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Research Article

Mean Iterative Approach for Multiple Polynomial Zeros and Convergence

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Abstract. In this study, we propose a mean iterative approach of third order for solving a polynomial equation that has multiple type zeros. We used three prominent third-order algorithms for this build: *Chebyshev, Halley and Super-Halley* (CHS). For this CHS Combined Mean Method, we developed two forms of local convergence theorems to determine the convergence. We used the gauge function to determine the convergence of our technique. We employed two distinct forms of initial conditions on a field with norm to prove the local convergence theorems for the CHS combined mean technique. Our convergence analysis includes error estimates.

Keywords. Local convergence, Polynomial zeros, Multiple zeros, Initial conditions, Normed field, Chebyshev method, Halley method, Super-Halley method

Mathematics Subject Classification (2020). 65H04, 12Y05

1. Introduction

A very fundamental and a centuries-old topic in the field of numerical analysis is to find the zeros of a single variable polynomial equations. As Galois theorem states that for the polynomial equation that have a general solution are those of degree less or equal to four. Thus, we turn to iterative approaches to find zeros of a non-linear polynomial. In the literature of finding the roots of a higher degree equations, *Halley, Chebyshevand Super-Halley* method are very effective third order iterative methods. Osada [10], Neta [8], Chun and Neta [2], Ren and Argyros [13], Ivanov [7], Das and Kumar [3] and numerous other researchers have established iteration procedure to solve a non-linear equation with multiple zeros.

For multiple zeros, Chebyshev method ([1,9,14]) has the following form:

$$\mathcal{C}(s)=\begin{cases} s-\frac{p^2}{2}\frac{h(s)}{h'(s)}\left(\frac{3-p}{p}+\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right), & \text{if } h'(s)\neq 0,\\ s, & \text{otherwise.}\end{cases} \tag{1.1}$$

For multiple zeros, Halley method ([5,9,14]) has the following form:

$$\mathcal{H}(s)=\begin{cases} s-\left(\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\right)^{-1}, & \text{if } h(s)\neq 0,\\ s, & \text{otherwise.}\end{cases} \tag{1.2}$$

For multiple zeros, Super-Halley method ([4]) has the following form:

$$\mathcal{S}(s)=\begin{cases} s-\frac{h(s)}{2h'(s)}\left(p+\frac{1}{1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}}\right), & \text{if } h'(s)\neq 0,\\ s, & \text{otherwise.}\end{cases} \tag{1.3}$$

In this article, we have merged the above three third order iterative techniques to form our *CHS Mean Method*.

Convergence of the Picard iterative technique

$$s_{k+1}=Ts_k, \quad k=0,1,2,\ldots, \tag{1.4}$$

where $T:D\to M$ is the function of iteration, has been studied by Proinov [11,12] and thereafter Ivanov [6]. Also, they have used two kinds of initial conditions in establishing the convergence theorem. Here in this article we observe convergence of our mean iterative approach with help of same initial conditions.

This article is constructed as follows: In Section 2, we have structured our CHS Mean iterative method. In Section 3, we have showed local convergence of our proposed CHS mean iterative approach for a polynomial which has multiple type zeros using two different kinds of initial conditions.

2. Construction of the Method

In this section, we have structured our method by using three third order methods Chebyshev, Halley method and Super-Halley method. We have structured our CHS Mean Methods in the following form:

$$T(s)=\begin{cases} s-\frac{p^2}{6}\frac{h(s)}{h'(s)}\left(\frac{3-p}{p}+\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right)-\frac{1}{3}\left(\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\right)^{-1} & \\ \quad -\frac{h(s)}{6h'(s)}\left(p+\frac{1}{1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}}\right), & \text{if } h(s) \text{ and } h'(s)\neq 0,\\ s, & \text{otherwise.}\end{cases} \tag{2.1}$$

The set D is the domain of our mean iterative function T (2.1) and given by following:

$$D=\left\{s\in F: h(s)\neq 0, h'(s)\neq 0, 1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\neq 0 \text{ and } \frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\neq 0\right\}. \tag{2.2}$$

3. Local Convergence

Let F be a field, which has the norm $|\cdot|$ and $F[s]$ be the ring of polynomials. Let $h(s)$ be a non-linear polynomial of degree q. Let F has all the zeros of $h(s)$. Suppose ζ is a zero of $h(s)$ having multiplicity p. We will use the function of initial conditions I to prove the convergence of the CHS mean iterative approach (2.1) and is defined as follows:

$$I(s) = I_h(s) = \frac{|(s-\zeta)|}{d}, \tag{3.1}$$

where d is the distance from ζ to the closest zero of h, if ζ is the sole zero of h, then we assign $I(s) = 0$.

