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Research Article

Orthogonal Generalized (σ,τ)-Derivations on Semiprime Γ-Semirings

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Abstract. In this study, we regard M as a semiprime Γ-semiring and introduce the notion of orthogonal (σ,τ)-derivations within such structures. We explore various characterizations of semiprime Γ-semirings and determine the conditions under which two (σ,τ)-derivations are orthogonal.

Keywords. (σ,τ)-Derivation, Generalized (σ,τ)-Derivation, Orthogonal (σ,τ)-Derivation, Γ-Semiring, Semiprime Γ-Semiring

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1. Introduction

H. S. Vandiver [16] was the pioneer in introducing the concept of semirings. Later, Nobusawa [10] proposed Γ-ring to generalize the notion of a ring. Sen [14] introduced the idea of a Γ-semigroup, which was further developed into Γ-semiring by Rao [11, 12]. Huang [2] examined orthogonal generalized (σ,τ)-derivations in semiprime near-rings, and Dey *et al.* [1] investigated these derivations in semiprime Γ-near-rings. Javed *et al.* [3] focused on derivations in prime Γ-semirings. Orthogonal derivations in semirings were introduced by Suganthameena and Chandramouleeswaran [15] while Venkateswarlu *et al.* [17, 18] established the conditions for the orthogonality of derivations and reverse derivations in semiprime Γ-semirings. Majeed and Hamil [4] explored orthogonal generalized derivations in semiprime Γ-semirings. Murty *et al.* [5–9], and Reddy and Murty [13] have proved results on the orthogonality of generalized symmetric reverse bi-(σ,τ)-derivations in semiprime rings, on the orthogonality of generalized reverse (σ,τ)-derivations, (σ,τ)-derivations in semiprime Γ-rings, Γ-near rings and Γ-semirings.

In our current work, we explore the concept of generalized (σ,τ)-derivations in the context of semiprime Γ-semirings. We also determine the necessary and sufficient conditions for the orthogonality of two such generalized (σ,τ)-derivations on semiprime Γ-semirings, extending the previously established results by Majeed and Hamil [4].

2. Preliminaries

A set M is called a semiring with two associative binary operations, addition $(+)$ and multiplication $(\cdot)$, if the following conditions are met:

(i) The addition operation is commutative.

(ii) The multiplication operation distributes over addition from both the left side and the right side.

(iii) An element $0 \in M$ exists such that $a+0=a$ and $a\cdot 0 = 0\cdot a = 0$, for every $a \in M$.

If $(M,+)$ and $(\Gamma,+)$ are two abelian semigroups with identity elements 0 and θ of M and Γ, respectively and if there exists a mapping of $M\times\Gamma\times M \to M$ satisfying the following properties for $a,b,c \in M$ and $\alpha,\beta\in\Gamma$:

(1) $(a+b)\alpha c = a\alpha c + b\alpha c$,

(2) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

(3) $a\alpha(b+c) = a\alpha b + a\alpha c$,

(4) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

(5) $a\alpha 0 = 0\alpha a = 0$ and $a\theta b = 0$, then M is termed a Γ-semiring.

Let M be a Γ-semiring. M is said to be prime if for every $a,b\in M$, $a\Gamma M\Gamma b = 0$ suggests $a=0$ or $b=0$. M is said to be semiprime if for every $a\in M$, $a\Gamma M\Gamma a = 0$ implies $a=0$. A Γ-semiring M is said to be 2 torsion free if for every $a\in M$, $2a=0$ implies $a=0$. An additive mapping $d_1 : M\to M$ is called a (σ,τ)-derivation if for every $a,b\in M$, $d_1(a\alpha b) = d_1(a)\alpha\sigma(b) + \tau(a)\alpha d_1(b)$. An additive mapping $D_1 : M\to M$ is called a generalized (σ,τ)-derivation if for every $a,b\in M$, $\alpha\in\Gamma$, $D_1(a\alpha b) = D_1(a)\alpha\sigma(b)+\tau(a)\alpha d_1(b)$, where d_1 is an associated (σ,τ)-derivation. Let M be a Γ-semiring. Two additive mappings d_1 and d_2 of M into M are said to be orthogonal if for every $a,b\in M$, $d_1(a)\Gamma M\Gamma d_2(b) = \{0\} = d_2(a)\Gamma M\Gamma d_1(b)$.

