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Research Article

Natural Cubic Spline for Hyperbolic Equations With Constant Coefficients

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Abstract. In this paper, we have considered second-order hyperbolic equations by implementing *Natural Cubic Spline* (NCS) method. Wave propagation and dynamic systems represents hyperbolic equations. We have considered the class of hyperbolic *partial differential equations* (PDEs) with constant coefficients and implemented NCS method both explicitly and implicitly. In our implementation we replaced spatial derivatives by second derivative of Natural cubic spline and time derivatives by central finite difference operator. To show the effectiveness of proposed method we have considered numerical examples having both Dirichlet and Neumann conditions. In order to evaluate the NCS method's performance in managing various boundary behaviours which are frequently seen in real-world applications like heat conduction issues or wave propagation in bounded domains. The results are represented graphically exhibiting the accuracy of proposed NCS method. To check the efficiency of the NCS method we compared the results with analytical solution. The research not only demonstrates the Natural Cubic Spline's flexibility in resolving hyperbolic PDEs, but also emphasizes its benefits, including its capacity to smooth spatial discretization, adjust to different boundary conditions, and work with both explicit and implicit time integration schemes. The findings verify that the NCS approach is a reliable tool for numerical simulations in domains where hyperbolic equations are used, including fluid dynamics, acoustics, and other domains.

Keywords. Second-order hyperbolic equation, Natural cubic spline, Central finite difference operator, Analytical solution

Mathematics Subject Classification (2020). 65N30, 35J05

1. Introduction

In many branches of science and engineering, the hyperbolic partial differential equations appear. They serve as the foundation for the basic equations of atomic physics and simulate the vibrations of structures (such as buildings, beams, and machinery). The telegraph, Klein-Gordon, and sine-Gordon equations are examples of hyperbolic PDEs. The telegraph equation is crucial for simulating a number of pertinent issues, including wave propagation (Banasiak and Mika [2]), random theory (El-Azab and El-Gamel [4]), and signal analysis (Weston and He [8]). Numerous applications, including solid state physics, nonlinear optics, quantum field theory, and mathematical physics, depend on the Klein-Gordon equation. The sine-Gordon equation can be found in many scientific domains, including the motion of a rigid pendulum, solid state physics, nonlinear optics, and the stability of fluid motions. Other uses for hyperbolic equations include dissipative nonlinear wave equations and nonlinear waves of the Vander Pol type. Additionally, the study of viscoelasticity, thermo elasticity, plasma physics, medical research and chemical heterogeneity all include PDEs. They also occur in non-Newtonian fluid flows, radioactive nuclear decay, semi-conductor modelling, sub surface water flows in porous media, and non-local reactive transport.

The 2nd order hyperbolic PDE solutions have received a great deal of interest recently in the literature. Dehghan and Shokri [3] suggested a numerical method to approximate the solution of the one-dimensional hyperbolic telegraph problem using thin plate splines and radial basis function. The Taylor polynomial approximation was used by Bülbül and Sezer [1] to solve hyperbolic PDEs with constant coefficients. An improved Taylor matrix method has been described for solving integral and integro-differential equations as well as ordinary differential equations. Karunanithi *et al*. [6] used the Lax-Wendroff scheme to solve the 2nd-order linear wave equation in one dimension. Finite difference techniques were used by Esmailzadeh *et al*. [5] to solve hyperbolic PDEs with piecewise constant parameters and variable coefficients. For the approximate solution of one-dimensional PDEs, Singh *et al*. [7] created an effective Wavelet computational approach based on the operational matrices of integration of Legendre and Chebyshev wavelets.

Here, we have considered Natural Cubic Spline in solving linear hyperbolic equations. The type of procedures such as explicit and implicit is discussed in detail with different types of boundary conditions. In Section 2 we have explained NCS procedure and Section 3 comprises of examples of hyperbolic PDE with NCS explicit and implicit method. Here the change of matrix form in solving hyperbolic PDE with different types of boundary conditions using NCS explicit and implicit method is given in detail.

2. Natural Cubic Spline

Let the cubic spline $S(x)$ interpolates $y(x)$ at the mesh $a = x_0 < x_1 < \ldots < x_n = b$.

Since $S(x)$ is piecewise cubic spline, its second order derivative $S''(x)$ is piecewise linear on the interval $[x_{i-1}, x_i]$.