Lemma 3.1. *Let F contains all zeros of h, where $h \in F[s]$ is a non-linear polynomial of degree q (≥ 2). Let the zeros of $h(s)$ be $\zeta_1,\dots,\zeta_s$ and $p_1,\dots,p_s$, respectively be the multiplicities of the zeros. Then*

(i) *For any $i = 1,\dots,s$, we have the following:*

$$\frac{h'(s)}{h(s)} = \frac{p_i + \alpha_i}{s-\zeta_i},$$

where $\alpha_i = (s-\zeta_i)\sum_{j\neq i}\frac{p_j}{s-\zeta_j}$, $h(s)\neq 0$, $s \in F$.

(ii) *For any $i = 1,\dots,s$, we have*

$$\frac{h''(s)}{h'(s)} = \frac{(p_i+\alpha_i)^2 - (p_i+\beta_i)}{(s-\zeta_i)(p_i+\alpha_i)},$$

where $\beta_i = (s-\zeta_i)^2\sum_{j\neq i}\frac{p_j}{(s-\zeta_j)^2}$, $h(s)\neq 0$ and $h'(s)\neq 0$, $s \in F$.

Proof. (i): From

$$\frac{h'(s)}{h(s)} = \sum_{j=1}^{s}\frac{p_j}{s-\zeta_j},$$

we have

$$\frac{h'(s)}{h(s)} = \sum_{j=1}^{s}\frac{p_j}{s-\zeta_j} = \frac{p_i}{s-\zeta_i} + \sum_{j\neq i}\frac{p_j}{s-\zeta_j} = \frac{p_i+\alpha_i}{s-\zeta_i},$$

where $\alpha_i = (s-\zeta_i)\sum_{j\neq i}\frac{p_j}{s-\zeta_j}$.

This proves (i).

(ii) We will use the following identities to prove (ii).

$$\frac{h''(s)}{h'(s)} = \frac{h'(s)}{h(s)} - \frac{h(s)}{h'(s)}\sum_{j=1}^{s}\frac{p_j}{(s-\zeta_j)^2} \quad\text{and}\quad \sum_{j=1}^{s}\frac{p_j}{(s-\zeta_j)^2} = \frac{p_i+\beta_i}{(s-\zeta_i)^2}.$$

Using the above identities and the identity in (i), we get the following:

$$\frac{h''(s)}{h'(s)} = \frac{(p_i+\alpha_i)^2 - (p_i+\beta_i)}{(s-\zeta_i)(p_i+\alpha_i)},$$

where $\beta_i = (s-\zeta_i)^2\sum_{j\neq i}\frac{p_j}{(s-\zeta_j)^2}$. □

Lemma 3.2. *Let $s, \zeta \in F$ and $\zeta_1, \dots, \zeta_s \in F$ be the zeros of h which are other than ζ. Then*

$$|s - \zeta_j| \geq (1 - I(s))d, \quad \text{for any } j = 1, \dots, s, \tag{3.2}$$

where $I : F \to R_+$ is given in (3.1).

Proof. Clearly $d \leq |\zeta - \zeta_j|$, for all $j = 1, \dots, s$.
Consequently, following can be obtained by employing the triangle and the aforementioned inequality.

$$|s - \zeta_j| = |\zeta - \zeta_j + s - \zeta| \geq |\zeta - \zeta_j| - |s - \zeta| \geq (1 - I(s))d.$$ □

3.1 Local Convergence Theorem of First Kind

Let h be a non-linear polynomial of degree q in the ring of polynomial $F(s)$. We use the previously defined function of initial condition (3.1), $I : D \to R_+$ to prove the first kind convergence of the CHS Mean Iterative approach (2.1).

We now define two functions ϕ_c and ϕ_h and are defined as following:

$$\phi_c(u) = \frac{2(q-p)^3 u + p(q-p)(3q-2p)(1-u)}{2(p-qu)^3} u^2, \tag{3.3}$$

$$\phi_h(u) = \frac{q(q-p)}{q(3p-q)u^2 - 2p(p+q)u + 2p^2} u^2. \tag{3.4}$$

One can easily show that ϕ_c is a second degree quasi-homogeneous on $\left[0, \frac{p}{q}\right)$. Clearly, ϕ_h is a second degree quasi-homogeneous on the interval $\left[0, \frac{2p}{q+p+\sqrt{(q-p)(5q-p)}}\right)$.