We assume throughout the paper that M is a 2-torsion-free semiprime Γ-semiring, while σ and τ are automorphisms of M and d_1,d_2 are (σ,τ)-derivations on M such that $\tau d_2 = d_2\tau$, $\tau d_1 = d_1\tau$, $\sigma d_1 = d_1\sigma$, $\sigma d_2 = d_2\sigma$. We denote two generalized (σ,τ)-derivations $D_1 : M\to M$ and $D_2 : M\to M$ determined by the (σ,τ)-derivations d_1,d_2 of M such that $D_1\tau = \tau D_1$, $D_2\tau = \tau D_2$, $D_1\sigma = \sigma D_1$, $\sigma D_2 = D_2\sigma$.

Lemma 2.1 ([17, Lemma 3.1])**.** *Let a and b be two elements of a 2 torsion-free semiprime Γ-semiring M. Then the following statements are equivalent:*

(i) $a\Gamma x\Gamma b = 0$.

(ii) $b\Gamma x\Gamma a = 0$.

(iii) $a\Gamma x\Gamma b + b\Gamma x\Gamma a = 0$, *for every $x\in M$.*

If one of these conditions is fulfilled then $a\Gamma b = b\Gamma a = 0$.

Lemma 2.2 ([17, Lemma 3.2])**.** *Let M be a 2-torsion free semiprime Γ-semiring. If additive mappings d_1 and d_2 of M into itself satisfy $d_1(a)\Gamma M\Gamma d_2(a)=\{0\}$, for every $a\in M$ then $d_1(a)\Gamma M\Gamma d_2(b)=\{0\}$, for every $a,b\in M$.*

Lemma 2.3. *Given a 2-torsion free semiprime Γ-semiring M. Then two (σ,τ)-derivations d_1 and d_2 of M into itself are considered orthogonal if and only if for any $a,b\in M$ and $\alpha\in\Gamma$, the expression $d_1(a)\alpha d_2(b)+d_2(a)\alpha d_1(b)=0$, for every $a,b\in M$, $\alpha\in\Gamma$ holds.*

Proof. Suppose that

$$d_1(a)\alpha d_2(b)+d_2(a)\alpha d_1(b)=0,\quad \text{for every } a,b\in M,\ \alpha\in\Gamma. \tag{2.1}$$

Replacing b by $b\beta a$, for all $a\in M$, $\beta\in\Gamma$ in (2.1), we get

$$(d_1(a)\alpha d_2(b)+d_2(a)\alpha d_1(b))\beta\sigma(a)+d_1(a)\alpha\tau(b)\beta d_2(a)+d_2(a)\alpha\tau(b)\beta d_1(a)=0.$$

Using the equation (2.1), we obtain

$$d_1(a)\alpha\tau(b)\beta d_2(a)+d_2(a)\alpha\tau(b)\beta d_1(a)=0.$$

Since τ is an automorphism on M and using Lemma 2.1, we get

$$d_1(a)\alpha\tau(b)\beta d_2(a)=0=d_2(a)\alpha\tau(b)\beta d_1(a).$$

Since τ is an automorphism on M and using Lemma 2.2, we get

$$d_1(a)\alpha\tau(b)\beta d_2(b)=0=d_2(a)\alpha\tau(b)\beta d_1(b).$$

Using Lemma 2.1, we get

$$d_1(a)\alpha d_2(b)=0=d_2(a)\alpha d_1(b).$$

Thus, d_1 and d_2 are orthogonal.
Conversely, suppose that d_1 and d_2 are orthogonal.
Then,

$$d_1(a)\Gamma M\Gamma d_2(b)=\{0\},\quad \text{for every } a,b\in M,$$

$$d_1(a)\Gamma d_2(b)=\{0\}=d_2(a)\Gamma d_2(b),\quad \text{for every } a,b\in M. \qquad \text{(By Lemma 2.1)}$$

Hence, the theorem is proved. □

Lemma 2.4. *Let M be a 2-torsion free semiprime Γ-semiring. Suppose d_1 and d_2 are (σ,τ)-derivations of M into M. Then, d_1 and d_2 are orthogonal if and only if $d_1d_2=0$.*

