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Research Article

Some Common Fixed-Point Theorems in Fuzzy Metric Space in the Context of Single and Set-Valued Mapping via OWC Mapping

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Abstract. This study aims to construct some common theorems in FMS for two pairs of single and set-valued OCM mappings that satisfy integral type contractive requirements. Our findings expand upon and generalize a number of related findings from previous research for independent of continuity and completeness.

Keywords. Fuzzy Metric Space (FMS), Common Fixed Point (CFP), Occasionally Weakly Compatible Mapping (OWC), Single Valued Mapping (SVM), Set-Valued Mapping (SEVM)

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1. Introduction

Other writers were able to provide fixed point solutions for fuzzy mapping, but Zadeh's concept of a fuzzy set, first introduced in 1965 [30], proved to be a turning moment in the history of mathematics and established the groundwork for fuzz mathematics. In 1975, Kramosil and Michálek [18] presented the new idea of FMS using continuous t-norms. common fixed point theorems for non-Archimedean FMS that apply to SVM and SEVM were proven by Samanta and Mohinta [24]. Bouhadjera and Djoudi [6] demonstrated a few fixed point theorems for common fixed point theorems for maps with SVM and SEVM meeting a rigorous contractive requirement without continuity. The Banach Type fixed Point Theorem for SEVM on a FMS was demonstrated by Sastry *et al*. [25]. Rezapour and Samet [23] offered the (α-ψ)-contractive and α-admissible mapping, also constructed some FPT. Hong [12] presented the (α-ψ)-contractive for set valued mapping in FMS. There is currently a sizable and extensive body of research in this field. Several fixed point results for SEM and SEVM have been proven in recent years and have a wide range of applications. Theorems for typical fixed locations for maps with SVM and SEVM are fascinating and essential in numerous fields. Jinakul *et al*. [14] demonstrated fixed point and common fixed point findings for multi-valued mapping in b-metric space. The notion of compatibility was recently undermined by Gupta *et al*. [11] by demonstrating several fixed point findings for SVM and SEVM. This led to the concept of OWC is the simple one of all types of commutativity views

The presence of fuzzy fixed points in metric and FMS of SEVM was recently demonstrated by Kanwal *et al*. [17]. In order to our study aims to propose the notion that single-valued and set-valued maps in FMS can occasionally be weakly compatible and to show in fixed point theory, common fixed point outcomes. We loosened the space's completeness and continuity in this paper. In this study, we use various unique incorporating integral type generalized contractions to FMS. Examples and applications that demonstrate and corroborate our obtained results have been included.

2. Preludes

Definition 2.1 ([18]). A map $*:[0,1]\times[0,1]\to[0,1]$ is called continuous triangular norm, if it is satisfied: $a,b,c,d\in[0,1]$:

(i) (Symmetry) $a*b=b*a$;

(ii) (Monotonicity) $a*b\le c*d$ if $a\le c$ and $b\le d$;

(iii) (Associativity) $a*(b*c)=(a*b)*c$;

(iv) (Boundary condition) $a*1=a$.

Definition 2.2 ([7]). The 3-tuple $(X,\mathcal{M},*)$ is known as FMS if X is an arbitrary set, $*$ is t-norm and $\mathcal{M}$ is a fuzzy set on $X\times X\times[0,\infty)$ such that for all $x,y,z\in X$ and $p,q\ge 0$, then

(i) $\mathcal{M}(x,y,0)=0$;

(ii) $\mathcal{M}(x,y,t)=1$, for all $t>0$ if and only if $x=y$;

(iii) $\mathcal{M}(x,y,t)=\mathcal{M}(y,x,t)$;

(iv) $\mathcal{M}(x,z,p+q)\ge\mathcal{M}(x,y,p)*\mathcal{M}(y,z,q)$;

(v) $\mathcal{M}(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.

Example 2.1 ([18]). Let (X,d) be a metric space. Define $u*v=\min\{u,v\}$ (or $u*v=uv$), for all $u,v\in[0,1]$. Then fuzzy metric may define as

$$M_0(u,v,t)=\frac{t}{t+d(x_0,y)},\quad \text{for all } x,y\in X \text{ and } t>0.$$

Example 2.2 ([18]). Let $X[0,\infty)$, $u*v=uv$ for every, $u,v\in[0,1]$ and d is usual metric defined on X. Define a function $M_0(x,y,t)=e^{-\frac{d(x_0,y)}{t}}$; t and $x,\ y\in X$, $t>0$ then $(X,\mathcal{M},*)$ is a FMS.

Note 4′. $M_0(x,z,\max\{p,q\})\geq M_0(x,y,p)*M_0(y,z,q)$; if the condition (iv) of Definition 2.2 is replaced by (4′) then FMS $(X,\mathcal{M},*)$ is called non-Archimedean FMS. All non-Archimedean FMSs are FMSs as well.