Using linear Lagrange interpolating formula, we have

$$S''(x) = S''(x_{i-1})\frac{x_i - x}{x_i - x_{i-1}} + S''(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}. \tag{2.0}$$

Putting $M_i = S''(x_i)$ and $M_{i-1} = S''(x_{i-1})$, the above expression becomes

$$S''(x) = \frac{1}{h}(M_{i-1}(x_i - x) + M_i(x - x_{i-1})). \tag{2.1}$$

Integrating (2.1) twice, we get

$$S(x) = M_{i-1}\frac{(x_i - x)^3}{6h} + M_i\frac{(x - x_{i-1})^3}{6h} + C_1 x + C_2, \tag{2.2}$$

where C_1 and C_2 are constants of integration to be determined.

Evaluating $S(x)$ at x_i and x_{i-1}, we have

$$y_{i-1} = M_{i-1}\frac{h^2}{6} + C_1 x_{i-1} + C_2, \tag{2.3}$$

$$y_i = M_i\frac{h^2}{6} + C_1 x_i + C_2. \tag{2.4}$$

Solving (2.3) and (2.4) for C_1 and C_2, we get

$$C_1 = \left(\frac{h}{6}M_{i-1} - \frac{h}{6}M_i\right) + \frac{(y_i - y_{i-1})}{h},$$

$$C_2 = y_i - \frac{h^2}{6}M_i - \left[\frac{h}{6}(M_{i-1} - M_i) + \frac{y_i - y_{i-1}}{h}\right]x_i.$$

Substituting the values of C_1 and C_2 in (2.4), we have

$$S(x) = \frac{1}{6h}(M_{i-1}(x_i - x)^3 + M_i(x - x_{i-1})^3) + \left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)\left(\frac{x_i - x}{h}\right)$$
$$+ \left(y_i - \frac{h^2}{6}M_i\right)\left(\frac{x - x_{i-1}}{h}\right). \tag{2.5}$$

The function $S(x)$in the interval $[x_i, x_{i+1}]$ is obtained by replacing i by $i+1$ in eq. (2.5). Thus,

$$S(x) = M_i\frac{(x_{i+1} - x)^3}{6h} + M_i\frac{(x - x_i)^3}{6h} + \left(y_i - \frac{h^2}{6}M_i\right)\left(\frac{x_{i+1} - x}{h}\right) + \left(y_{i+1} - \frac{h^2}{6}M_{i+1}\right)\left(\frac{x - x_i}{h}\right). \tag{2.6}$$

Differentiating (2.5) and (2.6),

$$S'(x) = \frac{1}{2h}(-M_{i-1}(x_i - x)^2 + M_i(x - x_{i-1})^2) + \frac{y_i - y_{i-1}}{h} - \frac{(M_i - M_{i-1})}{6}h \tag{2.7}$$

$$= -M_i\frac{(x_{i+1} - x)^2}{2h} + M_{i+1}\frac{(x - x_i)^2}{2h} + \frac{y_{i+1} - y_i}{h} - \frac{(M_i + M_{i+1})}{6}h. \tag{2.8}$$

Calculating $S'(x)$ at $x = x_i$,

$$S'(x_i^-) = \frac{h}{6}M_{i-1} + \frac{h}{3}M_i + \frac{y_i - y_{i-1}}{h}, \qquad i = 1, 2, \dots, n, \tag{2.9}$$

$$S'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{y_{i+1} - y_i}{h}, \quad i = 0, 1, \dots, n-1. \tag{2.10}$$

Using continuity condition of the cubic spline, we have

$$\frac{h^2}{6}(M_{i-1} + 4M_i + M_{i+1}) = (y_{i-1} - 2y_i + y_{i+1}), \quad i = 1, 2, \dots, n. \tag{2.11}$$

The relation in eq. (2.11) is called the continuity or consistency relations of the cubic spline.

The cubic spline can be assumed as

$$S_j(x) = M_{i-1}^j\frac{(x_i - x)^3}{6h} + M_i^j\frac{(x - x_{i-1})^3}{6h} + \left(u_{i-1}^j - \frac{h^2}{6}M_{i-1}^j\right)\left(\frac{x_i - x}{h}\right)$$

$$+\left(u_i^j - \frac{h^2}{6}M_i^j\right)\left(\frac{x-x_{i-1}}{h}\right), \quad i=1,2,\dots,n.$$

Let

$$L_i^j = S'(x_i^+) = -\frac{h}{3}M_i^j - \frac{h}{6}M_{i+1}^j + \frac{u_{i+1}^j - u_i^j}{h}, \quad i=0,1,\dots,n-1, \tag{2.12}$$

$$L_i^j = S'(x_i^-) = \frac{h}{3}M_i^j + \frac{h}{6}M_{i-1}^j + \frac{u_i^j - u_{i-1}^j}{h}, \quad i=0,1,\dots,n. \tag{2.13}$$