Proof. Let M be a 2 torsion-free semiprime Γ-semiring.
Suppose $d_1d_2=0$,

$$d_1d_2(a\alpha b)=0,$$

$$d_1(d_2(a))\alpha\sigma^2(b)+\tau((d_2(a))\alpha d_1(\sigma(b))+d_1(\tau(a))\alpha\sigma(d_2(b))+\tau^2(a)\alpha d_1d_2(b)=0.$$

Since σ,τ are automorphism and $\tau d_2=d_2\tau$, $\sigma d_2=d_2\sigma$, $\tau d_1=d_1\tau$, $\sigma d_1=d_1\sigma$, we get

$$d_1d_2(a)\alpha\sigma(b)+d_2(u)\alpha d_1(v)+d_1(u)\alpha d_2(v)+\tau(u)\alpha d_1d_2(v)=0,\quad \text{for } a,b\in M,\ \alpha\in\Gamma,$$

$$d_2(a)\alpha d_1(b)+d_1(a)\alpha d_2(b)=0.\quad \text{(Since } d_1d_2=0).$$

Therefore, d_1 and d_2 are orthogonal (by Lemma 2.3).
Conversely, suppose that d_1 and d_2 are orthogonal.

Then

$$d_1(a)\Gamma M\Gamma d_2(b)=\{0\}, \quad \text{for every } a,b\in M,$$

$$d_1(a)\alpha c\beta d_2(b)=0, \quad \text{for every } a,b,c\in M,\ \alpha,\beta\in\Gamma.$$

Then, we have

$$d_1(d_1(a)\alpha c\beta d_2(b))=0,$$

$$d_1(d_1(a))\alpha\sigma(c)\beta\sigma(d_2(b))+\tau(d_1(a))\alpha(d_1(c)\beta\sigma(d_2(b))+\tau(c)\beta d_1d_2(b))=0.$$

Since σ,τ are automorphism of M and using $\sigma d_2=d_2\sigma$, $\tau d_1=d_1\tau$, we get

$$d_1(d_1(a))\alpha\sigma(c)\beta d_2(b)+d_1(a)\alpha d_1(c)\beta d_2(b)+d_1(a)\alpha\tau(c)\beta d_1d_2(b)=0.$$

The first and second summands are zeros as d_1 and d_2 are orthogonal.
Hence, we get

$$d_1(a)\alpha\tau(c)\beta d_1d_2(b)=0. \tag{2.2}$$

Replace a by $d_2(b)$ in eq. (2.2),

$$d_1d_2(b)\alpha\tau(c)\beta d_1d_2(b)=0.$$

Since τ is an automorphism and using the semiprimeness of M, we get

$$d_1d_2(b)=0.$$

Hence the result is proved. □

3. Main Results

Theorem 3.1. *Assume that M is a semiprime Γ-semiring which is 2-torsion free and suppose that D_1,D_2 are two generalized (σ,τ)-derivations on M associated with the (σ,τ)-derivations d_1,d_2 of M into itself. The following conditions are to be satisfied if D_1 and D_2 are orthogonal:*

(a) (i) $D_1(a)\Gamma D_2(b)=D_2(a)\Gamma D_1(b)=\{0\}$, *for every $a,b\in M$.*
(ii) $D_1(a)\Gamma D_2(b)+D_2(a)\Gamma D_1(b)=\{0\}$, *for every $a,b\in M$.*

(b) *d_1 and D_2 are orthogonal and $d_1(a)\Gamma D_2(b)=D_2(a)\Gamma d_1(b)=\{0\}$, for every $a,b\in M$.*

(c) *d_2 and D_1 are orthogonal and $d_2(a)\Gamma D_1(b)=D_1(a)\Gamma d_2(b)=\{0\}$, for every $a,b\in M$.*

(d) *d_1 and d_2 are orthogonal.*

Proof. (a): It is given that D_1 and D_2 are orthogonal generalized (σ,τ)-derivations on M.
Then by the definition of orthogonality, we have

$$D_1(a)\Gamma M\Gamma D_2(b)=D_2(a)\Gamma M\Gamma D_1(b)=\{0\}, \quad \text{for every } a,b\in M. \tag{3.1}$$

By Lemma 2.1, we can write

$$D_1(a)\Gamma D_2(b)=\{0\}=D_2(a)\Gamma D_1(b), \quad \text{for every } a,b\in M.$$

Hence, the result is evident.