Definition 2.3 ([4]). A mapping that has a single value for each point in the domain within its range is called a SVM. It is therefore many-to-one or one-to-one.

Definition 2.4 ([4]). A mathematical function called a SEVM, or correspondence, is a kind of mapping that moves elements from one function domain (a set) to sub-domains of another set. Another word for it is multi-valued mapping.

Definition 2.5 ([22]). Let $CB(X)\neq\emptyset$ bounded $\subset FMS(X,\Omega,*)$. For $A,B\in CB(X)$ and $t>0$. Define

$$H(A,B,t)=\sup\{\Omega(x_0,y,t):x_0\in A,\ y_0\in B\},$$

$$\text{and } \delta_H(A,B,t)=\inf\{\Omega(x,y,t):x\in A,\ y\in B\},$$

if $A=x$ consist of single point, then $\delta_H(A,B,t)=\Omega(x,B,t)$,

if $A=x,\ B=y,$ then $\delta_H(A,B,t)=\Omega(x,y,t)$.

It follows immediately from definition that

$$\delta_H(A,B,t)=\delta_H(B,A,t)\geq 0,$$

$$\delta_H(A,B,t)=1\iff A=B=\{x\},\quad \text{for all } A,B\in CB(X),$$

and let δ_H be the Hausdorff FMS on $CB(X)$ for every A,B in $CB(X)$,

$$\delta_H(A,B,t)=\min\Big\{\inf_{y\in B}\Omega(A,y,t),\inf_{x\in A}\Omega(x,B,t)\Big\}.$$

Definition 2.6 ([22]). Given a FMS $(X,d,*)$, a sequence $\{x_n\}$ in a FMS $(X,d,*)$ to a point $x\in X$ if $\lim_{n\to\infty}\mathcal{M}(x_n,x,t)=1$, for all $t>0$.

Definition 2.7 ([22]). Given a FMS $(X,d,*)$, a sequence $\{x_n\}$ in a (X,d,t) is referred to be the Cauchy sequence if and only if all $\epsilon\in(0,1)$ and $t>0$ there exists $n_0\in N$ such that

$$\lim_{n\to\infty}\mathcal{M}(x_n,x_m,t)=1-\epsilon,\quad \text{for all } n,m>n_0.$$

Definition 2.8 ([11]). Every Cauchy sequence converges in a complete FMS.

Definition 2.9 ([6]). $A:X_0\to X$ and $B:X\to CB(X)$ are compatible. If $ABx\in CB(X)$ for all $x\in X,\ t>0$ and $\lim_{n\to\infty}H(ABx_n,BAx_n,t)=1$, and whenever $\{x_n\}$ is a series with in X that $Ax_n\to x\in M$ and $Bx_n\to M\in CB(X)$.

Definition 2.10 ([6]). The maps $B:X\to CB(X)$ and $A:X\to X$, $Ax\in Bx$ then a point $x\in X$ is referred to as a coincidence point (respectively, $x=Ax\in Bx$).

Definition 2.11 ([6])**.** The maps $A : X \to X$ and $B : X \to CB(X)$ commute at coincidence points, i.e., $ABx = BAx$ whenever $Ax \in Bx$, then they are considered weakly compatible.

Definition 2.12 ([6])**.** If there is a point x in X such that $Ax \in Bx$ and $ABx \subseteq BAx$, then the maps $A : X \to X$ and $B : X \to CB(X)$ are said to be sometimes weakly compatible (OWC).

Example 2.3 ([7])**.** Let R be the usual metric space. Let $A : X \to X$ and $B : X \to CB(X)$ by $Ax = 3x$ and $Bx = x^2$, for all $x \in R$. Then $Ax = Bx$ for $x = 0,3$, but $AB(0) = BA(0)$ and $AB(3) \neq BA(3)$. Because of this, A and B are sometimes weakly compatible but not always so.

Example 2.4 ([7])**.** Let $X = [0,\infty)$ with $a * b = \min\{a,b\}$ for all $a,b \in [0,1]$ and $\Omega(x,y,t) = \frac{t}{t+d(x,y)}$, $t > 0$. Let the maps $A : X \to X$ and $B : X \to CB(X)$ by

$$Ax = \begin{cases} 0, & 0 \le x < 1, \\ x+1, & 1 \le x < \infty \end{cases} \quad \text{and} \quad Bx = \begin{cases} \{0\}, & 0 \le x < 1, \\ [1,x+2], & 1 \le x < \infty. \end{cases}$$

Then $Ax = Bx$ for $x = 1$ but $AB(1) = [2,4] \neq BA(1) = [1,4]$, $A(0) \in B(0)$ and $AB(0 \subseteq BA(0))$, that is, $A\{0\} = 0 \subseteq B(0) = \{0\}$, indicate that A and B are not weakly compatible. Therefore, A and B are weakly compatible when $x = 0$, they are also OWC.