From (2.12) and (2.13), we have

$$-L_i^j - \frac{h}{6}M_{i+1}^j + \frac{u_{i+1}^j - u_i^j}{h} = \frac{h}{3}M_i^j, \quad i=0,1,\dots,n-1, \tag{2.14}$$

$$L_i^j - \frac{h}{6}M_{i-1}^j - \frac{u_i^j - u_{i-1}^j}{h} = \frac{h}{3}M_i^j, \quad i=0,1,\dots,n. \tag{2.15}$$

Equating (2.15) and (2.14),

$$-L_i^j - \frac{h}{6}M_{i+1}^j + \frac{u_{i+1}^j - u_i^j}{h} = L_i^j - \frac{h}{6}M_{i-1}^j - \left(\frac{u_i^j - u_{i-1}^j}{h}\right),$$

$$-L_i^j - \frac{1}{2}\left[L_{i+1}^j - \frac{h}{6}M_i^j - \left(\frac{u_{i+1}^j - u_i^j}{h}\right)\right] + \left(\frac{u_{i+1}^j - u_i^j}{h}\right)$$

$$= L_i^j + \frac{1}{2}\left[L_{i-1}^j + \frac{h}{6}M_i^j - \left(\frac{u_i^j - u_{i-1}^j}{h}\right)\right] - \left(\frac{u_i^j - u_{i-1}^j}{h}\right),$$

$$2L_i^j + \frac{1}{2}L_{i-1}^j + \frac{1}{2}L_{i+1}^j - \frac{1}{2}\left(\frac{u_i^j - u_{i-1}^j}{h}\right) - \frac{1}{2}\left(\frac{u_{i+1}^j - u_i^j}{h}\right) - \left(\frac{u_i^j - u_{i-1}^j}{h}\right) - \left(\frac{u_{i+1}^j - u_i^j}{h}\right) = 0,$$

$$2L_i^j + \frac{1}{2}L_{i-1}^j + \frac{1}{2}L_{i+1}^j - \frac{1}{h}\left(\frac{u_i^j}{2} - \frac{u_{i-1}^j}{2} + u_i^j - u_{i-1}^j\right) - \frac{1}{h2+}\left(\frac{u_{i+1}^j}{2} - \frac{u_i^j}{2} + u_{i+1}^j - u_i^j\right) = 0,$$

$$2L_i^j + \frac{1}{2}L_{i-1}^j + \frac{1}{2}L_{i+1}^j - \frac{1}{h}\left(\frac{3u_i^j}{2} - \frac{3u_{i-1}^j}{2}\right) - \frac{1}{h}\left(\frac{3u_{i+1}^j}{2} - \frac{3u_i^j}{2}\right) = 0,$$

implies

$$4L_i^j + L_{i-1}^j + L_{i+1}^j - \frac{1}{h}[3u_i^j - 3u_{i-1}^j + 3u_{i+1}^j - 3u_i^j] = 0,$$

$$L_{i-1}^j + L_{i+1}^j + 4L_i^j = \frac{1}{h}[3u_{i+1}^j - 3u_{i-1}^j].$$

Dividing by 6 throughout

$$\frac{1}{6}L_{i-1}^j + \frac{2}{3}L_i^j + \frac{1}{6}L_{i+1}^j = \frac{1}{2h}(u_{i+1}^j - u_{i-1}^j).$$

This is called recurrence relation in L_i^j.

3. Numerical Results

In this section, we have considered hyperbolic PDEs with both Dirichlet and both Neumann conditions.

Example 3.1. Consider the following wave equation:

$$u_{tt} = u_{xx}, \quad 0 < x < \pi, \ t > 0 \tag{3.1}$$

subject to: $u(x,0) = \sin x$, $u_t(x,0) = 0$, $0 \le x \le \pi$ and $u(0,t) = u(\pi,t) = 0$, $t \ge 0$.

Exact solution: $u(x,t) = \sin x \cos t$.

Explicit Method

Let x denote the space variable and t denote the time variable. The time and space derivatives are replaced by central finite difference operator and second derivative of natural cubic spline in (3.1), we have

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{k^2} = M_i^j. \tag{3.2}$$

From (2.11),

$$\frac{1}{6}M_{i-1}^j + \frac{2}{3}M_i^j + \frac{1}{6}M_{i+1}^j = \left(\frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}\right), \quad i = 1,2,\dots,n-1. \tag{3.3}$$

Using (2.11), eq. (3.2) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = (2+6r^2)u_{i-1}^j + (8-12r^2)u_i^j + (2+6r^2)u_{i+1}^j - u_{i-1}^{j-1} - 4u_i^{j-1} - u_{i+1}^{j-1}. \tag{3.4}$$

