Communications in Mathematics and Applications
Vol. 16, No. 1, pp. 327–337, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i1.2965

http://www.rgnpublications.com

Research Article

A Note on Linear Multiplier Fractional *q*-Differintegral Operator With Varying Arguments

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Received: December 23, 2024 **Revised:** April 17, 2025 **Accepted:** June 2, 2025

Abstract. We introduce new subclasses of analytic functions with varying arguments by making use of linear multiplier fractional q-differintegral operator. For functions belonging to these classes, we obtain coefficient estimates, distortion theorems, extreme points, q-Bernardi integral operator, and many more properties.

Keywords. Analytic function, Univalent function, Fractional q-differintegral operator, q-Bernardi operator

Mathematics Subject Classification (2020). Primary 30C45; Secondary 30C50

1. Introduction and Preliminaries

Suppose $\mathscr{A}$ represent the class of the functions of the form

$$\eta(\tau) = \tau + \sum_{\mathfrak{s}=2}^{\infty} a_{\mathfrak{s}} \tau^{\mathfrak{s}}, \tag{1.1}$$

where the functions are analytic in the open unit disc $\mathfrak{U} = \{\tau \in \mathfrak{C} : |\tau| < 1\}$, which satisfies the normalization conditions $\eta(0) = 0$ and $\eta'(0) = 1$. Suppose $\mathfrak{S}$ be the family of all subclasses of $\mathscr{A}$ that exhibit univalence in $\mathfrak{U}$.

Let $\mathfrak{P}[\mathfrak{X},\mathfrak{Y}]$ represents a family of functions $\mathfrak{P}(\tau)=1+\mathfrak{c}_1\tau+\cdots$, which are analytic in $\mathfrak{U}$ and satisfy the subordination condition

$$\mathfrak{P}(\tau)\prec\frac{1+\mathfrak{X}\tau}{1+\mathfrak{Y}\tau},\quad -1\leq\mathfrak{Y}<\mathfrak{X}\leq 1.$$

That family, referred to as the Janowski [8] class of functions, includes numerous other sets.

The q-gamma function, denoted as Γ_q, it is defined through the following recurrence relation:

$$\Gamma_q(\mathrm{t}+1)=[\mathrm{t}]_q\Gamma_q(\mathrm{t}),$$

where $[\mathrm{t}]_q=\frac{1-q^{\mathrm{t}}}{1-q}$ representing the q-analogue of t and $\Gamma_q(1)=1$.

Jackson's [9] q-derivative and q-integral of a function η, defined on a subset of $\mathfrak{C}$, are expressed as (see also, Gasper and Rahman [6]):

$$\mathfrak{D}_q\eta(y)=\frac{\eta(y)-\eta(qy)}{(1-q)y}$$

and

$$\int_0^z\eta(y)\mathfrak{d}_q y=(1-q)y\sum_{n=0}^{\infty}q^n\eta(q^n y),$$

where $y\neq 0$, $q\neq 1$ and $0<q<1$, note that $\mathfrak{D}_q\eta(0)=\eta'(0)$ and $\mathfrak{D}_q^2\eta(\tau)=\mathfrak{D}(\mathfrak{D}_q\eta(\tau))$.

The q-derivative for the function given by (1.1) is

$$\mathfrak{D}_q\eta(\tau)=1+\sum_{\mathfrak{s}=2}^{\infty}[\mathfrak{s}]_q a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}.$$

We readily observe that $[\mathfrak{s}]_q\to\mathfrak{s}$ as $q\to 1^-$.

Definition 1.1 ([11])**.** The definition of the fractional q-differintegral operator $\Omega^{\sigma}_{q,\tau}$ is presented as follows. For a function $\eta(\tau)$ of the form given in equation (1.1), we define

$$\Omega^{\sigma}_q\eta(\tau)=\Gamma_q(2-\sigma)\tau^{\sigma}\mathfrak{D}^{\sigma}_{q,\tau}\eta(\tau),$$

where $\mathfrak{D}^{\sigma}_{q,\tau}$ represents the fractional q-integral of order σ, when $-\infty<\sigma<0$, and the fractional q-derivative of order σ, when $0\leq\sigma<2$.