Lemma 2.1 ([16])**.** *Let $\{A_n\}$ and $\{B_n\}$ in $CB(X)$ to A and B in $CB(X)$. Then*

$$\delta_H(A_n,B_n,t) \to \delta_H(A,B,t) \text{ as } n \to \infty, \text{ for all } t > 0.$$

Lemma 2.2 ([16])**.** *Let $\{A_n\}$ and $\{B_n\}$ in $CB(X)$, then $\Omega(x,B,t) \ge \delta_H(A,B,t)$ for any $x \in A$.*

3. Main Result

Theorem 3.1. *Assume that $F,G : X \to CB(X)$ is a SEVM, $f,g : X \to X$ is a SVM. The pairs $\{f,F\}$ and $\{g,G\}$ are sometimes weakly compatible. Let $\varphi : R^5 \to R$ such that $\varphi(t,1,1,t*t) > 1$ and $0 < t < 1$ and satisfies the condition:*

$$\int_0^{\delta_H(Fx,Gy,t)} \varphi(t)dt \ge \int_0^{M(x,y,t)} \varphi(t)dt, \tag{3.1}$$

$$M(x,y,t) = \varphi\{H(fx,gy,t),\Omega(fx,Fx,t),\Omega(gy,Gy,t),\Omega(fx,Gy,t) * \Omega(gy,Fx,t)\}.$$

If a function is non-negative, summable, and Lebesgue integrable, then $\int_0^{\varepsilon} \varphi(t)dt$, for each $\varepsilon > 0$, for all $x,y \in X$, $t > 0$, $\exists$ a unique common fixed point of f,g,F and G.

Proof. Given that $\{f,F\}$ and $\{g,G\}$ are owc pairs. Thus, points $p,q \in X$ such that $fp \in Fp$, $gq \in Gq$, $fFp \subseteq Ffx$ and $gGq \subseteq Ggq$. Also, by Lemma 2.2, we obtain as $fp \in Fp$ thus $ffp \subset fF \subset Ffp$, $gq \in Gq$ thus $ggq \subset gGq \subset Ggq$. Hence

$$\Omega(fp,gq,t) \ge \delta_H(Fp,Gq,t) \tag{3.2}$$

and

$$\Omega(f^2p,g^2q,t) \ge \delta_H(Ffp,Ggq,t). \tag{3.3}$$

Now we shall show that $fp = gq$. If not, then $\delta_H(Ffp,Ggq,t) < 1$, put $x = fp$, $y = gq$, we have

$$\begin{aligned} M(fp,gq,t) &= \varphi\{H(ffp,ggq,t),\Omega(ffp,Ffp,t),\Omega(ggq,Ggq,t),\Omega(ffp,Ggq,t) * \Omega(ggq,Ffp,t)\} \\ &= \varphi\{H(f^2p,g^2q,t),\Omega(f^2p,Ffp,t),\Omega(g^2q,Ggq,t),\Omega(f^2p,Ggq,t) * \Omega(g^2q,Ffp,t)\}. \end{aligned}$$

From (3.2), we have

$$M(fp,gq,t)=\varphi\{\delta_H(Ffp,Ggq,t),1,1,\delta_H(Ffp,Ggq,t)*\delta_H(Ggq,Ffp,t)\},$$

$$M(fp,gq,t)=\delta_H(Ffp,Ggq,t).$$

Then from (3.1)

$$\int_0^{\delta_H(Ffp,Ggq,t)}\varphi(t)dt\ge\int_0^{M(fp,gq,t)}\varphi(t)dt\ge\int_0^{\delta_H(Ffp,Ggq,t)}\varphi(t)dt.$$

This is a contradiction

$\Rightarrow\quad \delta_H(Ffp,Ggq,t)=1$

$\Rightarrow\quad Ffp=Ggq$

i.e.,

$$fp=gq.$$

Also,

$$\Omega(f^2p,gp,t)\ge\delta_H(Ffp,Gp,t)$$

and

$$\Omega(f^2p,Gp,t)\ge\delta_H(Ffp,Gp,t).$$

Now, we claim $fp=p$. If not, then $\delta_H(Ffp,Gp,t)<1$.
Put $x=fp$, $y=p$, we have

$$\begin{aligned}M(fp,p,t)&=\varphi\{H(ffp,gp,t),\Omega(ffp,Ffp,t),\Omega(gp,Gp,t),\Omega(ffp,Gp,t)*\Omega(gp,Ffp,t)\}\\&=\varphi\{H(f^2p,gp,t),\Omega(f^2p,Ffp,t),\Omega(gp,Gp,t),\Omega(f^2p,Gp,t)*\Omega(gp,Ffp,t)\}\\&=\varphi\{H(f^2p,gp,t),H(f^2p,Ffp,t),H(gp,Gp,t),H(f^2p,Gp,t)*H(gp,Ffp,t)\}\\&=\varphi\{\delta_H(Ffp,Gp,t),1,1,H(Ffp,Gp,t)*H(Gp,Ffp,t)\}\\&=\delta_H(Ffp,Gp,t).\end{aligned}$$