Given $u_t(x,0) = 0$

$$\implies \quad \frac{u_i^{j+1} - u_i^{j-1}}{2k} = 0$$

$$\implies \quad u_i^{j-1} = u_i^{j+1}, \qquad i = 0,1,2,\dots,N_x.$$

Thus, equation (3.4) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = (1+3r^2)u_{i-1}^j + (4-6r^2)u_i^j + (1+3r^2)u_{i+1}^j, \quad i = 1,2,\dots,n-1, \tag{3.5}$$

where $r^2 = \frac{k^2}{h^2}$. Matrix form of (3.5) is

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} l & m & n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & l & m & n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & l & m & n & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & l & m & n \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^j \\ u_1^j \\ \vdots \\ u_{N_x-1}^j \\ u_{N_x}^j \end{bmatrix},$$

for $j = 0,1,2,\dots,N_t$, where $l = 1+3r^2$, $m = 4-6r^2$ and $n = 1+3r^2$.

Given $u(0,t) = 0 \implies u_0^j = 0$, $j \ge 0$,

$$u_0^{j+1} + 4u_1^{j+1} + u_2^{j+1} = (1+3r)u_0^j + (4-6r^2)u_1^j + (1+3r^2)u_2^j, \quad i = 1$$

$$\implies \quad 4u_1^{j+1} + u_2^{j+1} = (4-6r^2)u_1^j + (1+3r^2)u_2^j$$

and $u(\pi,t)=0 \implies u^j_{N_x}=0,\ j\ge 0,$

$$u^{j+1}_{n-2}+4u^{j+1}_{n-1}+u^{j+1}_{n}=(1+3r^2)u^j_{n-2}+(4-6r^2)u^j_{n-1}+(1+3r^2)u^j_n, \quad i=n-1$$

$$\implies \quad u^{j+1}_{n-2}+4u^{j+1}_{n-1}=(4-6r^2)u^j_{n-1}+(1+3r^2)u^j_{n-2}.$$

Thus, the above matrix representation simplifies to

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 4 & 1\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} u^{j+1}_1\\ u^{j+1}_2\\ \vdots\\ u^{j+1}_{N_x-2}\\ u^{j+1}_{N_x-1}\end{bmatrix}=\begin{bmatrix} m & n & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ l & m & n & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & l & m & n & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & l & m & n\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & l & m \end{bmatrix}\begin{bmatrix} u^{j}_1\\ u^{j}_2\\ \vdots\\ u^{j}_{N_x-2}\\ u^{j}_{N_x-1}\end{bmatrix}.$$

In short form

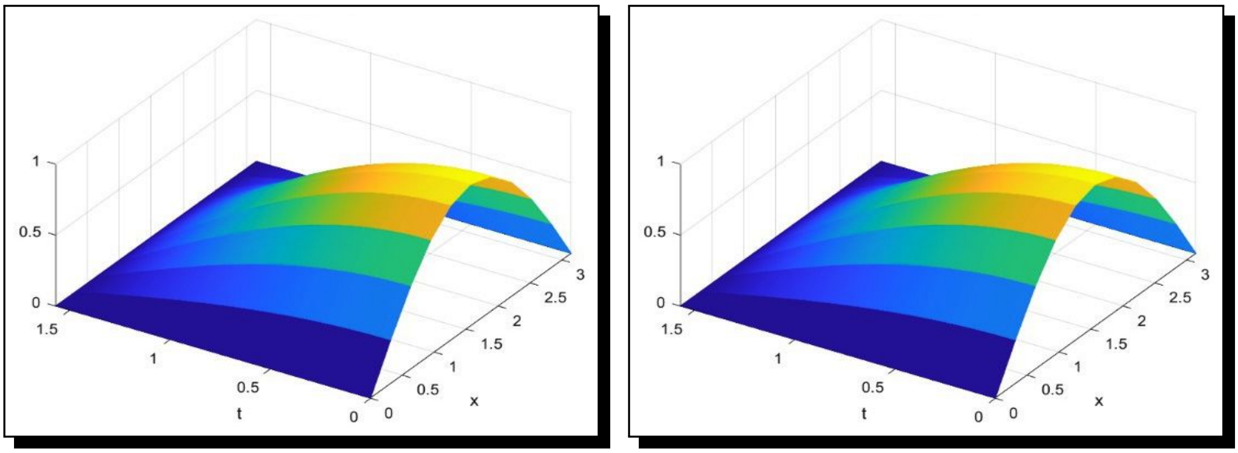
$$M_L X^{j+1}=M_R X^j.$$

Therefore, the required solution is

$$X^{j+1}=(M_L)^{-1}[M_R X^j]. \tag{3.6}$$

By the inverse operation, solution is obtained and presented in Figure 1. The NCS solution is comparing with analytical solution. It shows that NCS method results correlate with analytical solution. To check the accuracy of the NCS method, absolute error at $t=0.5$ is calculated and presented in Figure 1 at different step sizes along space coordinates. It is also observed that at step size 10^{-3}, accuracy of 10^{-7} is obtained for NCS method.