The expression for $\Omega^{\sigma}_q\eta(\tau)$ in terms of the coefficients $a_{\mathfrak{s}}$ from the power series expansion of $\eta(\tau)$ is given by

$$\Omega^{\sigma}_q\eta(\tau)=\tau+\sum_{\mathfrak{s}=2}^{\infty}\frac{\Gamma_q(\mathfrak{s}+1)\Gamma_q(2-\sigma)}{\Gamma_q(\mathfrak{s}+1-\sigma)}a_{\mathfrak{s}}\tau^{\mathfrak{s}}.$$

Definition 1.2 ([12])**.** A linear multiplier fractional q-differintegral operator is defined as follows:

$$\mathfrak{L}^{\sigma,0}_{q,\beta}\eta(\tau)=\eta(\tau),$$
$$\mathfrak{L}^{\sigma,1}_{q,\beta}\eta(\tau)=(1-\beta)\Omega^{\sigma}_q\eta(\tau)+\beta\tau\mathfrak{L}_q(\Omega^{\sigma}_q\eta(\tau)),$$
$$\mathfrak{L}^{\sigma,2}_{q,\beta}\eta(\tau)=\mathfrak{L}^{\sigma,1}_{q,\beta}(\mathfrak{L}^{\sigma,1}_{q,\beta}\eta(\tau)),$$
$$\vdots$$

and, more generally,

$$\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)=\mathfrak{L}^{\sigma,1}_{q,\beta}(\mathfrak{L}^{\sigma,n-1}_{q,\beta}\eta(\tau)).\tag{1.2}$$

If $\eta \in \mathscr{A}$ is expressed as in equation (1.1), then according to equation (1.2), we obtain

$$\mathfrak{L}_{q,\beta}^{\sigma,n}\eta(\tau) = \tau + \sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)a_{\mathfrak{s}}\tau^{\mathfrak{s}},$$

where

$$\mathfrak{h}\left(\mathfrak{s},\sigma,q,n,\beta\right) = \left(\frac{\Gamma_q(\mathfrak{s}+1)\Gamma_q(2-\sigma)}{\Gamma_q(\mathfrak{s}+1-\sigma)}[([\mathfrak{s}]_q-1)\beta+1]\right)^n,$$

and $0 \le \sigma < 2$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\beta > 0$, $0 < q < 1$.

Definition 1.3. Let $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ denotes the subclass of $\mathscr{A}$ consisting of functions $\eta(\tau)$ of the form given in equation (1.1) and satisfying the following analytic criterion:

$$(1-\delta)\frac{\mathfrak{L}_{q,\beta}^{\sigma,n}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}_{q,\beta}^{\sigma,n}\eta(\tau)) \prec \frac{1+\mathfrak{X}\tau}{1+\mathfrak{Y}\tau}, \tag{1.3}$$

where $-1 \le \mathfrak{X} < \mathfrak{Y} \le 1$, $n \in \mathbb{N} \cup \{0\}$, $q \in (0,1)$, $0 \le \sigma < 2$, $\tau \in \mathfrak{U}$, $\beta > 0$, and $\delta \ge 0$. Several notable subclasses have been established through specific parameter selections of $\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X}$, and $\mathfrak{Y}$. These specialized cases have been extensively investigated by various researchers in the field. The following significant subclasses emerge:

(1) For $\sigma = 0$, $\beta = 1$, and $q \to 1^-$, the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, reduces to class was investigated by Aouf *et al.* [1].

(2) For $\sigma = 0$, $\beta = 1$, and $q \to 1^-$, the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, reduces to class studied by Sivasubramanian *et al.* [15], with $b = 1$.

(3) For $\sigma = 0$, $\beta = 1$, $q \to 1^-$, $\mathfrak{X} = 2\alpha - 1$, and $\mathfrak{Y} = 1$, the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, reduces to the class studied by Aouf *et al.* [2], under the conditions $b = 1$ and $\delta \ge 0$.