Here, we will define two functions ϕ_{sn} and ϕ_{sd} and they are define as following:

$$\phi_{sn}(u) = (q-p)\left(\frac{2(q-p)u}{1-u} + q\right) u^2, \tag{3.5}$$

$$\phi_{sd}(u) = \frac{2(p-qu)((2p-q)u^2 - 2pu + p)}{1-u}. \tag{3.6}$$

Clearly, the function ϕ_{sn} is positive on $[0,1)$. One can easily show that ϕ_{sn} second degree quasi-homogeneous function on $[0,1)$. The function ϕ_{sd} is decreasing and positive on the interval $[0,\tau)$, where τ is defined below:

$$\tau = \begin{cases} \frac{p}{q}, & \text{if } q \geq 2p, \\ \frac{p}{p+\sqrt{p(q-p)}}, & \text{if } q < 2p. \end{cases} \tag{3.7}$$

With the help of (3.5), (3.6) and (3.7), we have defined a function $\phi_s : [0,\tau) \to R_+$ and which is defined as follows:

$$\phi_s(u) = \frac{\phi_{sn}}{\phi_{sd}} = \frac{(q-p)(q+(q-2p)u)u^2}{2(p-qu)((2p-q)u^2 - 2pu + p)}. \tag{3.8}$$

Using the properties of quasi-homogeneous functions mentioned in [12], we can say that ϕ_s is a second degree quasi-homogeneous function on $[0,\tau)$.

Using (3.3), (3.4) and (3.8), we have defined a function $\phi : \left[0, \frac{p}{p+\sqrt{(q-p)p}}\right) \to R_+$, which is basically the mean of the three functions $\phi_c(u), \phi_h(u)$ and $\phi_s(u)$ and is defined by

$$\phi(u) = \frac{\phi_c(u)}{3} + \frac{\phi_h(u)}{3} + \frac{\phi_s(u)}{3}. \tag{3.9}$$

As $\phi_c(u)$, $\phi_h(u)$ and $\phi_s(u)$ all are quasi homogeneous functions of second degree, therefore ϕ is also second degree quasi-homogeneous on the interval $\left[0, \frac{p}{p+\sqrt{(q-p)p}}\right)$.

Lemma 3.3. *Let F contains all zeros of the polynomial $h(s) \in F[s]$ and ζ be one such multiple type zero, which has the multiplicity p. Let $s \in F$ satisfies the following*

$$I(s) < \tau_1 = \frac{p}{p + \sqrt{p(q-p)}}, \tag{3.10}$$

where I is given by equation (3.1) *and the value of τ_1 is defined by the above equation* (3.10). *Then, the following holds:*

(i) *s lies in D, where D is the domain set* (2.2) *of the method.*

(ii) *$|Ts - \zeta| \le \phi(I(s))|s - \zeta|$, the function ϕ given by equation* (3.9).

Proof. Let $s \in F$ such that the inequality (3.10) holds. If $s = \zeta$ or $p = q$ or both $s = \zeta$ and $p = q$ holds, then $Ts = \zeta$. Therefore, both of the lemma's claims are true. Now, we will consider the remaining case, i.e., $s \ne \zeta$ and $p \ne q$. Let us suppose that $\zeta_1, \dots, \zeta_m$ is the collection of all different zeros of the non-linear polynomial h and let $p_1, \dots, p_m$ be the respective multiplicity of the roots. Let $\zeta = \zeta_i$, $p = p_i$, $\alpha = \alpha_i$ and $\beta = \beta_i$ for some i $(1 \le i \le m)$, where α_i and β_i defined in Lemma 3.1.

In order to establish (i), we have to show that $h(s) \ne 0$ and $h'(s) \ne 0$ implies $1 - \frac{h(s)}{h'(s)} \frac{h''(s)}{h'(s)} \ne 0$ and $\frac{p+1}{2p} \frac{h'(s)}{h(s)} - \frac{1}{2} \frac{h''(s)}{h'(s)} \ne 0$.

Using Lemma 3.2 and (3.10), we get

$$|s - \zeta_j| \ge (1 - I(s))d > 0, \tag{3.11}$$

for all $j \ne i$. The above inequality (3.11) guarantees that $h(s) \ne 0$. Hence from Lemma 3.1, we can have

$$\frac{h'(s)}{h(s)} = \frac{p + \alpha}{s - \zeta}, \tag{3.12}$$

where $\alpha = (s - \zeta) \sum\limits_{j \ne i} \frac{p_j}{s - \zeta_j}$.

Consequently, the following can be obtained by employing the triangle and the aforementioned inequality (3.11),

$$|\alpha| \le |s - \zeta| \sum_{j \ne i} \frac{p_j}{|s - \zeta_j|} \le \frac{|s - \zeta|}{(1 - I(s))d} \sum_{j \ne i} p_j = \frac{(q-p)I(s)}{1 - I(s)}. \tag{3.13}$$

With the help of triangle inequality and (3.13) and as $I(s) < \tau_1 \le \frac{p}{q}$, we get the following:

$$|p + \alpha| \ge p - |\alpha| \ge p - \frac{(q-p)I(s)}{1 - I(s)} = \frac{p - qI(s)}{1 - I(s)} > 0. \tag{3.14}$$

This shows that $p + \alpha \ne 0$. Hence $h'(s) \ne 0$.

