Then $p(z)$ is analytic in U, $Re\,p(z) > 0$ $(z \in U)$, and can be written by

$$p(z) = 1 + \sum_{k=n+1}^{\infty} c_k z^{k-1}. \tag{3.12}$$

It follows from (3.11) that

$$e^{i\alpha}L_{j-1}f(z) - \beta\cos\alpha L_jf(z) = (e^{i\alpha} - \beta\cos\alpha)p(z)L_jf(z). \tag{3.13}$$

We also note that

$$e^{i\alpha}L_{j-1}f(z) - \beta\cos\alpha L_jf(z) = (e^{i\alpha} - \beta\cos\alpha)z + \sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j\left(\frac{k+1}{2}e^{i\alpha} - \beta\cos\alpha\right)a_k z^k \tag{3.14}$$

and

$$(e^{i\alpha} - \beta\cos\alpha)p(z)L_j(z) = (e^{i\alpha} - \beta\cos\alpha)\left(1 + \sum_{k=n+1}^{\infty} c_k z^{k-1}\right)\left(z + \sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j a_k z^k\right). \tag{3.15}$$

Let us consider the coefficients of z^k in analytic in (3.14) and (3.15). Then, we have that

$$\begin{aligned}\left(\frac{2}{k+1}\right)^j\left(\frac{k-1}{2}\right)a_k = (e^{i\alpha} - \beta\cos\alpha)\Bigg\{ & c_k + \left(\frac{2}{n+2}\right)^j c_{k-n}a_{n+1} + \left(\frac{2}{n+3}\right)^j c_{k-n-1}a_{n+2} \\ & + \left(\frac{2}{n+4}\right)^j c_{k-n-2}a_{n+3} + \ldots + \left(\frac{2}{k-n}\right)^j c_{n+2}a_{k-n-1} \\ & + \left(\frac{2}{k-n+1}\right)^j c_{n+1}a_{k-n}\Bigg\}.\end{aligned} \tag{3.16}$$

Since

$$|c_k| \le 2, \quad k = n+1, n+2, n+3, \ldots \tag{3.17}$$

by Lemma 3.2, we obtain that

$$\begin{aligned}\left(\frac{2}{k+1}\right)^j\left(\frac{k-1}{2}\right)|a_k| \le 2|e^{i\alpha} - \beta\cos\alpha|\Bigg\{ & 1 + \left(\frac{2}{n+2}\right)^j |a_{n+1}| + \left(\frac{2}{n+3}\right)^j |a_{n+2}| \\ & + \left(\frac{2}{n+4}\right)^j |a_{n+3}| + \ldots + \left(\frac{2}{k-n}\right)^j |a_{k-n-1}| + \left(\frac{2}{k-n-1}\right)^j |a_{k-n}|\Bigg\}.\end{aligned} \tag{3.18}$$

If $k = n+1$, then

$$\left(\frac{2}{n+2}\right)^j\left(\frac{n}{2}\right)|a_{n+1}| \le 2|e^{i\alpha} - \beta\cos\alpha| \tag{3.19}$$

or

$$|a_{n+1}| \le \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n}\left(\frac{n+2}{2}\right)^j. \tag{3.20}$$

If $k=n+2$, then

$$\left(\frac{2}{n+3}\right)^j\left(\frac{n+1}{2}\right)|a_{n+2}|\le 2|e^{i\alpha}-\beta\cos\alpha|\left\{1+\left(\frac{2}{n+2}\right)^j|a_{n+1}|\right\}$$

and

$$\begin{aligned}|a_{n+2}| &\le \frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+1}\left(\frac{n+3}{2}\right)^j\left\{1+\left(\frac{2}{n+2}\right)^j\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\left(\frac{n+2}{2}\right)^j\right\}\\ &= \frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+1}\left(\frac{n+3}{2}\right)^j\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\right).\end{aligned} \tag{3.21}$$