(b): From condition (i), we have

$$D_1(a)\Gamma D_2(b)=\{0\}, \quad \text{for every } a,b\in M.$$

Then

$$D_1(a)\beta D_2(b)=0. \tag{3.2}$$

Replacing a by $c\alpha a$, $c \in M$ in (3.2), we get

$$D_1(c\alpha a)\beta D_2(b) = 0,$$
$$(D_1(c)\alpha\sigma(a) + \tau(c)\alpha d_1(a))\beta D_2(b) = 0,$$
$$D_1(c)\alpha\sigma(a)\beta D_2(b) + \tau(c)\alpha d_1(a)\beta D_2(b) = 0.$$

Since σ is an automorphism on M and using (3.1), we have

$$\tau(c)\alpha d_1(a)\beta D_2(b) = 0, \quad \text{for every } a,b,c \in M,\ \alpha,\beta \in \Gamma. \tag{3.3}$$

Premultiplying (3.3) by $d_1(a)\beta D_2(b)\alpha$, we get

$$d_1(a)\beta D_2(b)\alpha\tau(c)\alpha d_1(a)\beta D_2(b) = 0.$$

Since τ is an automorphism on M, we obtain

$$d_1(a)\beta D_2(b) = 0, \quad \text{for every } a,b \in M,\ \beta \in \Gamma,$$
$$d_1(a)\Gamma D_2(b) = \{0\}, \quad \text{for every } a,b \in M. \tag{3.4}$$

Replacing a by $a\alpha c$, $c \in M$, $\alpha \in \Gamma$ in (3.4) and using it, we get

$$d_1(a)\alpha\sigma(c)\Gamma D_2(b) = \{0\}.$$

Since σ is an automorphism on M, we get

$$d_1(a)\Gamma M\Gamma D_2(b) = \{0\}. \tag{3.5}$$

Using Lemma 2.1, we get

$$d_1(a)\Gamma D_2(b) = \{0\}, \quad \text{for every } a,b \in M. \tag{3.6}$$

From the result (i), we can have

$$D_2(a)\Gamma D_1(b) = \{0\}, \quad \text{for every } a,b \in M. \tag{3.7}$$

Replacing b by $c\beta b$, where $c \in M, \beta \in \Gamma$ in (3.7) and using it, we get

$$D_2(a)\Gamma\tau(c)\alpha d_1(b) = \{0\}.$$

Since τ is an automorphism on M,

$$D_2(a)\Gamma M\Gamma d_1(b) = \{0\}. \tag{3.8}$$

By Lemma 2.1, we get

$$D_2(a)\Gamma d_1(b) = \{0\}, \quad \text{for every } a,b \in M. \tag{3.9}$$

From (3.5) and (3.8), we can conclude that d_1 and D_2 are orthogonal.

(c): By proceeding in the similar manner as in (b), we can easily prove the result (c).

(d): From the condition (i), we have

$$D_1(a)\Gamma D_2(b) = \{0\}, \quad \text{for every } a,b \in M. \tag{3.10}$$

Replacing a by $a\alpha c$ and b by $b\beta u$, where $c,u \in M$, $\alpha,\beta \in \Gamma$ in eq. (3.10), we get

$$D_1(a)\alpha\sigma(c)\Gamma D_2(b)\beta\sigma(u) + \tau(a)\alpha d_1(c)\Gamma D_2(b)\beta\sigma(u) + D_1(a)\alpha\sigma(c)\Gamma\tau(b)\beta d_2(u)$$
$$+ \tau(a)\alpha d_1(c)\Gamma\tau(b)\beta d_2(u) = 0.$$

Using (3.1) and conditions (b) and (c), we get

$$\tau(a)\alpha d_1(c)\Gamma\tau(b)\beta d_2(u) = \{0\}, \quad \text{for every } a,b,c,u \in M,\ \alpha \in \Gamma. \tag{3.11}$$

Premultiplying (3.11) by $d_1(c)\Gamma\tau(b)\beta d_2(u)\alpha$, we get

$$d_1(c)\Gamma\tau(b)\beta d_2(u)=\{0\}. \quad \text{(Since } M \text{ is semiprime.)}$$