The accuracy of the NCS explicit with analytical solution for Example 3.1 is shown in Figure 1.



(a) (b)

Figure 1. Solution of Example 3.1 using (a) NCS explicit method and (b) analytical solution

Natural Cubic Spline Implicit Method

Let x denote the space variable ant t denote the time variable. Replacing time derivative by central finite difference operator and space derivatives by second derivative of natural cubic spline in equation (3.1), we have

$$\frac{u^{j+1}_i-2u^j_i+u^{j-1}_i}{k^2}=M^j_i.$$

Now writing M_i^j implicitly as $\frac{1}{2}(M_i^{j-1}+M_i^{j+1})$, then

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{k^2}=\frac{1}{2}(M_i^{j-1}+M_i^{j+1}). \tag{3.7}$$

From (2.11), eqn. (3.7) becomes

$$\begin{aligned}&(1-3r^2)u_{i-1}^{j+1}+(4+6r^2)u_i^{j+1}+(1-3r^2)u_{i+1}^{j+1}\\ &\quad=2(u_{i+1}^j+4u_i^j+u_{i-1}^j)+(3r^2-1)u_{i+1}^{j-1}-(6r^2+4)u_i^{j-1}+(3r^2-1)u_{i+1}^{j-1},\quad r^2=\frac{k^2}{h^2}.\end{aligned} \tag{3.8}$$

Given $\quad u_t(x,0)=0$,

$$\implies \quad \frac{u_i^{j+1}-u_i^{j-1}}{2k}=0$$

$$\implies \quad u_i^{j+1}=u_i^{j-1}.$$

Eq. (3.8) becomes

$$(1-3r^2)u_{i-1}^{j+1}+(4+6r^2)u_i^{j+1}+(1-3r^2)u_{i+1}^{j+1}=u_{i+1}^j+4u_i^j+u_{i-1}^j,\quad r^2=\frac{k^2}{h^2}. \tag{3.9}$$

Matrix form of (3.9) is

$$\begin{bmatrix} a & b & c & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & a & b & c & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & a & b & c & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & a & b & c\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & b \end{bmatrix} \begin{bmatrix} u_0^{j+1}\\ u_1^{j+1}\\ \vdots\\ u_{N_x-1}^{j+1}\\ u_{N_x}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j}\\ u_1^{j}\\ \vdots\\ u_{N_x-1}^{j}\\ u_{N_x}^{j} \end{bmatrix},$$

for $j=0,1,2,\dots,N_t$, where $a=1-3r^2$, $b=4+6r^2$ and $c=1-3r^2$.

Since $u(0,t)=0 \implies u_0^j=0,\ j\ge 0$,

$$(1-3r^2)u_0^{j+1}+(4+6r^2)u_1^{j+1}+(1-3r^2)u_2^{j+1}=u_0^j+4u_1^j+u_2^j,\quad \text{for } i=1$$

$$\implies \quad (4+6r^2)u_1^{j+1}+(1-3r^2)u_2^{j+1}=4u_1^j+u_2^j$$

and $u(\pi,t)=0 \implies u_{N_x}^j=0,\ j\ge 0$,

$$(1-3r^2)u_{N_x-2}^{j+1}+(4+6r^2)u_{N_x-1}^{j+1}+(1-3r^2)u_{N_x}^{j+1}=u_{N_x-2}^j+4u_{N_x-1}^j+u_{N_x}^j,\quad \text{for } i=N_x-1$$

$$\implies \quad (1-3r^2)u_{N_x-2}^{j+1}+(4+6r^2)u_{N_x-1}^{j+1}=u_{N_x-2}^j+4u_{N_x-1}^j$$

Therefore, the above matrix (eq. (3.9)) reduces to

$$\begin{bmatrix} b & c & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a & b & c & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & a & b & c & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & a & b & c \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a & b \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{N_x-2}^{j+1} \\ u_{N_x-1}^{j+1} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ \ddots \\ u_{N_x-2}^{j} \\ u_{N_x-1}^{j} \end{bmatrix}.$$

In short form,

$$M_L X^{j+1} = M_R X^j .$$

Therefore, the required solution is

$$X^{j+1} = (M_L)^{-1}[M_R X^j]. \tag{3.10}$$

The accuracy of the NCS implicit with analytical solution for Example 3.1 is shown in Figure 2.