Therefore, Lemma 3.1 gives us

$$\frac{h''(s)}{h'(s)} = \frac{(p+\alpha)^2 - (p + \beta)}{(s - \zeta)(p + \alpha)}, \tag{3.15}$$

where $\beta = (s - \zeta)^2 \sum\limits_{j \ne i} \frac{p_j}{(s - \zeta_j)^2}$.

Using the equation (3.12) and equation (3.15), we get

$$1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}=1-\frac{(s-\zeta)}{p+\alpha}\frac{(p+\alpha)^2-(p+\beta)}{(s-\zeta)(p+\alpha)}=\frac{p+\beta}{(p+\alpha)^2}. \tag{3.16}$$

Following estimate can be obtained with the help of triangle inequality, equation (3.11) and $I(s)<\tau_1$,

$$|\beta|\leq\frac{(q-p)I(s)^2}{(1-I(s))^2} \quad \text{and} \quad |p+\beta|\geq p-|\beta|\geq\frac{\phi_{sd}(I(s))}{2(p-qI(s))(1-I(s))}>0. \tag{3.17}$$

From above, we conclude that

$$\left|1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right|>0.$$

Now it remains to prove that $\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\neq 0$,

$$\begin{aligned}\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}&=\frac{1}{2}\left(\frac{p+\alpha}{p}+\frac{p+\beta}{p+\alpha}\right)\frac{1}{s-\zeta}\\&=\left(1+\frac{\alpha^2+p\beta}{2p(p+\beta)}\right)\frac{1}{s-\zeta}\\&=\frac{1+\sigma_h}{s-\zeta},\end{aligned}$$

where

$$\sigma_h=\frac{\alpha^2+p\beta}{2p(p+\beta)}.$$

Now,

$$|\sigma_h|=\left|\frac{\alpha^2+p\beta}{2p(p+\beta)}\right|\leq\frac{|\alpha|^2+p|\beta|}{2p|(p+\beta)|}\leq\frac{q(q-p)I(s)^2}{2p(1-I(s))(p-qI(s))}.$$

Now,

$$|1+\sigma_h|\geq 1-|\sigma_h|\geq 1-\frac{1}{2p}\frac{q(q-p)I(s)^2}{(1-I(s))(p-qI(s))}=\frac{q(3p-q)I(s)^2-2p(p+q)I(s)+2p^2}{2p(1-I(s))(p-qI(s))}>0.$$

The above inequality assures that $1+\sigma_h\neq 0$.

Consequently, $s\in D$. This ensures the proof of the first portion of Lemma 3.3.

To prove (ii), we use the recurrence relation of our method,

$$\begin{aligned}Ts-\zeta&=s-\zeta-\frac{p^2}{6}\frac{h(s)}{h'(s)}\left(\frac{3-p}{p}+\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right)-\frac{1}{3}\left(\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\right)^{-1}-\frac{h(s)}{6h'(s)}\left(p+\frac{1}{1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}}\right)\\&=s-\zeta-\frac{(s-\zeta)}{3}\left[1-\frac{p}{2}\frac{3(p+\alpha)^2-p(p+\beta)}{(p+\alpha)^3}\right]-\frac{1}{3}\frac{s-\zeta}{1+\sigma_h}-\frac{(s-\zeta)}{6(p+\alpha)}\left[p+\frac{(p+\alpha)^2}{p+\beta}\right]\\&=\frac{s-\zeta}{3}\frac{\sigma_h}{1+\sigma_h}+\frac{(s-\zeta)}{3}\frac{2(p+\alpha)^3-3p(p+\alpha)^2+p^2(p+\beta)}{2(p+\alpha)^3}+\frac{(s-\zeta)}{3}\left[1-\frac{2p^2+p\beta+2p\alpha+\alpha^2}{2(p+\alpha)(p+\beta)}\right]\\&=\frac{s-\zeta}{3}\frac{\sigma_h}{1+\sigma_h}+\frac{(s-\zeta)}{3}\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]+\frac{(s-\zeta)}{3}\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\\&=\sigma(s-\zeta),\end{aligned}$$

where

$$\sigma = \frac{1}{3}\left(\left[\frac{\sigma_h}{1+\sigma_h}\right]+\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]+\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\right). \tag{3.18}$$