Since τ is an automorphism of M, we get

$$d_1(c)\Gamma M\Gamma d_2(u)=\{0\}. \tag{3.12}$$

From the hypothesis, we have

$$D_2(a)\Gamma D_1(b)=\{0\}, \quad \text{for every } a,b\in M.$$

Replacing a by $a\alpha c$ and b by $b\beta u$, where $c,u\in M$, $\alpha,\beta\in\Gamma$ in eq. (3.7), we get

$$D_2(a\alpha c)\Gamma D_1(b\beta u)=\{0\},$$

$$(D_2(a)\alpha\sigma(c)+\tau(a)\alpha d_2(c))\Gamma(D_1(b)\beta\sigma(u)+\tau(b)\beta d_1(u))=\{0\},$$

$$D_2(a)\alpha\sigma(c)\Gamma D_1(b)\beta\sigma(u)+D_2(a)\alpha\sigma(c)\Gamma\tau(b)\beta d_1(u)+\tau(a)\alpha d_2(c)\Gamma D_1(b)\beta\sigma(u)$$
$$+\tau(a)\alpha d_2(c)\Gamma\tau(b)\beta d_1(u)=\{0\}, \quad \text{for every } a,b,c,u\in M,\ \alpha,\beta\in\Gamma.$$

Using (3.1) and conditions (b) and (c), we get

$$\tau(a)\alpha d_2(c)\Gamma\tau(b)\beta d_1(u)=\{0\}, \quad \text{for every } a,b,c,u\in M,\ \alpha,\beta\in\Gamma.$$

Premultiplying by $d_2(c)\Gamma\tau(b)\beta d_1(u)\alpha$, we get

$$d_2(c)\Gamma\tau(b)\beta d_1(u)=\{0\}. \quad \text{(Since } M \text{ is semiprime)}$$

$$d_2(c)\Gamma M\Gamma d_1(u)=\{0\}, \quad \text{for every } c,u\in M. \quad \text{(Since } \tau \text{ is an automorphism)} \tag{3.13}$$

From (3.12) and (3.13), we can conclude that d_1 and d_2 are orthogonal. □

Theorem 3.2. *Assume that M is a 2-torsion free semiprime Γ-semiring and D_1 is a generalized (σ,τ)-derivation on M associated with the (σ,τ)-derivation d_1 of M into M. If $D_1(a)\Gamma D_1(b)=\{0\}$, then, $D_1=d_1=0$.*

Proof. By hypothesis,

$$D_1(a)\Gamma D_1(b)=\{0\}, \quad \text{for every } a,b\in M. \tag{3.14}$$

Replacing b by $b\alpha c$, where $c\in M$ in (3.14) and using it, we get $D_1(a)\Gamma\tau(b)\alpha d_1(c)=\{0\}$. Hence,

$$D_1(a)\Gamma M\Gamma d_1(c)=\{0\}. \quad \text{(Since } \tau \text{ is an automorphism)} \tag{3.15}$$

Using Lemma 2.1, we get

$$D_1(a)\Gamma d_1(c)=\{0\}=d_1(c)\Gamma D_1(a), \quad \text{for every } a,c\in M. \tag{3.16}$$

Replacing a by $a\alpha c$, where $c\in M$ in (3.16) and using it, $d_1(c)\Gamma\tau(a)\alpha d_1(c)=\{0\}$ and so $d_1(c)\Gamma M\Gamma d_1(c)=\{0\}$.

Since τ is an automorphism on M and by Lemma 2.1, we get

$$d_1(c)\Gamma d_1(c)=\{0\}, \quad \text{for every } c\in M.$$

Since M is semiprime, we can have

$$d_1(c)=0, \quad \text{for every } c\in M.$$

Again replacing a by $b\alpha a$, $b\in M,\alpha\in\Gamma$ in (3.14) and using (3.16), we obtain

$$D_1(b)\alpha\sigma(a)\Gamma D_1(b)=\{0\}.$$

Since σ is an automorphism on M, we can write

$$D_1(b)\Gamma M\Gamma D_1(b)=\{0\}, \quad \text{for every } b\in M.$$

By using Lemma 2.1, we get

$$D_1(b)\Gamma M\Gamma D_1(b)=\{0\}.$$

Using the semiprimeness of M, we get

$$D_1(b)=0.$$

Hence, we get $D_1=d_1=0$. □

Theorem 3.3. *Assume that M be a 2-torsion free semiprime Γ-semiring. Let D_1,D_2 be two generalized (σ,τ)-derivations on M associated with the (σ,τ)-derivations d_1,d_2 of M into itself. If the relations*

(i) $D_1(a)\Gamma D_2(b)+D_2(a)\Gamma D_1(b)=\{0\}$, *for every $a,b\in M$,*

(ii) $d_1(a)\Gamma D_2(b)+d_2(a)\Gamma D_1(b)=\{0\}$, *for every $a,b\in M$, are satisfied,*

then D_1 and D_2 are orthogonal on M.