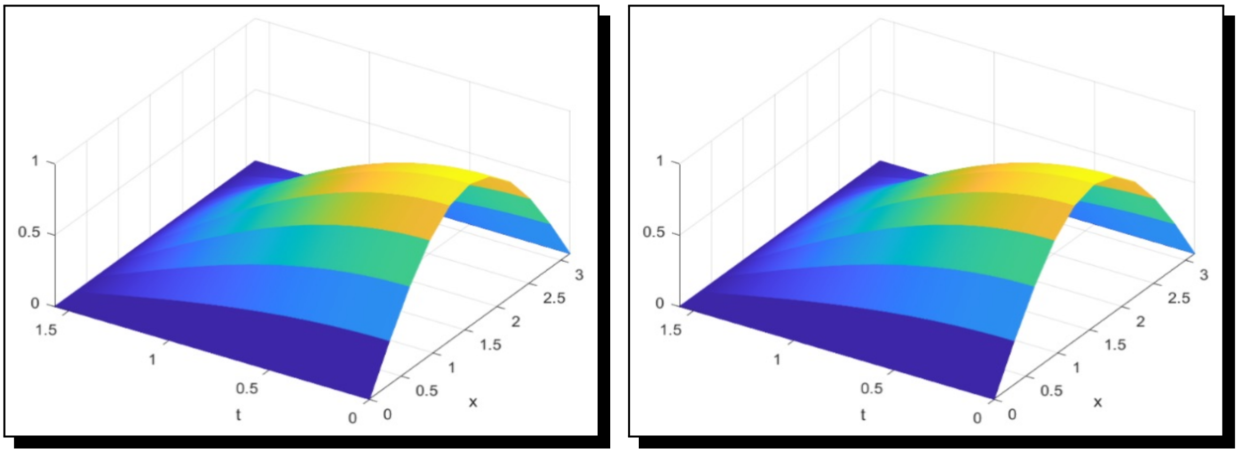


Figure 2. Solution of Example 3.1 using (a) NCS implicit method and (b) analytical solution

Example 3.2. Consider the following wave equation

$$u_{tt} = u_{xx}, \quad 0 < x < 1,\ t > 0 \tag{3.11}$$

subject to: $u(x,0) = \sin 2\pi x$, $u_t(x,0) = 2\pi \sin 2\pi x$, $0 \le x \le 1$ and $u(0,t) = u(1,t) = 0$, $t \ge 0$

Exact solution: $u(x,t) = \sin 2\pi x(\cos 2\pi t + \sin 2\pi t)$.

Explicit Method

By NCS explicit method, (3.11) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = (2+6r^2)u_{i-1}^j + (8-12r^2)u_i^j + (2+6r^2)u_{i+1}^j - u_{i-1}^{j-1} - 4u_i^{j-1} - u_{i+1}^{j-1}. \tag{3.12}$$

Given $u_t(x,0) = 2\pi \sin 2\pi x$

$$\implies \quad \frac{u_i^{j+1} - u_i^{j-1}}{2k} = 2\pi \sin 2\pi x_i$$

$$\implies \quad u_i^{j+1} = u_i^{j-1} + 2k(2\pi \sin 2\pi x_i), \quad i = 0,1,2,\dots,N_x$$

Therefore, eq. (3.12) becomes

$$u_{i-1}^{j+1}+4u_i^{j+1}+u_{i+1}^{j+1}=(1+3r^2)u_{i-1}^j+(4-6r^2)u_i^j+(1+3r^2)u_{i+1}^j+6k\pi(2\pi\sin 2\pi x_i), \tag{3.13}$$

where $r^2=\frac{k^2}{h^2}$. The matrix form of eq. (3.13) is

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1}\\ u_1^{j+1}\\ \vdots\\ u_{N_x-1}^{j+1}\\ u_{N_x}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} l & m & n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & l & m & n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & l & m & n & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & l & m & n\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j}\\ u_1^{j}\\ \vdots\\ u_{N_x-1}^{j}\\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 6k\pi(\sin 2\pi x)\\ \vdots\\ 6k\pi(\sin 2\pi x) \end{bmatrix},$$

for $j=0,1,2,\dots,N_t$, where $l=1+3r^2$, $m=4-6r^2$ and $n=1+3r^2$.