Then from (3.1),

$$\int_0^{\delta_H(Ffp,Gp,t)}\varphi(t)dt\ge\int_0^{M(fp,p,t)}\varphi(t)dt\ge\int_0^{\delta_H(Ffp,Gp,t)}\varphi(t)dt.$$

This is a contradiction

$\Rightarrow\quad \delta_H(Ffp,Gp,t)=1$

$\Rightarrow\quad Ffp=Gp$

i.e.,

$$fp=p.$$

Similarly, we can show that, $gq=q$.
Therefore, $ffp=fp=ggq=gq=gfp$ and $fp=f^2p\in fFu\subseteq Ffu$ so that $fp\in Ffp$ and $fp=gfp\in Gfu$. Then fp is common fixed point of f,g,F and G.

Uniqueness: Let z' be f,g,F, and G's other common fixed point.
Put $fx=z$ then we have

$$\Omega(z,z',t)=\Omega(fz,gz',t)\ge\delta_H(Fz,Gz',t).$$

Now,

$$\begin{aligned} M(z,z',t) &= \varphi\{H(fz,gz',t),\Omega(fz,Fz,t),\Omega(gz',Gz',t),\Omega(fz,Gz',t)*\Omega(gz',Fz,t)\} \\ &= \varphi\{H(fz,gz',t),1,1,\Omega(fz,Gz',t)*\Omega(Fz,gz',t)\} \\ &= \varphi\{H(fz,gz',t),1,1,H(fz,gz',t)*H(fz,gz',t)\} \\ &= H(fz,gz',t). \end{aligned}$$

Then from (3.1),

$$\int_0^{\delta_H(Fz,Gz',t)} \varphi(t)dt \geq \int_0^{M(z,z',t)} \varphi(t)dt \geq \int_0^{\delta_H(Fz,Gz',t)} \varphi(t)dt.$$

This is a contradiction

$\Rightarrow \quad \delta_H(Fz,Gz',t) = 1$

$\Rightarrow \quad Fz = Gz'$

i.e.,

$$z = z'.$$

Thus f,g,F and G have unique common fixed point.

The proof is now complete. □

Example 3.1. Let $X = [0,4]$ with metrc d is defined $d = |x-y|$ for all $t \in [0,1]$, and $\Omega = (x,y,t) = \frac{t}{t+|x-y|}$.

Set the SEVM $F,G : X \to CB(X)$. Define the SVM $f,g : X \to X$.

$$f(x) = \begin{cases} x, & 0 \leq x \leq 2, \\ 3, & 2 < x \leq 4, \end{cases} \qquad g(x) = \begin{cases} 2, & 0 \leq x \leq 2, \\ \frac{x}{4}, & 2 < x \leq 4, \end{cases}$$

$$F(x) = \begin{cases} \{2\}, & 0 \leq x \leq 2, \\ \{0\}, & 2 < x \leq 4, \end{cases} \qquad G(x) = \begin{cases} \{2\}, & 0 \leq x \leq 2, \\ \{4\}, & 2 < x \leq 4, \end{cases}$$

that is,

$$f(2) = \{2\} \in F(2) \text{ and } Ff(2) = \{2\} = fF(2)$$

and

$$g(2) = \{2\} \in G(2) \text{ and } Gg(2) = \{2\} = gG(2).$$

Hence, there are times when $\{f,F\}$ and $\{g,G\}$ are poorly compatible. Additionally, f,g,F, and G have a unique shared fixed point of 2.

Example 3.2. Let $X = [0,4]$ with the metric d defined $d(x,y) = |x-y|$, and $a*b = \min\{a,b\}$ for each $t > 0$, define $\Omega(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$

Define the maps f,g,F and G:

$$fx = \begin{cases} 2x-1, & x \leq 5, \\ 2x, & x > 5, \end{cases} \qquad gx = \begin{cases} 3-2x, & x \leq 1, \\ x+1, & x > 1, \end{cases}$$

$$Fx = \begin{cases} \{1\}, & x<2, \\ [2x,2x+5], & x \geq 2, \end{cases} \qquad Gx = \begin{cases} \{1\}, & x=1, \\ [x,x+2], & \text{otherwise}. \end{cases}$$

Hence $\{f,F\}$ and $\{g,G\}$ be occasionally weakly compatible.
Define $\varphi:[0,1]\to[0,1]$ as $\varphi(0)=0$, $\varphi(1)=1$ and $\varphi(t)=t^{1/2}$, for $0<t<1$, then condition (3.1) is satisfied for all $t>1$.