Proof. By the condition (i) of the hypothesis, we have

$$D_1(a)\Gamma D_2(b)+D_2(a)\Gamma D_1(b)=\{0\},\quad \text{for every } a,b\in M. \tag{3.17}$$

Replacing a by $a\alpha c$, $c\in M$, $\alpha\in\Gamma$ in (3.17), we get

$$D_1(a)\alpha\sigma(c)\Gamma D_2(b)+D_2(a)\alpha\sigma(c)\Gamma D_1(b)+\tau(a)\alpha(d_1(c)\Gamma D_2(b)+d_2(c)\Gamma D_1(b))=\{0\},$$

$$D_1(a)\alpha\sigma(c)\Gamma D_2(b)+D_2(a)\alpha\sigma(c)\Gamma D_1(b)=\{0\}.\quad \text{(Using condition (ii))} \tag{3.18}$$

Replacing b by a in (3.18), we get

$$D_1(a)\alpha\sigma(c)\Gamma D_2(a)+D_2(a)\alpha\sigma(c)\Gamma D_1(a)=\{0\},$$

$$D_1(a)\Gamma M\Gamma D_2(a)+D_2(a)\Gamma M\Gamma D_1(a)=\{0\},\quad \text{(Since } \sigma \text{ is an automorphism)}$$

$$D_1(a)\Gamma M\Gamma D_2(a)=D_2(a)\Gamma M\Gamma D_1(a)=\{0\},\quad \text{for every } a\in M.\quad \text{(By Lemma 2.1)}$$

By Lemma 2.2, we get

$$D_1(a)\Gamma M\Gamma D_2(b)=D_2(a)\Gamma M\Gamma D_1(b)=\{0\},\quad \text{for every } a,b\in M.$$

Therefore, D_1 and D_2 are orthogonal. □

Theorem 3.4. *Assume that M is a 2-torsion free semiprime Γ-semiring and let D_1,D_2 be two generalized (σ,τ)-derivations on M associated with the (σ,τ)-derivations d_1,d_2 of M into itself. If $D_1(a)\Gamma D_2(b)=d_1(a)\Gamma D_2(b)=\{0\}$, then D_1,D_2 are orthogonal, for every $a,b\in M$.*

Proof. Given that

$$D_1(a)\Gamma D_2(b)=\{0\},\quad \text{for every } a,b\in M, \tag{3.19}$$

$$d_1(a)\Gamma D_2(b)=\{0\},\quad \text{for every } a,b\in M. \tag{3.20}$$

Replacing a by $a\alpha c$, where $c\in M$, $\alpha\in\Gamma$ in (3.19) and by (3.20), we get

$$D_1(a)\alpha\sigma(c)\Gamma D_2(b)=\{0\}.$$

Since σ is an automorphism on M, we obtain

$$D_1(a)\Gamma M\Gamma D_2(b)=\{0\}, \tag{3.21}$$

$$D_2(b)\Gamma M\Gamma D_1(a)=\{0\}.\quad \text{(By Lemma 2.1)} \tag{3.22}$$

Replacing b by a in (3.22), we get

$$D_2(a)\Gamma M\Gamma D_1(a)=\{0\},$$

$$D_2(a)\Gamma M\Gamma D_1(b)=\{0\}, \quad \text{for every } a,b\in M. \quad \text{(By Lemma 2.2)} \tag{3.23}$$

From (3.21) and (3.23), we can conclude that D_1,D_2 are orthogonal. □

Theorem 3.5. *Assume that M is a 2-torsion free semiprime Γ-semiring and let D_1,D_2 be two generalized (σ,τ)-derivations on M associated with the (σ,τ)-derivations d_1,d_2 of M into itself. If $D_1(a)\Gamma D_2(b)=\{0\}$, for every $a,b\in M$ and $d_1D_2=d_1d_2=0$, then D_1,D_2 are orthogonal.*