To find the estimate of $|\sigma|$, we use (3.13) and (3.14),

$$\begin{aligned}
|\sigma| &\le \frac{1}{3}\left(\left|\left[\frac{\sigma_h}{1+\sigma_h}\right]\right|+\left|\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]\right|+\left|\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\right|\right)\\
&\le \frac{1}{3}\left(\left[\frac{|\sigma_h|}{1-|\sigma_h|}\right]+\left[\frac{2|\alpha|^3+3p|\alpha|^2+p^2|\beta|}{2|(p+\alpha)|^3}\right]+\left[\frac{2|\alpha||\beta|+p|\beta|+|\alpha|^2}{2|(p+\alpha)||(p+\beta)|}\right]\right)\\
&\le \frac{1}{3}\frac{q(q-p)I(s)^2}{q(3p-q)I(s)^2-2p(p+q)I(s)+2p^2}+\frac{2\left(\frac{(q-p)I(s)}{1-I(s)}\right)^3+3p\left(\frac{(q-p)I(s)}{1-I(s)}\right)^2+p^2\frac{(q-p)I(s)^2}{(1-I(s))^2}}{6\left(\frac{p-qI(s)}{1-I(s)}\right)^3}\\
&\quad+\frac{2\frac{(q-p)I(s)}{1-I(s)}\frac{(q-p)I(s)^2}{(1-I(s))^2}+p\frac{(q-p)I(s)^2}{(1-I(s))^2}+\left(\frac{(q-p)I(s)}{1-I(s)}\right)^2}{6\frac{p-qI(s)}{1-I(s)}\frac{\phi_{sd}(I(s))}{2(p-qI(s))(1-I(s))}}\\
&=\frac{1}{3}[\phi_h(I(s))+\phi_c(I(s))+\phi_s(I(s))]\\
&=\phi(I(s)),
\end{aligned}$$

which proves (ii). □

Theorem 3.1. *Let F contains all zeros of the polynomial $h(s)\in F[s]$ and ζ be one such multiple type zero, which has the multiplicity p. Let $s_0\in F$ satisfies the following initial condition*

$$I(s_0)<\tau_1 \ \text{ and } \ \phi(I(s_0))<1. \tag{3.19}$$

Equation (3.1) *defines function $I:D\to R_+$ and equation* (3.9) *defines function ϕ. Then following holds.*

(i) *The method is defined, has an order of convergence of three, and converges to ζ.*

(ii) *The error estimations are as follows:*

$$|s_{m+1}-\zeta|\le\mu^{3^m}|s_m-\zeta| \text{ and } |s_m-\zeta|\le\mu^{(3^m-1)/2}|s_0-\zeta|, \quad \text{for all } m\ge 0, \tag{3.20}$$

where $\mu=\phi(I(s_0))$.

(iii) *Following provides a posteriori error estimate,*

$$|s_{m+1}-\zeta|<\frac{1}{(Rd)^2}|s_m-\zeta|^3, \quad \text{for all } m\ge 0, \tag{3.21}$$

here R is the unique solution of $\phi(t)=1$ in $(0,\tau_1)$.

Proof. Lemma 3.3 and [6, Theorem 1] gives the proof. □

3.2 Local Convergence Theorem of Second Kind

Let F be a field, which has the norm $|\cdot|$ and $F[s]$ be the ring of polynomials. Let $h(s)$ be a non-linear polynomial of degree q. Let F has all the zeros of $h(s)$. Suppose ζ is a zero of $h(s)$ having multiplicity p. We will use the function of initial conditions I to prove the convergence of the CHS mean iterative approach (2.1) and is defined as follows:

$$I(s)=I_h(s)=\frac{|(s-\zeta)|}{\rho(s)}, \tag{3.22}$$

where $\rho(s)$ is the distance from s to the closest zero of h other than ζ, if ζ is the sole zero of h then we assign $I(s)=0$.

We have defined three real valued functions ϑ_c, ϑ_h and ϑ_s. For $q>p\geq 1$, the functions are defined as follows:

$$\vartheta_c(u)=\frac{2(q-p)^3u^3+p(q-p)(3q-2p)u^2}{2(p-(q-p)u)^3}, \tag{3.23}$$

$$\vartheta_h(u)=\frac{q(q-p)u^2}{2p^2-2p(q-p)u-q(q-p)u^2} \tag{3.24}$$

and

$$\vartheta_s(u)=\frac{(q-p)(q+2(q-p)u)u^2}{2(p-(q-p)u)(p-(q-p)u^2)}. \tag{3.25}$$

Above defined three functions are quasi-homogeneous functions of second degree on $[0,\tau_2)$, where τ_2 is defined by

$$\tau_2=\frac{2p}{q+\sqrt{q^2+4(q-p)^2}}. \tag{3.26}$$

Therefore, a real valued function can be defined using the above three real valued functions. Let $\vartheta:[0,\tau_2)\to R_+$ be the function defined as follows:

$$\vartheta(u)=\frac{\vartheta_c(u)}{3}+\frac{\vartheta_h(u)}{3}+\frac{\vartheta_s(u)}{3}. \tag{3.27}$$

As all the functions $\vartheta_c(u)$, $\vartheta_h(u)$ and $\vartheta_s(u)$ are quasi-homogeneous of second degree, hence ϑ is also a second degree quasi-homogeneous in $[0,\tau_2)$.