(4) For $\sigma = 0$, $\beta = 1$, $q \to 1^-$, $\mathfrak{X} = 2\alpha - 1$, $\mathfrak{Y} = 1$ and $n = 0$, we have the class presented by Chunyi and Owa [5], under the conditions $0 \le \delta \le 1$.

(5) For $\sigma = 0$, $\beta = 1$, $q \to 1^-$, $\mathfrak{X} = 2\alpha - 1$, $\mathfrak{Y} = 1$, $n = 0$ and $\delta = 0$ the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, reduces to the classes studied by Chen [3,4] and Goel [7].

(6) For $\sigma = 0$, $\beta = 1$, $q \to 1^-$, $\mathfrak{X} = 2\alpha - 1$, $\mathfrak{Y} = 1$, $n = 1$ and $\delta = 0$ the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, reduces to the class studied by Srivastava and Owa [14].

Definition 1.4 ([13]). A function $\eta(\tau)$, defined by equation (1.1), is said to belong to the class $\mathfrak{T}(\theta_{\mathfrak{s}})$ if $\eta(\tau) \in \mathscr{A}$ and $arg(a_{\mathfrak{s}}) = \theta_{\mathfrak{s}}$, for all $\mathfrak{s} \ge 2$. Furthermore, if there exists a real number γ such that

$$\theta_{\mathfrak{s}} + (\mathfrak{s}-1)\gamma \equiv \pi (\text{mod } 2\pi),$$

then the function $\eta(\tau)$ is considered to be in the class $\mathfrak{T}(\theta_{\mathfrak{s}};\gamma)$. The union of $\mathfrak{T}(\theta_{\mathfrak{s}};\gamma)$ over all possible sequences $\{\theta_{\mathfrak{s}}\}$ and all real numbers γ is denoted by $\mathfrak{T}$.

Suppose $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ represents the subclass of $\mathfrak{T}$ that consists of functions $\eta(\tau)$ with the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$.

2. Coefficient Estimates

Theorem 2.1. *Let the function $\eta(\tau) \in \mathscr{A}$ be represented in the form given by equation* (1.1). *If*

$$\sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)\left[\delta([\mathfrak{s}]_q-1)+1\right](\mathfrak{Y}+1)|a_{\mathfrak{s}}| \le (\mathfrak{Y}-\mathfrak{X}), \tag{2.1}$$

then $\eta(\tau)$ belongs to the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, where $-1 \le \mathfrak{X} < \mathfrak{Y} \le 1$, $n \in \mathbb{N}\cup\{0\}$, $q \in (0,1)$, $0 \le \sigma < 2$, $\beta > 0$, and $\delta \ge 0$.

Proof. A function $\eta(\tau)$ of the form given in equation (1.1) be in the class $\mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ if and only if there is a function w, where $|w(\tau)| \le \tau$, such that

$$(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)) = \frac{1+\mathfrak{X}w(\tau)}{1+\mathfrak{Y}w(\tau)}. \tag{2.2}$$

Alternatively,

$$\left|\frac{(1-\delta)\left(\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau}\right) + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)) - 1}{\mathfrak{Y}\left[(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau))\right] - \mathfrak{X}}\right| < 1. \tag{2.3}$$

Therefore, it suffices to establish that

$$\left|(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)) - 1\right| - \left|\mathfrak{Y}\left[(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau))\right] - \mathfrak{X}\right| < 0. \tag{2.4}$$