Corollary 3.2. *Assume that $F,G:X\to CB(X)$ is a SEVM and $f,g:X\to X$ is SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC, and $0<t<1$ and satisfies the condition:*

$$\int_0^{\delta_H(Fx,Gy,t)} \varphi(t)dt \geq \int_0^{M(x,y,t)} \varphi(t)dt, \tag{3.4}$$

$$M(x,y,t) = \{\Omega(fx,gy,t),\Omega(fx,Fx,t),\Omega(gy,Gy,t),\Omega(fx,Gy,t),\Omega(gy,Fx,t)\}.$$

If a function is non-negative, summable, and Lebesgue integrable such that $\int_0^{\varepsilon}\varphi(t)dt$, for each $\varepsilon>0$, for all $x,y\in X$, $t>0$ then there exists a unique common fixed point of f,g,F and G.

Corollary 3.3. *Assume that $F,G:X\to CB(X)$ is a SEVM and $f,g:X\to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC. Let $\varphi:R^5\to R$ such that $\varphi(t)>1$ and $0<t<1$ and satisfies the condition:*

$$\int_0^{\delta_H(Fx,Gy,t)} \varphi(t)dt \geq \int_0^{M(x,y,t)} \varphi(t)dt, \tag{3.5}$$

$$M(x,y,t) = \varphi[\min\{\Omega(fx,gy,t),\Omega(fx,Fx,t),\Omega(gy,Gy,t),\Omega(fx,Gy,t),\Omega(gy,Fx,t)\}.$$

If a function is non-negative, summable, and Lebesgue integrable such that $\int_0^{\varepsilon}\varphi(t)dt$, for each $\varepsilon>0$ for all $x,y\in X$, $t>0$ then there exists a unique common fixed point of f,g,F and G.

Corollary 3.4. *Assume that $F,G:X\to CB(X)$ is a SEVM and $f,g:X\to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC. Let $\varphi:R\to R$ such that for every, $0\leq\alpha\leq1$, $\psi(\alpha)>\alpha$ and satisfies the condition:*

$$\int_0^{\delta_H^{\beta}(Fx,Gy,t)} \varphi(t)dt \geq \int_0^{M(x,y,t)} \varphi(t)dt, \tag{3.6}$$

where

$$M(x,y,t) = \psi[\eta\Omega^{\beta}(fx,gy,t)+(1-\eta)\Omega^{\frac{\beta}{2}}(gy,Fx,t).\Omega^{\frac{\beta}{2}}(fx,Gy,t)]$$

is a function is sumable, Lebesgue integrable, non-negative such that $\int_0^{\varepsilon}\varphi(t)dt$, for each $\varepsilon>0$ for all $x,y\in X$, $0<\alpha<1$ and $\beta\geq1$ then f,g,F and G has unique common fixed point.

Proof. Given that $\{f,F\}$ and $\{g,G\}$ are OWC pairs. Thus, points $p,q\in X$ such that $fp\in Fp$, $gq\in Gq$, $fFp\subseteq Ffp$ and $gGq\subseteq Ggq$. Also, by Lemma 2.2, we obtain as $fp\in Fp$ thus $ffp\subset fFp\subset Ffp$, $gq\in Gq$ thus $ggq\subset gGq\subset Ggq$,

$$\Omega(fp,gq,t)\geq\delta_H(Fp,Gq,t) \tag{3.7}$$

and

$$\Omega(f^2p,gq,t)\geq\delta_H(Ffp,Gqt). \tag{3.8}$$

Now to claim $f^2p=fp$.

Using (3.7) and Lemma 2.2, we have

$$M(p,q,t) = \psi[\eta\delta_H^\beta(Fp,Gq,t) + (1-\eta)\delta_H^\beta(Gq,Fp,t)] = \psi[\delta_H^\beta(Fp,Gq,t)].$$

Since $0 \leq \alpha \leq 1$, $\psi(\alpha) > \alpha$,

$$0 \leq \delta_H^\beta(Fp,Gq,t) < 1,\ \delta_H^\beta(Fp,Gq,t) \geq \psi(\delta_H^\beta(Fp,Gq,t)) > \delta_H^\beta(Fp,Gq,t).$$

So by inequality (3.6).
We have,

$$\int_0^{\delta_H^\beta(Fp,Gq,t)} \varphi(t)dt \geq \int_0^{\delta_H^\beta(Fp,Gq,t)} \varphi(t)dt.$$

This is contradiction, thus we get

$$\delta_H(Fp,Gq,t) = 1$$
$$\Rightarrow \quad Fp = Gq$$
$$\Rightarrow \quad fp = gq.$$