Proof. By hypothesis, we have

$$d_1d_2=0.$$

By Lemma 2.4, we have that d_1 and d_2 orthogonal.
Hence,

$$d_1(a)\Gamma M\Gamma d_2(b)=\{0\}=d_2(a)\Gamma M\Gamma d_1(b), \quad \text{for every } a,b\in M. \tag{3.24}$$

By using Lemma 2.1, we can write

$$d_1(a)\Gamma d_2(b)=d_2(a)\Gamma d_1(b)=\{0\}, \quad \text{for every } a,b\in M. \tag{3.25}$$

It is also given that $d_1D_2=0$ (by hypothesis),

$$d_1D_2(a\alpha b)=0, \quad \text{for every } a,b\in M,\ \alpha\in\Gamma,$$

$$d_1D_2(a)\alpha\sigma^2(b)+\tau(D_2(a))\alpha d_1(\sigma(b))+d_1(\tau(a))\alpha\sigma(d_2(b))+\tau^2(a)\alpha d_1d_2(b)=0.$$

Since $d_1D_2=d_1d_2=0$ and $\tau D_2=D_2\tau$, $\tau d_1=d_1\tau$, $\sigma d_1=d_1\sigma$, $\sigma d_2=d_2\sigma$, where σ,τ are automorphisms on M and using (3.25), we get

$$D_2(a)\alpha d_1(b)=0, \quad \text{for every } a,b\in M. \tag{3.26}$$

Replacing a by $a\beta c$, for $c\in M,\beta\in\Gamma$ in (3.26) and using (3.25), we get

$$D_2(a)\beta\sigma(c)\alpha d_1(b)=0.$$

Since σ is an automorphism on M and using Lemma 2.1, we can write

$$D_2(a)\Gamma M\Gamma d_1(b)=\{0\}=d_1(b)\Gamma M\Gamma D_2(a), \quad \text{for every } a,b\in M.$$

Replacing b by a in the above equation, we get

$$d_1(a)\Gamma M\Gamma D_2(a)=\{0\}, \quad \text{for every } a\in M,$$

$$d_1(a)\Gamma M\Gamma D_2(b)=\{0\}, \quad \text{for every } a,b\in M, \quad \text{(By Lemma 2.2)}$$

$$d_1(a)\Gamma D_2(b)=\{0\}, \quad \text{for every } a,b\in M. \quad \text{(By Lemma 2.1)}$$

Also, we have

$$D_1(a)\Gamma D_2(b)=\{0\}, \quad \text{for every } a,b\in M.$$

Thus, we have

$$D_1(a)\Gamma D_2(b)=\{0\}=d_1(a)\Gamma D_2(b), \quad \text{for every } a,b\in M.$$

By Theorem 3.4, we can conclude that D_1 and D_2 are orthogonal. □

Theorem 3.6. *Assume that M is a 2-torsion free semiprime Γ-semiring. Let D_1,D_2 be two generalized (σ,τ)-derivations on M related to the (σ,τ)-derivations d_1,d_2 of M into itself, respectively such that D_1D_2 is a generalized (σ,τ)-derivation on M related to the (σ,τ)-derivation d_1d_2 and $D_1(a)\Gamma D_2(b)=\{0\}$, for every $a,b\in M$, then D_1,D_2 are orthogonal.*

Proof. Given that D_1,D_2 are two generalized (σ,τ)-derivations on M associated with the (σ,τ)-derivations d_1,d_2 of M.

Also, D_1D_2 is a generalized (σ,τ)-derivation on M associated with (σ,τ)-derivation d_1d_2 and

$$D_1(a)\Gamma D_2(b)=\{0\},\quad \text{for every } a,b\in M. \tag{3.27}$$

Let $a,b\in M$, $\alpha\in\Gamma$, then

$$\begin{aligned}D_1D_2(a\alpha b)&=D_1(D_2(a\alpha b))=D_1(D_2(a)\alpha\sigma(b)+\tau(a)\alpha d_2(b))\\&=D_1D_2(a)\alpha\sigma^2(b)+\tau(D_2(a))\alpha d_1(\sigma(b))+D_1(\tau(a)\alpha\sigma(d_2(b))+\tau^2(a)\alpha d_1d_2(b).\end{aligned}$$