Further, if $k=n+3$, then

$$\left(\frac{2}{n+4}\right)^j\left(\frac{n+2}{2}\right)|a_{n+3}|\le 2|e^{i\alpha}-\beta\cos\alpha|\left\{1+\left(\frac{2}{n+2}\right)^j|a_{n+1}|+\left(\frac{2}{n+3}\right)^j|a_{n+2}|\right\} \tag{3.22}$$

and

$$\begin{aligned}|a_{n+3}| &\le \frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+2}\left(\frac{n+4}{2}\right)^j\left\{1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+1}\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\right)\right\}\\ &= \frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+2}\left(\frac{n+4}{2}\right)^j\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\right)\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n+1}\right).\end{aligned} \tag{3.23}$$

Therefore, the theorem holds for $k=n+1,n+2,n+3,\dots$. By applying mathematical induction, we obtain that

$$|a_k|\le \frac{4|e^{i\alpha}-\beta\cos\alpha|}{k-1}\left(\frac{k+1}{2}\right)^j\prod_{l=n}^{k-2}\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{l}\right), \tag{3.24}$$

for if $k=n+1,n+2,n+3,\dots$.

Finally, we consider a function $p(z)$ given by

$$p(z)=\frac{1+z^{n-1}}{1-z^{n-1}}. \tag{3.25}$$

Then, we see that

$$\frac{L_{j-1}f(z)}{L_jf(z)}=\frac{1+(1-2\beta\cos\alpha e^{-i\alpha})z^{n-1}}{1-z^{n-1}} \tag{3.26}$$

and that

$$\frac{(L_jf(z))'}{L_jf(z)}=\frac{1}{z}+\frac{4(1-\beta\cos\alpha e^{-i\alpha})z^{n-2}}{1-z^{n-1}} \tag{3.27}$$

by (1.8). Therefore, we have

$$\log(L_jf(z))=\log z-\frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1}\log(1-z^{n-1}) \tag{3.28}$$

and

$$L_jf(z)=\frac{z}{(1-z^{n-1})^t} \tag{3.29}$$

with

$$t=\frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1}. \tag{3.30}$$

□

4. Subordination Problems

Let functions $f(z)$ and $g(z)$ be analytic in U. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, $z \in U$, if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ and such that $f(z) = g(w(z))$ (see, Miller and Mocanu [3]).

For subordinations, we have

Theorem 4.1. *If $f(z) \in A(n)$ satisfies*

$$\frac{z(L_j f(z))'}{L_j f(z)} \prec \frac{1-(1-4(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}, \quad z \in U \tag{4.1}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ and β $(0 \le \beta < 1)$, then $f(z) \in A_j(n,\alpha,\beta)$, $j = 1,2,3,\dots$.

Proof. We note that

$$Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_j f(z)}\right) = Re\left\{\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right)\right\} > \beta\cos,\alpha \quad z \in U \tag{4.2}$$

for $f(z) \in A_j(n,\alpha,\beta)$. Since

$$\begin{aligned}\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right) &\prec \frac{e^{i\alpha}}{2}\left\{2\frac{1-(1-2(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}\right\} \\ &= e^{i\alpha}\left\{\frac{1-(1-2(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}\right\},\end{aligned} \tag{4.3}$$

we have that

$$\begin{aligned}Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_j f(z)}\right) &= Re\left\{\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right)\right\} \\ &= Re\left\{e^{i\alpha}+2(1-\beta)\cos\alpha\frac{z^n}{1-z^n}\right\} \\ &> \cos\alpha-(1-\beta)\cos\alpha = \beta\cos\alpha, \quad z \in U.\end{aligned} \tag{4.4}$$

This completes the proof of the theorem.