Again using (3.7) and Lemma 2.2, we have

$$\begin{aligned} M(fp,q,t) &= \psi[\eta\Omega^\beta(f^2p,gq,t) + (1-\eta)\Omega^{\frac{\beta}{2}}(gq,Ffp,t).\Omega^{\frac{\beta}{2}}(f^2p,Gq,t)] \\ &= \psi[\eta\delta_H^\beta(Ffp,Gq,t) + (1-\eta)\delta_H^\beta(Ffp,Gq,t)] \\ &= \delta_H^\beta(Fp,Gq,t). \end{aligned}$$

If $0 \leq \delta_H^\beta(Fp,Gq,t) < 1$, then by we have

$$\delta_H^\beta(Ffp,Gq,t) \geq \psi(\delta_H^\beta(Ffp,Gq,t)) > \delta_H^\beta(Ffp,Gp,t).$$

So by inequality (3.6).
We have

$$\int_0^{\delta_H^\beta(Ffp,Gq,t)} \varphi(t)dt \geq \int_0^{\delta_H^\beta(Ffp,Gq,t)} \varphi(t)dt.$$

This is contradiction, thus we get

$$\delta_H(Ffp,Gq,t) = 1$$
$$\Rightarrow \quad Ffp = Gq$$
$$\Rightarrow \quad f^2p = fp.$$

Similarly, $\{f,F\}$ and $\{g,G\}$ have the same role so we can show $gq = g^2q$.
Suppose $fp = z$ then $fz = z = gz$.

Uniqueness: Let z' be f,g,F, and G's other common fixed point then by inequality (**??**).
Put $fx = z$, then we have

$$\Omega(z,z',t) = \Omega(fz,gz',t) \geq \delta_H(Fz,Gz',t). \tag{3.9}$$

Now

$$\begin{aligned} M(z,z',t) &= \psi[\eta\Omega^\beta(fz,gz',t) + (1-\eta)\Omega^{\frac{\beta}{2}}(gz',Fz,t).\Omega^{\frac{\beta}{2}}(fz,Gz',t)] \\ &= \psi[\delta_H^\beta(Fz,Gz',t)] > \delta_H^\beta(Fz,Gz',t). \end{aligned}$$

Then by inequality (3.6),

$$\int_0^{\delta_H^\beta(Fz,Gz',t)} \varphi(t)dt \geq \int_0^{\delta_H^\beta(Fz,Gz',t)} \varphi(t)dt.$$

This is contradiction, thus

$$\delta_H(Fz,Gz',t) = 1$$

$$\Rightarrow \quad Fz = Gz.$$

Since z and z' are common fixed point of f,g,F and G.
We have

$$\Omega(fz,gz',t) \geq \Omega(fz,Fz,t) * \delta_H(Fz,Gz',t) * \Omega(gz',Gz',t) \geq \delta_H(Fz,Gz',t).$$

So $z = fz = gz' = z'$ and there exists a unique common fixed point of f,g,F and G. This completes the proof. □

Theorem 3.5. *Assume that $F,G : X \to CB(X)$ is a SEVM and $f,g : X \to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC. Let $\varphi : [0,1] \to [0,1]$ such that for all $t \in [0,1)$, $\psi(t) = 1$ iff $t = 1$, and satisfies the condition:*

$$\psi\{\delta_H(Fx,Gy,t)\} \geq l\{\Omega(fx,gy,t)\}\psi\{\Omega(fx,gy,t)\} + m\{\Omega(fx,gy,t)\} \min\{\psi\{\Omega(fx,Gy,t)\},\psi\{\Omega(gy,Fx,t)\}\}, \tag{3.10}$$

for all $x,y \in X$, where $l,m : [0,1] \to [0,1]$ are satisfying the conditions:
$l(t)+m(t) > 1$, for all $t > 0$ and $l(t)+m(t) = 1$ iff $t = 1$, then f,g,F and G has unique common fixed point.

Proof. Since the pairs $\{f,F\}$ and $\{g,G\}$) be OWC. So, there are points $x,y \in X$ such that $fx \in Fx$, $gy \in Gy$, $fFx \subseteq Ffx$ and $gGy \subseteq Ggy$. Also, by Lemma 2.2, we obtain as $fx \in Fx$ thus $ffx \subset fFx \subset Ffx$, $gy \in Gy$ thus $ggy \subset gGq \subset Ggy$,

$$\Omega(fx,gy,t) \geq \delta_H(Fx,Gy,t) \tag{3.11}$$

and

$$\Omega(f^2x,gy,t) \geq \delta_H(Ffx,Gy,t). \tag{3.12}$$

Now we shall show that $fx = gy$. If not then applying above conditions in inequality (3.10) then, we have