By setting $|\tau| = \mathfrak{r}$, where $0 \le \mathfrak{r} < 1$, yields the following

$$\begin{aligned}
&\left|(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)) - 1\right| - \left|\mathfrak{Y}\left[(1-\delta)\frac{\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau)}{\tau} + \delta\mathfrak{D}_q(\mathfrak{L}^{\sigma,n}_{q,\beta}\eta(\tau))\right] - \mathfrak{X}\right| \\
&\quad = \left|\sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1]a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}\right| \\
&\qquad - \left|(\mathfrak{Y}-\mathfrak{X}) + \sum_{\mathfrak{s}=2}^{\infty} \mathfrak{Y}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1]a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}\right| \\
&\quad \le \left[\sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1} - (\mathfrak{Y}-\mathfrak{X})\right. \\
&\qquad \left. + \sum_{\mathfrak{s}=2}^{\infty} \mathfrak{Y}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}\right] \\
&\quad < \sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1](1+\mathfrak{Y})|a_{\mathfrak{s}}| - (\mathfrak{Y}-\mathfrak{X}).
\end{aligned}$$

In light of equation (2.1), the preceding inequality is less than zero, which consequently implies that $\eta(\tau) \in \mathfrak{R}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$. □

Theorem 2.2. *Let the function $\eta(\tau) \in \mathscr{A}$ be represented in the form given by equation* (1.1)*. Then, $\eta(\tau)$ belongs to the class $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ if and only if*

$$\sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1](\mathfrak{Y}+1)|a_{\mathfrak{s}}| \le (\mathfrak{Y}-\mathfrak{X}), \tag{2.5}$$

where $-1 \le \mathfrak{X} < \mathfrak{Y} \le 1$, $n \in \mathbb{N}\cup\{0\}$, $q \in (0,1)$, $0 \le \sigma < 2$, $\beta > 0$, and $\delta \ge 0$.

Proof. Given Theorem 2.1, it suffices to show that each function $\eta(\tau)$ from the class $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ satisfies the inequality given in equation (2.1).

Consider $\eta(\tau)\in\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y}))$. Then, according to equations (2.3) and (1.1), we obtain

$$\left|\frac{\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}}{(\mathfrak{Y}-\mathfrak{X})+\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{Y}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}}\right|<1. \tag{2.6}$$

Since $\eta(\tau)\in\mathfrak{T}$, it follows that $\eta(\tau)$ belongs to the class $\mathfrak{T}(\theta_{\mathfrak{s}},\gamma)$ for some sequence $\{\theta_{\mathfrak{s}}\}$ and a real number γ, such that $\theta_{\mathfrak{s}}+(\mathfrak{s}-1)\gamma\equiv\pi(\text{mod }2\pi)$ for $\mathfrak{s}\geq 2$. By setting $\tau=\mathfrak{r}\mathfrak{e}^{j\gamma}$ into the inequality (2.6), and noting that $\Re\mathfrak{e}\{w(\tau)\}\leq|w(\tau)|<1$, we have

$$\frac{\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}}{(\mathfrak{Y}-\mathfrak{X})-\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{Y}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}}<1.$$

Obviously,

$$(\mathfrak{Y}-\mathfrak{X})-\mathfrak{Y}\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}\neq 0,\quad 0\leq\mathfrak{r}<1.$$

Furthermore,

$$(\mathfrak{Y}-\mathfrak{X})-\mathfrak{Y}\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1]|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}>0,\quad \mathfrak{r}=0.$$

Therefore, we deduce

$$\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[1+\delta([\mathfrak{s}]_q-1)](1+\mathfrak{Y})|a_{\mathfrak{s}}|\mathfrak{r}^{\mathfrak{s}-1}\leq(\mathfrak{Y}-\mathfrak{X}),\quad \mathfrak{r}\in[0,1).$$

Setting $\mathfrak{r}\to 1^-$ directly establishes the assertion presented in equation (2.5). □

Corollary 2.3. *Suppose the function* $\eta(\tau)\in\mathscr{A}$*, defined by equation* (1.1)*, belongs to the class* $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y}))$*. Then*

$$|a_{\mathfrak{s}}|\leq\frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)},\quad \mathfrak{s}\geq 2.$$

This result is optimal for the function given by

$$\eta(\tau)=\tau+\frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)}\mathfrak{e}^{j\theta_{\mathfrak{s}}}\tau^{\mathfrak{s}},\quad \mathfrak{s}\geq 2. \tag{2.7}$$