Since σ,τ are automorphisms and $\tau D_2=D_2\tau$, $d_1\sigma=\sigma d_1$, $D_1\tau=\tau D_1$, $d_2\sigma=\sigma d_2$,

$$D_1D_2(a\alpha b)=D_1D_2(a)\alpha\sigma(b)+D_2(a)\alpha d_1(b)+D_1(a)\alpha d_2(b)+\tau(a)\alpha d_1d_2(b). \tag{3.28}$$

But, it is given that D_1D_2 is a generalized (σ,τ)-derivation and so

$$D_1D_2(a\alpha b)=D_1D_2(a)\alpha\sigma(b)+\tau(a)\alpha d_1d_2(b). \tag{3.29}$$

From (3.28) and (3.29), we get

$$D_2(a)\alpha d_1(b)+D_1(a)\alpha d_2(b)=0. \tag{3.30}$$

Replacing b by $b\beta c$, $c\in M$, $\beta\in\Gamma$ in (3.27), we get

$$D_1(a)\alpha D_2(b)\beta\sigma(c)+D_1(a)\alpha\tau(b)\beta d_2(c)=0.$$

Using equation (3.27), we get

$$D_1(a)\alpha\tau(b)\beta d_2(c)=0.$$

Since τ is an automorphism of M,

$$D_1(a)\Gamma M\Gamma d_2(c)=\{0\},\quad \text{for every } a,c\in M.$$

By Lemma 2.1, we have

$$D_1(a)\Gamma d_2(c)=\{0\}=d_2(c)\Gamma D_1(a),\quad \text{for every } a,c\in M.$$

Consider

$$d_2(c)\Gamma D_1(a)=\{0\}. \tag{3.31}$$

Replacing c by $b\alpha c$, $b\in M$, $\alpha\in\Gamma$ in (3.31) and using it, we get

$$d_2(b)\alpha\sigma(c)\Gamma D_1(a)=\{0\}.$$

Since σ is an automorphism on M, we get

$$d_2(b)\Gamma M\Gamma D_1(a)=\{0\}.$$

By Lemma 2.1, we get

$$d_2(b)\Gamma D_1(a)=\{0\}=D_1(a)\Gamma d_2(b). \tag{3.32}$$

Using eq. (3.32) relation in (3.30), we get

$$D_2(a)\Gamma d_1(b)=\{0\},\quad \text{for every } a,b\in M. \tag{3.33}$$

Replacing b by $b\beta u$, $u\in M$, $\beta\in\Gamma$ in (3.33) and using it, we get

$$D_2(a)\Gamma\tau(b)\beta d_1(u)=\{0\}.$$

Hence,

$$D_2(a)\Gamma M\Gamma d_1(u)=\{0\},\quad \text{for every } a,u\in M,$$

$$D_2(a)\Gamma d_1(u)=\{0\}=d_1(u)\Gamma D_2(a).\quad \text{(By Lemma 2.1)}$$

Consider $d_1(u)\Gamma D_2(a) = \{0\}$.
Replacing u by a and a by b in the above equation, we get

$$d_1(a)\Gamma D_2(b) = \{0\}, \quad \text{for every } a,b \in M.$$

Thus, we have

$$d_1(a)\Gamma D_2(b) = \{0\} = D_1(a)\Gamma D_2(b), \quad \text{for every } a,b \in M.$$

Then, by Theorem 3.4, we can conclude that D_1, D_2 are orthogonal. □

4. Advantages of the Study

This study introduces and develops the concept of orthogonal (σ,τ)-derivations in semiprime Γ-semirings, extending the existing theory of derivations in algebraic structures. By providing various characterizations of semiprime Γ-semirings, the research opens up new lines of inquiry for future work in algebraic structures. The results of the study have potential applications in related fields such as mathematical physics, where understanding derivations can lead to new techniques and models.

5. Future Research

Researchers can explore how the concept of orthogonality can be extended to other types of generalized derivations in semiprime Γ-semirings. The study can be expanded to explore the properties of (σ,τ)-derivations in noncommutative semirings and their potential applications. Researchers can verify whether these concepts can be applied to fields such as cryptography or coding theory, where non commutative algebra plays a role.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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