Lemma 3.4. *Let F contains all zeros of the polynomial $h(s)\in F[s]$ and ζ be one such multiple type zero, which has the multiplicity p. Let $s\in F$ satisfies the following:*

$$I(s)<\tau_2, \tag{3.28}$$

here I is given in (3.22). *Then*

(i) $s\in D$. *Here D is the domain set* (2.2) *of the method.*

(ii) $|Ts-\zeta|\leq\vartheta(I(s))|s-\zeta|$, *where the function ϑ is defined in* (3.27).

Proof. Let $s\in F$ such that the inequality (3.28) hold. Now if $s=\zeta$ or $p=q$ or both $s=\zeta$ and $p=q$ are true, then $Ts=\zeta$. Therefore, both of the lemma's claims are true. Now, we will consider the remaining case, i.e., $s\neq\zeta$ and $p\neq q$. Suppose $\zeta_1,\dots,\zeta_m$ is the collection of all distinct zeros of the non-linear polynomial h and let $p_1,\dots,p_m$ be the respective multiplicity of the roots. Let $\zeta=\zeta_i$, $p=p_i$, $\alpha=\alpha_i$ and $\beta=\beta_i$ for some i $(1\leq i\leq m)$, where α_i and β_i defined in Lemma 3.1.

In order to establish (i), we have to show that $h(s)\neq 0$ and $h'(s)\neq 0$ implies $1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\neq 0$ and $\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\neq 0$.

For each $j\neq i$ we have the following

$$|s-\zeta_j|\geq\rho(s)>0, \tag{3.29}$$

which shows that $h(s)\neq 0$. Hence, Lemma 3.1 gives

$$\frac{h'(s)}{h(s)}=\frac{p+\alpha}{s-\zeta}, \tag{3.30}$$

where $\alpha = (s-\zeta)\sum_{j\neq i}\frac{p_j}{s-\zeta_j}$.

Applying the triangle inequality and (3.29), we get the following inequality,

$$|\alpha| \le |s-\zeta|\sum_{j\neq i}\frac{p_j}{|s-\zeta_j|} \le \frac{|s-\zeta|}{\rho(s)}\sum_{j\neq i}p_j = (q-p)I(s). \tag{3.31}$$

Applying the triangle inequality, inequality (3.31) and $I(s) < \tau_2$, we get the following:

$$|p+\alpha| \ge p - |\alpha| \ge p-(q-p)I(s) > 0. \tag{3.32}$$

This guarantees $p+\alpha \neq 0$. Hence $h'(s)\neq 0$.

Therefore, Lemma 3.1 gives us the following:

$$\frac{h''(s)}{h'(s)} = \frac{(p+\alpha)^2-(p+\beta)}{(s-\zeta)(p+\alpha)}, \tag{3.33}$$

where $\beta = (s-\zeta)^2\sum_{j\neq i}\frac{p_j}{(s-\zeta_j)^2}$.

Using (3.30) and (3.33), we deduce the following:

$$1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)} = 1-\frac{(s-\zeta)}{p+\alpha}\frac{(p+\alpha)^2-(p+\beta)}{(s-\zeta)(p+\alpha)} = \frac{p+\beta}{(p+\alpha)^2}. \tag{3.34}$$

Therefore, by triangle inequality, equation (3.29) and $I(s) < \tau_2$, we arrive at the following:

$$|\beta| \le (q-p)I(s)^2 \text{ and } |p+\beta| \ge p-|\beta| \ge p-(q-p)I(s)^2 \ge 0. \tag{3.35}$$

Applying inequality (3.35) in equation (3.34), we get

$$\left|1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right| > 0.$$

Now it remains to prove that $\frac{p+1}{2p}\frac{h'(s)}{h(s)} - \frac{1}{2}\frac{h''(s)}{h'(s)} \neq 0$,

$$\begin{aligned}\frac{p+1}{2p}\frac{h'(s)}{h(s)} - \frac{1}{2}\frac{h''(s)}{h'(s)} &= \frac{1}{2}\left(\frac{p+\alpha}{p}+\frac{p+\beta}{p+\alpha}\right)\frac{1}{s-\zeta}\\ &= \left(1+\frac{\alpha^2+p\beta}{2p(p+\beta)}\right)\frac{1}{s-\zeta}\\ &= \frac{1+\sigma_h}{s-\zeta},\end{aligned}$$

where

$$\sigma_h = \frac{\alpha^2+p\beta}{2p(p+\beta)}.$$

Now,

$$|\sigma_h| = \left|\frac{\alpha^2+p\beta}{2p(p+\beta)}\right| \le \frac{|\alpha|^2+p|\beta|}{2p|(p+\beta)|} \le \frac{q(q-p)I(s)^2}{2p(1-I(s))(p-qI(s))}.$$