3. Distortion Theorems

Theorem 3.1. *Let the function* $\eta(\tau)\in\mathscr{A}$*, defined by equation* (1.1)*, belongs to the class* $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y}))$*. Then*

$$|\tau|-\frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2$$
$$\leq|\eta(\tau)|\leq|\tau|+\frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2,$$

where

$$\mathfrak{h}(2,\sigma,q,n,\beta)=\left(\frac{\Gamma_q(3)}{[2-\sigma]_q}\right)^n\left[1+([2]_q-1)\beta\right]^n.$$

This result is considered sharp.

Proof. Given that

$$\varphi(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}_q]-1)+1](\mathfrak{Y}+1),$$

this expression is a non-decreasing function of $\mathfrak{s}$ for $\mathfrak{s} \geq 2$. According to Theorem 2.1, the following result is achieved:

$$\begin{aligned}\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}| &\leq \sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)|a_\mathfrak{s}| \\ &\leq (\mathfrak{Y}-\mathfrak{X}).\end{aligned}$$

In other words,

$$\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}| \leq \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}. \tag{3.1}$$

This latter inequality, in combination with the following inequality,

$$|\eta(\tau)| \leq |\tau| + |\tau|^2\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}|$$

implies

$$|\eta(\tau)| \leq |\tau| + \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2.$$

Likewise, we achieve

$$\begin{aligned}|\eta(\tau)| &\geq |\tau| - |\tau|^2\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}| \\ &\geq |\tau| - \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&|\tau| - \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2 \\ &\quad \leq |\eta(\tau)| \leq |\tau| + \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|^2.\end{aligned}$$

The result is precise for the function

$$\eta(\tau) = \tau + \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}\mathfrak{e}^{j\theta_2}\tau^2, \tag{3.2}$$

at $\tau = \pm|\tau|e^{-j\theta_2}$. □

Corollary 3.2. *Under the assumptions of Theorem* 3.1*, the function* $\eta(\tau)$ *contained within a disk centered at the origin, with a radius* $\mathfrak{r}_1$ *given by*

$$\mathfrak{r}_1 = 1 + \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}.$$

Theorem 3.3. *Suppose that* $\eta(\tau) \in \mathscr{A}$*, be of the form* (1.1)*, belongs to the class* $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$*. Then*

$$1 - \frac{(\mathfrak{Y}-\mathfrak{X})(1+q)}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}|\tau|$$

$$\le |\mathfrak{D}_q\eta(\tau)| \le 1 + \frac{(\mathfrak{Y}-\mathfrak{X})(1+q)}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1]q(\mathfrak{Y}+1)}|\tau|. \tag{3.3}$$

The result is exact for the given function (3.2)*.*

Proof. Given that $\eta(\tau)$ belongs to $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$, by using Theorem 2.1, we need to prove

$$\Phi(\mathfrak{s}) = \left(\frac{\Gamma_q(\mathfrak{s}+1)\Gamma_q(2-\sigma)}{\Gamma_q(\mathfrak{s}+1-\sigma)}[([\mathfrak{s}]_q-1)\beta+1]\right)^n, \quad \mathfrak{s}\ge 2$$

is a monotonic non-decreasing function of $\mathfrak{s}$ for $0\le\sigma<2$, and thus, we obtain

$$\frac{\Phi(1+\mathfrak{s})}{\Phi(\mathfrak{s})} = \left(\frac{\frac{\Gamma_q(\mathfrak{s}+2)\Gamma_q(2-\sigma)}{\Gamma_q(\mathfrak{s}+2-\sigma)}[1+\beta([\mathfrak{s}+1]_q-1)]}{\frac{\Gamma_q(\mathfrak{s}+1)\Gamma_q(2-\sigma)}{\Gamma_q(\mathfrak{s}+1-\sigma)}[1+\beta([\mathfrak{s}]_q-1)]}\right)^n$$
$$= \frac{[1+\beta([\mathfrak{s}+1]_q-1)]^n(1-q^{\mathfrak{s}+1})^n}{[1+\beta([\mathfrak{s}]_q-1)]^n(1-q^{\mathfrak{s}+1-\sigma})^n}.$$