$$\begin{aligned}\psi\{\delta_H(Fx,Gy,t)\} &\geq l\{\Omega(fx,gy,t)\}\psi\{\Omega(Fx,Gy,t)\} + m\{\Omega(fx,gy,t)\} \\ &\quad \min\{\psi\{\Omega(Fx,Gy,t)\},\psi\{\Omega(Gy,Fx,t)\}\} \\ &\geq l\{\Omega(fx,gy,t)\}\psi\{\Omega(Fx,Gy,t)\} + m\{\Omega(fx,gy,t)\}\psi\{\Omega(Fx,Gy,t)\} \\ &\geq l\{\Omega(fx,gy,t)\}\psi\{\delta_H(Fx,Gy,t)\} + m\{\Omega(fx,gy,t)\}\psi\{\delta_H(Fx,Gy,t)\} \\ &\geq [l\{\Omega(fx,gy,t)\} + m\{\Omega(fx,gy,t)\}]\psi\{\delta_H(Fx,Gy,t)\}.\end{aligned}$$

Since $[l\{\Omega(fx,gy,t)\} + m\{\Omega(fx,gy,t)\}] = 1$ iff $t = 1$ then

$$\psi\{\delta_H(Fx,Gy,t)\} > \psi\{\delta_H(Fx,Gy,t)\}.$$

This is a contradiction.

Thus

$$\delta_H(Fx,Gy,t)=1$$
$$\Rightarrow \quad Fx=Gy$$
$$\Rightarrow \quad fx=gy.$$

Again to prove, $f^2x=fx$. If not then applying above conditions in inequality (3.10) then, we have

$$\begin{aligned}\psi\{\delta_H(Ffx,Gy,t)\} &\geq l\{\Omega(f^2x,gy,t)\}\psi\{\Omega(Ffx,gy,t)\}+m\{\Omega(f^2x,gy,t)\}\\ &\quad \min\{\psi\{\Omega(Ffx,Gy,t)\},\psi\{\Omega(gy,Ffx,t)\}\}\\ &\geq l\{\Omega(f^2x,gy,t)\}\psi\{\Omega(Ffx,gy,t)\}+m\{\Omega(f^2x,gy,t)\}\\ &\quad \min\{\psi\{\Omega(Ffx,Gy,t)\},\psi\{\Omega(G,Ffx,t)\}\}\\ &\geq l\{\Omega(f^2x,gy,t)\}\psi\{\Omega(Ffx,gy,t)\}+m\{\Omega(f^2x,gy,t)\}\psi\{\Omega(Ffx,Gy,t)\}\\ &\geq [l\{\Omega(f^2x,gy,t)\}+m\{\Omega(f^2x,gy,t)\}]\psi\{\Omega(Ffx,Gy,t)\}.\end{aligned}$$

Since $[l\{\Omega(f^2x,gy,t)\}+m\{\Omega(f^2x,gy,t)\}]=1$ iff $t=1$ then

$$\psi\{\delta_H(Ffx,Gy,t)\}>\psi\{\delta_H(Ffx,Gy,t)\}.$$

This is a contradiction.
Thus

$$\delta_H(Ffx,Gy,t)=1$$
$$\Rightarrow \quad Ffx=Gy$$
$$\Rightarrow \quad f^2x=fx.$$

Similarly, $\{f,F\}$ and $\{g,G\}$ have the same role so we can show $gy=g^2y$.
Suppose $fz=z=gz$ then $fz=z=gz$ and there exists a distinct common fixed point exists of f,g,F and G. This completes the proof.

Uniqueness: Let z' be f,g,F, and G's other common fixed point then by inequality (3.10).
Put $fx=z$, then we have

$$\Omega(z,z',t)=\Omega(fz,gz',t)\geq \delta_H(Fz,Gz',t). \tag{3.13}$$

Then by inequality (3.10),

$$\begin{aligned}\psi\{\delta_H(Fz,Gz',t)\} &\geq l\{\Omega(fz,gz',t)\}\psi\{\Omega(fz,gz',t)\}+m\{\Omega(fz,gz',t)\}\\ &\quad \min\{\psi\{\Omega(fz,Gz',t)\},\psi\{\Omega(gz',Fz,t)\}\}\\ &\geq l\{\Omega(z,z',t)\}\psi\{\Omega(Fz,Gz',t)\}+m\{\Omega(z,z',t)\}\\ &\quad \min\{\psi\{\Omega(Fz,Gz',t)\},\psi\{\Omega(Gz',Fz,t)\}\}\\ &\geq l\{\Omega(z,z',t)\}\psi\{\Omega(Fz,Gz',t)\}+m\{\Omega(z,z',t)\}\psi\{\Omega(Fz,Gz',t)\}\\ &\geq [l\{\Omega(z,z',t)\}+m\{\Omega(z,z',t)\}]\psi\delta_H(Fz,Gz',t)\\ &> \psi\delta_H(Fz,Gz',t).\end{aligned}$$

Since $[l\{\Omega(z,z',t)\}+m\{\Omega(z,z',t)\}]=1$ iff $t=1$.
This is contradiction, so we get $Fz=Gz'$. Since z and z' are common fixed point of f,g,F and G.