Now,

$$|1+\sigma_h| \ge 1-|\sigma_h| \ge 1-\frac{q(q-p)I(s)^2}{2p(1-I(s))(p-qI(s))} = \frac{q(3p-q)I(s)^2-2p(p+q)I(s)+2p^2}{2p(1-I(s))(p-qI(s))} > 0.$$

Hence $1+\sigma_h \neq 0$.

Consequently, $s\in D$. This conclude the proof of (i).

To prove the second part we use the recurrence relation of our method,

$$\begin{aligned}
Ts-\zeta &= s-\zeta-\frac{p^2}{6}\frac{h(s)}{h'(s)}\left(\frac{3-p}{p}+\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}\right)-\frac{1}{3}\left(\frac{p+1}{2p}\frac{h'(s)}{h(s)}-\frac{1}{2}\frac{h''(s)}{h'(s)}\right)^{-1}-\frac{h(s)}{6h'(s)}\left(p+\frac{1}{1-\frac{h(s)}{h'(s)}\frac{h''(s)}{h'(s)}}\right)\\
&= s-\zeta-\frac{(s-\zeta)}{3}\left[1-\frac{p}{2}\frac{3(p+\alpha)^2-p(p+\beta)}{(p+\alpha)^3}\right]-\frac{1}{3}\frac{s-\zeta}{1+\sigma_h}-\frac{(s-\zeta)}{6(p+\alpha)}\left[p+\frac{(p+\alpha)^2}{p+\beta}\right]\\
&= \frac{s-\zeta}{3}\frac{\sigma_h}{1+\sigma_h}+\frac{(s-\zeta)}{3}\frac{2(p+\alpha)^3-3p(p+\alpha)^2+p^2(p+\beta)}{2(p+\alpha)^3}+\frac{(s-\zeta)}{3}\left[1-\frac{2p^2+p\beta+2p\alpha+\alpha^2}{2(p+\alpha)(p+\beta)}\right]\\
&= \frac{s-\zeta}{3}\frac{\sigma_h}{1+\sigma_h}+\frac{(s-\zeta)}{3}\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]+\frac{(s-\zeta)}{3}\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\\
&= \sigma(s-\zeta),
\end{aligned}$$

where σ is given by

$$\sigma=\frac{1}{3}\left(\left[\frac{\sigma_h}{1+\sigma_h}\right]+\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]+\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\right). \tag{3.36}$$

To estimate $|\sigma|$ we use the estimates (3.31), (3.32) and (3.35) and is given below:

$$\begin{aligned}
|\sigma| &\le \frac{1}{3}\left(\left|\left[\frac{\sigma_h}{1+\sigma_h}\right]\right|+\left|\left[\frac{2\alpha^3+3p\alpha^2+p^2\beta}{2(p+\alpha)^3}\right]\right|+\left|\left[\frac{2\alpha\beta+p\beta-\alpha^2}{2(p+\alpha)(p+\beta)}\right]\right|\right)\\
&\le \frac{1}{3}\left(\left[\frac{|\sigma_h|}{1-|\sigma_h|}\right]+\left[\frac{2|\alpha|^3+3p|\alpha|^2+p^2|\beta|}{2|(p+\alpha)|^3}\right]+\left[\frac{2|\alpha||\beta|+p|\beta|+|\alpha|^2}{2|(p+\alpha)||(p+\beta)|}\right]\right)\\
&\le \frac{2((q-p)I(s))^3+3p((q-p)I(s))^2+p^2(q-p)I(s)^2}{6(p-(q-p)I(s))^3}+\frac{1}{3}\frac{q(q-p)I(s)^2}{2p^2-2p(q-p)I(s)-q(q-p)I(s)^2}\\
&\quad+\frac{2(q-p)I(s)(q-p)I(s)^2+p(q-p)I(s)^2+((q-p)I(s))^2}{6(p-(q-p)I(s))(p-(q-p)I(s)^2)}\\
&= \frac{2(q-p)^3I(s)^3+p(q-p)(3q-2p)I(s)^2}{6(p-(q-p)I(s))^3}+\frac{1}{3}\frac{q(q-p)I(s)^2}{2p^2-2p(q-p)I(s)-q(q-p)I(s)^2}\\
&\quad+\frac{(q-p)(q+2(q-p)I(s))I(s)^2}{6(p-(q-p)I(s))(p-(q-p)I(s)^2)}\\
&= \frac{\vartheta_c(I(s))}{3}+\frac{\vartheta_h(I(s))}{3}+\frac{\vartheta_s(I(s))}{3}=\vartheta(I(s)),
\end{aligned}$$

which proves (ii). □

Next, we will state the second convergence theorem.