Clearly, that $\Phi(\mathfrak{s})$ is a non-decreasing function of $\mathfrak{s}$ if $\frac{\Phi(1+\mathfrak{s})}{\Phi(\mathfrak{s})}\ge 1$. This condition yields

$$\frac{[1+\beta([\mathfrak{s}+1]_q-1)]^n(1-q^{\mathfrak{s}+1})^n}{[1+\beta([\mathfrak{s}]_q-1)]^n(1-q^{\mathfrak{s}+1-\sigma})^n}\ge 1, \quad 0<q<1.$$

Therefore, $\Phi(\mathfrak{s})$ is an increasing function of $\mathfrak{s}$, for $0\le q\le 1$, $0\le\sigma\le 2$,

$$\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}| \le \sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)|a_\mathfrak{s}|$$
$$\le (\mathfrak{Y}-\mathfrak{X}).$$

To be precise,

$$\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}| \le \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(2,\sigma,q,n,\beta)[\delta([2]_q-1)+1](\mathfrak{Y}+1)}. \tag{3.4}$$

Hence,

$$|\mathfrak{D}_q\eta(\tau)| = \left|1+\sum_{\mathfrak{s}=2}^{\infty}[\mathfrak{s}]_q a_\mathfrak{s}\tau^{\mathfrak{s}-1}\right| \le 1+|\tau|[2]_q\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}|,$$

along with

$$|\mathfrak{D}_q\eta(\tau)| = \left|1-\sum_{\mathfrak{s}=2}^{\infty}[\mathfrak{s}]_q a_\mathfrak{s}\tau^{\mathfrak{s}-1}\right| \ge 1-|\tau|[2]_q\sum_{\mathfrak{s}=2}^{\infty}|a_\mathfrak{s}|.$$

By combining equation (3.4) with the preceding expression, the result directly leads to equation (3.3). For the function of the form (3.2), it is evident that the result is exact. □

4. Extreme Points

Theorem 4.1. *Consider the function* $\eta(\tau)\in\mathscr{A}$*, be of the form given by* (1.1)*, belong to the class* $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$*, with* $\arg(a_\mathfrak{s})=\theta_\mathfrak{s}$*, where* $\theta_\mathfrak{s}+(\mathfrak{s}-1)\gamma\equiv\pi(\text{mod } 2\pi)$*. Now define that* $\eta_1(\tau)=\tau$ *along with*

$$\eta_\mathfrak{s}(\tau) = \tau + \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)}\mathfrak{e}^{j\theta_\mathfrak{s}}\tau^\mathfrak{s}.$$

Then, $\eta(\tau) \in \mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$ *if and only if* $\eta(\tau)$ *can be written as*

$$\eta(\tau) = \sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} \eta_{\mathfrak{s}}(\tau),$$

where $\lambda_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \geq 1$ *with* $\sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} = 1$.

Proof. Consider that $\eta(\tau) = \sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} \eta_{\mathfrak{s}}(\tau)$ along with $\sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} = 1$ with $\lambda_{\mathfrak{s}} \geq 0$, thus

$$\begin{aligned}
&\sum_{\mathfrak{s}=2}^{\infty} \mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1](\mathfrak{Y}+1) \frac{(\mathfrak{Y}-\mathfrak{X})}{\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1](\mathfrak{Y}+1)} \lambda_{\mathfrak{s}} \\
&\quad = \sum_{\mathfrak{s}=2}^{\infty} (\mathfrak{Y}-\mathfrak{X}) \lambda_{\mathfrak{s}} \\
&\quad = (\mathfrak{Y}-\mathfrak{X})(1-\lambda_1) \\
&\quad \leq (\mathfrak{Y}-\mathfrak{X}).
\end{aligned}$$

Therefore, $\eta(\tau)$ belong to the class $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$.