A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{4.5}$$

is analytic, univalent and convex in U, we have

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z), \quad z \in U \tag{4.6}$$

with $a_1 = 1$. For such subordinating factor sequences, Wilf [4] gave the following result. □

Lemma 4.1. *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left(1+2\sum_{k=1}^{\infty} b_k z^k\right) > 0, \quad z \in U. \tag{4.7}$$

For functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A(n) \tag{4.8}$$

and

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \in A(n) \tag{4.9}$$

the convolution $(f * g)(z)$ *is given by*

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k. \tag{4.10}$$

By the relation (1.8)*, we know that*

$$L_{j-1} f(z) = \frac{1}{2}\left(z L_j f(z)\right)', \quad j \in \mathbb{N}. \tag{4.11}$$

We specially consider the case $j = 0$ *in* (4.11)*. Then, we have*

$$L_{-1} f(z) = \frac{1}{2}(z f(z))' = z + \sum_{k=n+1}^{\infty} \left(\frac{k+1}{2}\right) a_k z^k. \tag{4.12}$$

With (4.12)*, we introduce the subclass* $A_0(n,\alpha,\beta)$ *of* $A(n)$ *by*

$$Re\left(e^{i\alpha} \frac{L_{-1} f(z)}{L_0 f(z)}\right) = Re\left(e^{i\alpha}\left(1 + \frac{z f'(z)}{f(z)}\right)\right) > \beta \cos\alpha, \quad z \in U \tag{4.13}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ *and* β $(0 \le \beta < 1)$*.*

It is clear that Theorem 2.2 gives us the following lemma.

Lemma 4.2. *If* $f(z) \in A(n)$ *satisfies*

$$\sum_{k=n+1}^{\infty} \left(1 + \frac{k-1}{2(1-\beta)} \sec\alpha\right) |a_k| \le 1 \tag{4.14}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ *and* β $(0 \le \beta < 1)$*, then* $f(z) \in A_0(n,\alpha,\beta)$*.*

Now, we derive

Theorem 4.2. *If* $f(z) \in A(n)$ *be satisfy the coefficient inequality* (4.14) *and* $g(z) \in A(1)$ *be convex in* U*. Then*

$$\frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} (f * g)(z) \prec g(z), \quad z \in U. \tag{4.15}$$

In particular

$$Re f(z) > -\frac{4(1-\beta) + n\sec\alpha}{2(1-\beta) + n\sec\alpha}, \quad z \in U. \tag{4.16}$$

The constant in (4.15) *can not be replace by any lager one.*

Proof. Let us consider a function $f(z) \in A(n)$ given by (4.8) and a function $g(z) \in A(n)$ given by (4.9). Then, we know

$$\frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} (f * g)(z) = \frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} \left(z + \sum_{k=n+1}^{\infty} a_k b_k z^k\right) \tag{4.17}$$

It follows from Lemma 4.1,

$$Re\left\{1 + 2\sum_{k=1}^{\infty} \frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} a_k z^k\right\}$$

$$= Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}\left(z+\sum_{k=n+1}^{\infty}a_kz^k\right)\right\}>0, \quad z\in U. \tag{4.18}$$

We see that

$$\begin{aligned}
&Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}\left(z+\sum_{k=n+1}^{\infty}a_kz^k\right)\right\}\\
&= Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}z+\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}\sum_{k=n+1}^{\infty}\left(1+\frac{n}{2(1-\beta)}\sec\alpha\right)a_kz^k\right\}\\
&\geq 1-\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}|z|-\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}\sum_{k=n+1}^{\infty}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k||z|^k\\
&\geq 1-\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}|z|-\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}|z|^{n+1}\\
&\geq 1-|z|>0, \quad z\in U.
\end{aligned} \tag{4.19}$$

This implies the subordination (4.15).

Next, we take $g(z)$ given by

$$g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k\in A(1). \tag{4.20}$$

Then, $g(z)$ satisfies

$$Re\left(1+\frac{zg''(z)}{g'(z)}\right)=Re\left(\frac{1+z}{1-z}\right)>0, \quad z\in U. \tag{4.21}$$

Thus $g(z)$ is convex in U. Further, we take the function $f(z)\in A(n)$ given by

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k, \quad z\in U \tag{4.22}$$

with

$$a_k=\frac{2(n+1)(1-\beta)e^{i\theta}}{(2(1-\beta)+(k-1)\sec\alpha)k(k+1)}, \quad 0\leq\theta\leq 2\pi. \tag{4.23}$$

Then, $f(z)$ satisfies

$$\sum_{k=n+1}^{\infty}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k|=\sum_{k=n+1}^{\infty}\frac{n+1}{k(k+1)}=(n+1)\sum_{k=n+1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1. \tag{4.24}$$

This function, $f(z)\in A(n)$ satisfies (4.15).