Theorem 3.2. *Let F contains all zeros of the polynomial $h(s)\in F[s]$ and ζ be one such multiple type zero, which has the multiplicity p. Let $s_0\in F$ satisfies the following conditions*

$$I(s_0)<\tau_2 \text{ and } \vartheta(I(s_0))\le\psi(I(s_0)), \tag{3.37}$$

where the function I is defined in equation (3.22) *and ψ satisfies the following:*

$$\psi(u)=1-u(1+\vartheta(u)). \tag{3.38}$$

Then, this CHS Mean Iterative Method is well-defined and converges to ζ and the error estimates are the following:

$$|s_{m+1}-\zeta|\le\theta\mu^{3^m}|s_m-\zeta| \text{ and } |s_{m+1}-\zeta|\le\theta^m\mu^{(3^m-1)/2}|s_0-\zeta|, \quad \text{for all } m\ge 0, \tag{3.39}$$

where $\theta = \psi(I(s_0))$ *and* $\mu = \frac{\vartheta(I(s_0))}{\psi(I(s_0))}$.

Proof. Proof the theorem can be established using Lemma 3.4 and [6, Theorem 2]. □

4. Conclusion

The work is done in two parts, firstly, we have constructed a third order iterative method with the help of three efficient third order iterative approach namely Chebyshev, Halley and Super-Halley. Secondly, we have established the convergence of the method.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P. Chebychev, *Collected Works*, Academy of Sciences of the USSR, Moscow-Leningrad, (1944). (in Russian)

[2] C. Chun and B. Neta, A third-order modification of Newton's method for multiple roots, *Applied Mathematics and Computation* **211**(2) (2009), 474 – 479, DOI: 10.1016/j.amc.2009.01.087.

[3] R. S. Das and A. Kumar, Construction and convergence of h-s combined mean method for multiple polynomial zeros, *Communications in Mathematics and Applications* **14**(5) (2023), 1679 – 1692, DOI: 10.26713/cma.v14i5.2329.

[4] J. M. Gutiérrez and M. A. Hernandez, An acceleration of Newtons method: Super-Halley method, *Applied Mathematics and Computation* **117**(2-3) (2001), 223 – 239, DOI: 10.1016/S0096-3003(99)00175-7.

[5] E. Halley, Methodus nova accurata & facilis inveniendi radices æqnationum quarumcumque generaliter, sine praviæ reductione, *Philosophical Transactions of the Royal Society of London* **18**(210) (1964), 136 – 148, DOI: 10.1098/rstl.1694.0029.

[6] S. I. Ivanov, General local convergence theorems about the Picard iteration in arbitrary normed fields with applications to Super-Halley method for multiple polynomial zeros, *Mathematics* **8**(9) (2020), 1599, DOI: 10.3390/math8091599.

[7] S. I. Ivanov, Unified convergence analysis of Chebyshev-Halley methods for multiple polynomial zeros, *Mathematics* **10**(1) (2022), 135, DOI: 10.3390/math10010135.

[8] B. Neta, New third order nonlinear solvers for multiple roots, *Applied Mathematics and Computation* **202**(1) (2008), 162 – 170, DOI: 10.1016/j.amc.2008.01.031.

[9] N. Obreshkov, Sur la solution numerique des equations, *Annuaire de l'Université de Sofia Faculté de Physique i Matematika* **56** (1963), 73 – 83. (in Bulgarian)

[10] N. Osada, Asymptotic error constants of cubically convergent zero finding methods, *Journal of Computational and Applied Mathematics* **196**(2) (2006), 347 – 357, DOI: 10.1016/j.cam.2005.09.016.

[11] P. D. Proinov, General convergence theorems for iterative processes and applications to the Weierstrass root-finding method, *Journal of Complexity* **33** (2016), 118 – 144, DOI: 10.1016/j.jco.2015.10.001.

[12] P. D. Proinov, New general convergence theory for iterative processes and its applications to Newtons-Kantorovich type theorems, *Journal of Complexity* **26**(1) (2010), 3 – 42, DOI: 10.1016/j.jco.2009.05.001.

[13] H. Ren and I. K. Argyros, Convergence radius of the modified Newton method for multiple zeros under Holder continuous derivative, *Applied Mathematics and Computation* **217**(2) (2010), 612 – 621, DOI: 10.1016/j.amc.2010.05.098.

[14] J. F. Traub, *Iterative Methods for the Solution of Equations*, American Mathematical Society, USA, 310 pages (1982).