In contrast, suppose the function $\eta(\tau)$, be of the form given by (1.1), be in the class $\mathfrak{T}(\mathfrak{s},\sigma,q,\beta,n,\mathfrak{X},\mathfrak{Y})$. Let us define $\lambda_{\mathfrak{s}}$ as

$$\lambda_{\mathfrak{s}} = \frac{\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q - 1) + 1](\mathfrak{Y}+1)}{(\mathfrak{Y}-\mathfrak{X})} |a_{\mathfrak{s}}|$$

and

$$\lambda_1 = 1 - \sum_{\mathfrak{s}=2}^{\infty} \lambda_{\mathfrak{s}}.$$

Using Theorem 2.2, $\sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} \leq 1$ and hence $\lambda_{\mathfrak{s}} \geq 0$. Since $\lambda_{\mathfrak{s}} \eta_{\mathfrak{s}}(\tau) = \lambda_{\mathfrak{s}} \tau + a_{\mathfrak{s}} \tau^{\mathfrak{s}}$, thus

$$\sum_{\mathfrak{s}=1}^{\infty} \lambda_{\mathfrak{s}} \eta_{\mathfrak{s}}(\tau) = \tau + \sum_{\mathfrak{s}=2}^{\infty} a_{\mathfrak{s}} \tau^{\mathfrak{s}} = \eta(\tau).$$

□

5. q-Bernardi Integral Operator

The q-analogous of the Bernardi integral operator is defined as:

$$\mathfrak{B}_{\rho,q}\eta(\tau) = \frac{[\rho+1]_q}{\tau^{\rho}} \int_0^{\tau} \mathfrak{t}^{\rho-1} \eta(\tau) \mathfrak{d}_q \mathfrak{t}, \quad \rho = 1,2,3,\dots \tag{5.1}$$

was presented by Noor *et al*. [10].

Theorem 5.1. *Let* $\eta(\tau) \in \mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$. *Then* $\mathfrak{B}_{\rho,q}\eta(\tau) \in \mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$.

Proof. By using equation (5.1), we obtain

$$\begin{aligned}
\mathfrak{B}_{\rho,q}\eta(\tau) &= \frac{[\rho+1]_q}{\tau^{\rho}} \tau(1-q) \sum_{\iota=0}^{\infty} q^{\iota} (\tau q^{\iota})^{\rho-1} \eta(\tau q^{\iota}) \\
&= [\rho+1]_q (1-q) \sum_{\iota=0}^{\infty} q^{\iota\rho} \eta(\tau q^{\iota}) \\
&= [\rho+1]_q (1-q) \sum_{\mathfrak{s}=1}^{\infty} q^{\iota\rho} \sum_{\iota=0}^{\infty} q^{\iota\mathfrak{s}} |a_{\mathfrak{s}} \tau^{\mathfrak{s}}
\end{aligned}$$

$$= [\rho+1]_q \sum_{\iota=0}^{\infty}\sum_{\mathfrak{s}=1}^{\infty}(1-q)q^{\iota(\rho+\mathfrak{s})}|a_{\mathfrak{s}}|\tau^{\mathfrak{s}}$$
$$= \tau + \sum_{\mathfrak{s}=2}^{\infty}\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}a_{\mathfrak{s}}\tau^{\mathfrak{s}}.$$

Given that $\eta(\tau)$ belongs to the class $\mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\mathfrak{X},\mathfrak{Y})$ and noting that $\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q} < 1$ ($\forall\ \mathfrak{s} \ge 2$), we achieve

$$\sum_{\mathfrak{s}=2}^{\infty}\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)|a_{\mathfrak{s}}|\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q} \le (\mathfrak{Y}-\mathfrak{X}).$$ □