Now we have

$$\Omega(fz,gz',t) \geq \Omega(fz,Fz,t) * \delta_H(Fz,Gz',t) * \Omega(gz',Gz',t) \geq \delta_H(Fz,Gz',t).$$

So $z = fz = gz' = z'$ and there exists a unique common fixed point of f,g,F and G. This completes the proof. □

Theorem 3.6. *Assume that $F,G : X \to CB(X)$ is a SEVM and $f,g : X \to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC and satisfies the condition:*

$$\delta_H^{\beta}(Fx,Gy,t) \geq l\{\Omega(fx,gy,t)\}[\min\{\Omega(fx,gy,t).\Omega^{\beta-1}(fx,Fx,t),\Omega(fx,gy,t).\Omega^{\beta-1}(fx,Gy,t),$$
$$\Omega(fx,Fx,t).\Omega^{\beta-1}(gy,Gy,t),\Omega^{\beta-1}(fx,Gy,t).\Omega(gy,Fx,t)\}],$$

for all $x,y \in X$, where $\beta \geq 2$ and $l : [0,1] \to [0,\infty)$ are satisfying the conditions:
$l(t) > 1$, for all $0 \leq t < 1$ and $l(t) = 1$ iff $t = 1$. Then f,g,F and G has unique common fixed point.

Proof. Similar proof follows as Theorem 3.5. □

Applications. To define as $\Upsilon(\alpha) = \int_0^{\alpha} \beta(\alpha)d\alpha$, for all $\alpha > 0$, for each $\beta(\delta) > 0$, $\delta > 0$ and $\beta(\alpha) = 0$ if and only if $\alpha = 0$ and non-decreasing and continuous function $\Upsilon(\alpha) : [0,\infty) \to [0,\infty)$ then

Theorem 3.7. *Assume that $F,G : X \to CB(X)$ is a SEVM and $f,g : X \to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC. Let $\varphi : R^5 \to R$ such that $\varphi(t,1,1,t*t) > 1$ and $0 < t < 1$ and satisfies the condition:*

$$\int_0^{\delta_H(Fx,Gy,t)} \varphi(t)dt \geq \int_0^{M(x,y,t)} \varphi(t)dt, \tag{3.14}$$
$$M(x,y,t) = \varphi\{H(fx,gy,t),\Omega(fx,Fx,t),\Omega(gy,Gy,t),\Omega(fx,Gy,t)*\Omega(gy,Fx,t)\}$$

is a function is sumable, Lebesgue intregable, non-negative such that $\int_0^{\varepsilon} \varphi(t)dt$, for each $\varepsilon > 0$, for all $x,y \in X$, $t > 0$ then there exists a distinct common fixed point of f,g,F and G.

Proof. If we take $\varphi(t) = 1$ then we can easily proof by using Theorem 3.1. □

Theorem 3.8. *Assume that $F,G : X \to CB(X)$ is a SEVM and $f,g : X \to X$ is a SVM, this means that the pairs $\{f,F\}$ and $\{g,G\}$ are OWC. Let $\varphi : [0,1] \to [0,1]$ such that for all, $t \in [0,1)$, $\psi(t) = 1$ iff $t = 1$, and satisfies the condition:*

$$\int_0^{\psi\{\delta_H(Fx,Gy,t)\}} \varphi(t)dt \geq \int_0^{M(x,y,t)} \varphi(t)dt,$$

where

$$M(x,y,t) = l\{\Omega(fx,gy,t)\}\psi\{\Omega(fx,gy,t)\} + m\{\Omega(fx,gy,t)\}\min\{\psi\{\Omega(fx,Gy,t)\},\psi\{\Omega(gy,Fx,t)\}\},$$

for all $x,y \in X$, where $l,m : [0,1] \to [0,1]$ are satisfying the conditions:
$l(t) + m(t) > 1$, for all $t > 0$ and $l(t) + m(t) = 1$ iff $t = 1$. Then f,g,F and G has common fixed point.

Proof. If we take $\varphi(t) = 1$ then we can easily proof by using Theorem 3.5. □

4. Conclusion

There are many applications of fixed point theory in several field of science. In this paper, the main result is the improved and extended results of FMS with single and SEVM which can be further extended for multi-valued with occasionally weakly compatible (OWC) conditions. We use two generalized contractions with novel including integral approach in the context of FMS in this paper and can be used in the finding the solution of LPP, digital problems, economics population censes etc.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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