Further, we see that

$$\begin{aligned}
&\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}(f*g)(z)\right\}\\
&=\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}f(z)\right\}\\
&=\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}\left(z+\sum_{k=n+1}^{\infty}a_kz^k\right)\right\}\\
&=-\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}\left(1+\sum_{k=n+1}^{\infty}|a_k|\right)
\end{aligned}$$

$$= -\left\{ \frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}} + \frac{(n+1)(1-\beta)}{4(1-\beta)+n\sec\alpha} \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \right\}$$
$$= -\frac{1}{2}$$

with (4.23). Since the function $g(z) \in A(1)$ given by (4.2) satisfies $Re\, g(z) > -\frac{1}{2}$, $z \in U$, the constant $\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}$ is the best possible. □

Remark 4.1. We tried to get a suitable function $g(z) \in A(n)$ in Theorem 4.2. But we did not get any good $g(z) \in A(n)$ in our paper. Hence, we consider the convex function $g(z) \in A(1)$ defined in (4.20).

5. Appendix

We consider Libera integral operator $L_j f(z)$ for $f(z) \in A(n)$. Further, we introduce $L_{-1} f(z)$ by

$$L_{-1} f(z) = \frac{1}{2}(z f(z))' = z + \sum_{k=n+1}^{\infty} \left(\frac{k+1}{2}\right) a_k z^k. \tag{5.1}$$

In order to use (5.1)), we introduce

$$L_{-2} f(z) = \frac{1}{2}(z L_{-1} f(z))' = z + \sum_{k=n+1}^{\infty} \left(\frac{k+1}{2}\right)^2 a_k z^k \tag{5.2}$$

and

$$L_{-j-1} f(z) = \frac{1}{2}(z L_{-j} f(z))' = z + \sum_{k=n+1}^{\infty} \left(\frac{k+1}{2}\right)^{j+1} a_k z^k \tag{5.3}$$

for $j = 0, 1, 2, \ldots$. For such $L_{-j} f(z)$, we consider a subclass $A_{-j}(n,\alpha,\beta)$ of $A(n)$ by

$$Re\left(e^{i\alpha} \frac{L_{-j-1} f(z)}{L_{-j} f(z)}\right) > \beta\cos\alpha, \quad z \in U \tag{5.4}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ and β $(0 \leq \beta < 1)$.

With Theorem 2.2, we can prove the following result for $f(z) \in A_{-j}(n,\alpha,\beta)$.

Theorem 5.1. *If $f(z) \in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty} \left[\frac{k+1}{2}\right]^j \left(1 + \frac{k-1}{2(1-\beta)} \sec\alpha\right) |a_k| \leq 1 \tag{5.5}$$

for some real α $\left(|\alpha| < \frac{\pi}{2}\right)$ and β $(0 \leq \beta < 1)$, then the subclass $f(z) \in A_{-j}(n,\alpha,\beta)$.

Using Theorem 5.1, we derive

Theorem 5.2. *If $f(z) \in A(n)$ be satisfy the coefficient inequality* (5.5) *and $g(z) \in A(1)$ be convex in U. Then*

$$\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}} (f * g)(z) \prec g(z), \quad z \in U. \tag{5.6}$$

In particular

$$Re\, f(z) > -\frac{4(1-\beta)+n\sec\alpha}{2(1-\beta)+n\sec\alpha}, \quad z \in U. \tag{5.7}$$

The constant in (5.6) *cannot be replaced by any larger one.*

The manner of the proof is the same as the proof of Theorem 4.2.

6. Conclusion

Using Libera integral operator $Lf(z)$, we introduce the new subclass $A_j(n,\alpha,\beta)$ of $A(n)$. The coefficient and subordinate problem of class $A_j(n,\alpha,\beta)$ has been solved. Some results obtained by giving detailed proofs of theorems are presented.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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