Theorem 5.2. *If* $\eta \in \mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$. *Then* $\mathfrak{B}_{\rho,q}\eta(\tau)$ *is q-starlike of order* $0 \le \kappa \le 1$ *in* $|\tau| < \mathscr{Y}_1$ *where*

$$\mathscr{Y}_1 = \inf\left\{\left(\frac{[\rho+\mathfrak{s}]_q}{[1+\rho]_q}\frac{(1-\kappa)\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)}{([\mathfrak{s}]_q-\kappa)(\mathfrak{Y}-\mathfrak{X})}\right)^{\frac{1}{\mathfrak{s}-1}} : \mathfrak{s}\in\mathbb{N}\setminus\{0\}\right\}. \tag{5.2}$$

Proof. It is enough to show that

$$\left|\frac{\tau\mathfrak{D}_q(\mathfrak{B}_{q,\rho}\eta(\tau))}{\mathfrak{B}_{q,\rho}\eta(\tau)}-1\right| < 1-\kappa, \quad \tau\in\mathfrak{U},$$

$$\left|\frac{\tau\mathfrak{D}_q(\mathfrak{B}_{q,\rho}\eta(\tau))}{\mathfrak{B}_{q,\rho}\eta(\tau)}-1\right| = \left|\frac{\sum_{\mathfrak{s}=2}^{\infty}([\mathfrak{s}]_q-1)\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}a_{\mathfrak{s}}\tau^{\mathfrak{s}-1}}{1+\sum_{\mathfrak{s}=2}^{\infty}a_{\mathfrak{s}}\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}\tau^{\mathfrak{s}-1}}\right|$$
$$\le \frac{\sum_{\mathfrak{s}=2}^{\infty}([\mathfrak{s}]_q-1)\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}|a_{\mathfrak{s}}||\tau^{\mathfrak{s}-1}|}{1-\sum_{\mathfrak{s}=2}^{\infty}a_{\mathfrak{s}}\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}|\tau^{\mathfrak{s}-1}|}.$$

Since

$$\frac{\sum_{\mathfrak{s}=2}^{\infty}([\mathfrak{s}]_q-1)\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}|a_{\mathfrak{s}}||\tau^{\mathfrak{s}-1}|}{1-\sum_{\mathfrak{s}=2}^{\infty}a_{\mathfrak{s}}\frac{[1+\rho]_q}{[\rho+\mathfrak{s}]_q}|\tau^{\mathfrak{s}-1}|} \le 1-\kappa.$$

We conclude

$$|\tau^{\mathfrak{s}-1}| \le \left(\frac{[\rho+\mathfrak{s}]_q}{[1+\rho]_q}\right)\frac{(1-\kappa)\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[\delta([\mathfrak{s}]_q-1)+1](\mathfrak{Y}+1)}{([\mathfrak{s}]_q-\kappa)(\mathfrak{Y}-\mathfrak{X})},$$

which yields equation (5.2). □

Theorem 5.3. *Let* $\eta \in \mathfrak{T}(\mathfrak{s},\sigma,q,n,\beta,\delta,\mathfrak{X},\mathfrak{Y})$. *Then* $\mathfrak{B}_{\rho,q}\eta(\tau)$ *is q-convex of order* $0 \le \kappa \le 1$ *in* $|\tau| < \mathscr{Y}^*$, *where*

$$\mathscr{Y}^* = \inf\left\{\left(\frac{[\rho+\mathfrak{s}]_q}{[1+\rho]_q}\frac{(1-\kappa)\mathfrak{h}(\mathfrak{s},\sigma,q,n,\beta)[1+\delta([\mathfrak{s}]_q-1)](1+\mathfrak{Y})}{[\mathfrak{s}]_q([\mathfrak{s}]_q-\kappa)(\mathfrak{Y}-\mathfrak{X})}\right)^{\frac{1}{\mathfrak{s}-1}} : \mathfrak{s}\in\mathbb{N}\setminus\{0\}\right\}. \tag{5.3}$$

6. Conclusion

In this study, we introduced new subclasses of analytic functions with varying arguments, utilizing a linear multiplier fractional q-differintegral operator. For functions within these subclasses, we derived coefficient estimates, established distortion theorems, identified extreme points, and investigated the q-Bernardi integral operator, q-starlike of order κ and q-convex of order κ.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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