From the given boundary conditions $u(0,t)=0\Rightarrow u_0^j=0$, and $u(1,t)=0\Longrightarrow u_{N_x}^j=0$, the above matrix becomes

$$\begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 4 & 1\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_1^{j+1}\\ u_2^{j+1}\\ \vdots\\ u_{N_x-2}^{j+1}\\ u_{N_x-1}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} m & n & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ l & m & n & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & l & m & n & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & l & m & n\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} u_1^{j}\\ u_2^{j}\\ \vdots\\ u_{N_x-2}^{j}\\ u_{N_x-1}^{j} \end{bmatrix} + C_R$$

where $C_R=\begin{bmatrix} 6k\pi(\sin 2\pi x)\\ \vdots\\ 6k\pi(\sin 2\pi x) \end{bmatrix}$.

In short form,

$$M_LX^{j+1}=M_RX^j+C_R\,.$$

Thus, the required solution is

$$X^{j+1}=(M_L)^{-1}[M_RX^j+C_R]. \tag{3.14}$$

The accuracy of the NCS explicit with analytical solution for Example 3.2 is shown in Figure 3.

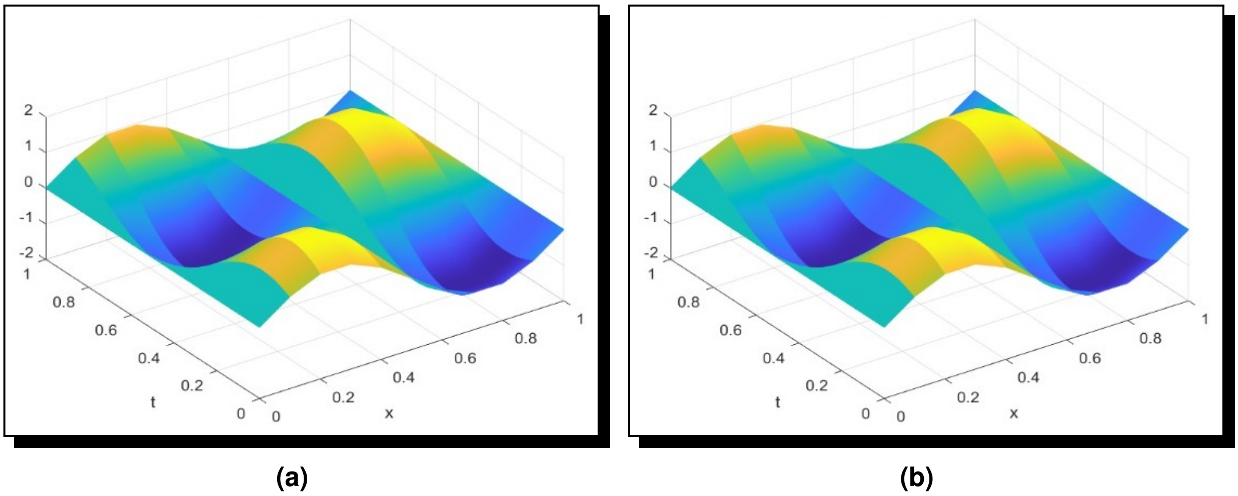


Figure 3. Solution of Example 3.2 using (a) NCS explicit method and (b) analytical solution

Implicit Method

By NCS implicit method, (3.11) becomes

$$\begin{aligned}&(1-3r^2)u_{i-1}^{j+1}+(4+6r^2)u_i^{j+1}+(1-3r^2)u_{i+1}^{j+1}\\ &\quad=(3r^2+1)u_{i+1}^{j-1}+(6r^2+4)u_i^{j-1}+(3r^2-1)u_{i+1}^{j-1}+2(u_{i+1}^j+4u_i^j+u_{i-1}^j),\end{aligned} \tag{3.15}$$

where $r^2=\frac{k^2}{h^2}$, given that

$$\begin{aligned}&u_t(x,0)=2\pi\sin 2\pi x\\ \Longrightarrow\quad &\frac{u_i^{j+1}-u_i^{j-1}}{2k}=2\pi\sin 2\pi x_i\\ \Longrightarrow\quad &u_i^{j+1}=u_i^{j-1}+2k(2\pi\sin 2\pi x_i),\quad i=0,1,2,\dots,N_x.\end{aligned}$$

Therefore, (3.15) becomes,

$$(1-3r^2)u_{i-1}^{j+1}+(4+6r^2)u_i^{j+1}+(1-3r^2)u_{i+1}^{j+1}=u_{i+1}^j+4u_i^j+u_{i-1}^j+6k\pi\sin 2\pi x_i\,. \tag{3.16}$$

Incorporating given conditions in eq. (3.16) and represents in matrix form as

$$\begin{bmatrix} b & c & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ a & b & c & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & a & b & c & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & a & b & c\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a & b\end{bmatrix}\begin{bmatrix} u_1^{j+1}\\ u_2^{j+1}\\ \vdots\\ u_{N_x-2}^{j+1}\\ u_{N_x-1}^{j+1}\end{bmatrix}$$

$$=\begin{bmatrix} 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 4 & 1\\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 4\end{bmatrix}\begin{bmatrix} u_1^j\\ u_2^j\\ \vdots\\ u_{N_x-2}^j\\ u_{N_x-1}^j\end{bmatrix}+\begin{bmatrix} c6k\pi\sin 2\pi x\\ \vdots\\ 6k\pi\sin 2\pi x\end{bmatrix}.$$

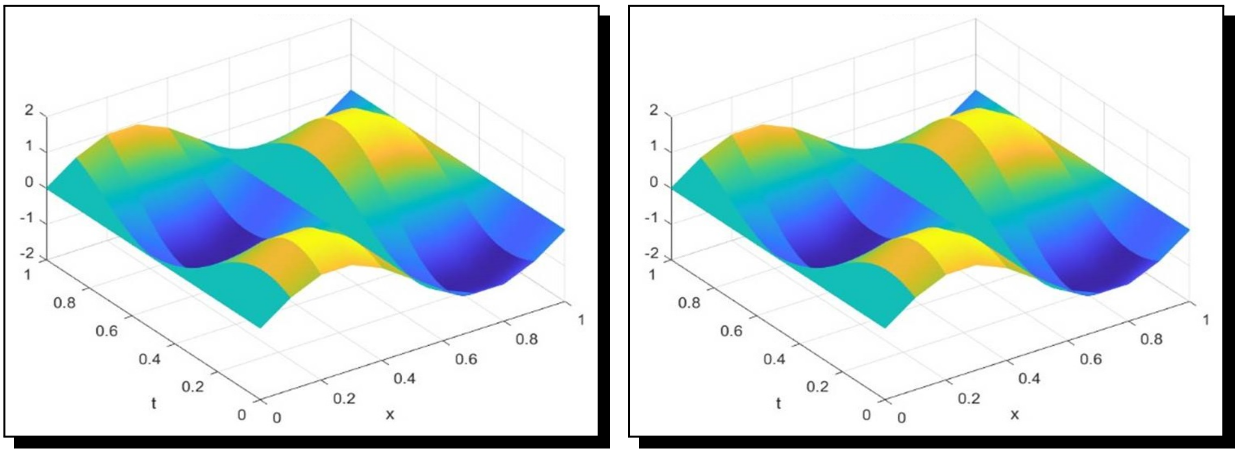
In short form

$$M_L X^{j+1} = M_R X^j + C_R\,.$$

Thus, the solution is

$$X^{j+1} = (M_L)^{-1}[M_R X^j + C_R].$$

The accuracy of the NCS implicit with analytical solution for Example 3.2 is shown in Figure 4.



(a) **(b)**

Figure 4. Solution of Example 3.2 using (a) NCS implicit method and (b) analytical solution

Example 3.3. Consider the following wave equation of the form:

$$u_{tt} = \frac{1}{4}u_{xx}, \quad 0 < x < 1,\ t > 0 \tag{3.17}$$

subject to: $u(x,0) = x$, $u_t(x,0) = e^x$, $0 < x < 1$

and $u_x(0,t) = 2\sinh(t/2)+1$, and $u_x(1,t) = 2e\sinh(t/2)+1$, $t > 0$

Exact solution: $u(x,t) = 2e^x\sinh(t/2)+x$

Explicit Method

By NCS explicit procedure, eq. (3.17) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = \left(2+\frac{3}{2}r^2\right)u_{i-1}^j + (8-3r^2)u_i^j + \left(2+\frac{3}{2}r^2\right)u_{i+1}^j - u_{i-1}^{j-1} - 4u_i^{j-1} - u_{i+1}^{j-1}\,. \tag{3.18}$$

Given $u_t(x,0) = e^x$

$$\implies \quad \frac{u_i^{j+1} - u_i^{j-1}}{2k} = e^{x_i}$$

$$\implies \quad u_i^{j+1} = u_i^{j-1} + 2k(e^{x_i}), \quad i = 0,1,2,\dots,N_x.$$

Equation (3.18) becomes

$$u_{i-1}^{j+1} + 4u_i^{j+1} + u_{i+1}^{j+1} = \left(1+\frac{3}{4}r^2\right)u_{i-1}^j + \left(4-\frac{3}{2}r^2\right)u_i^j + \left(1+\frac{3}{4}r^2\right)u_{i+1}^j + 3k(e^{x_i}), \tag{3.19}$$

where $r^2 = \frac{k^2}{h^2}$. The matrix form of (3.19) is

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} l & m & n & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & l & m & n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & l & m & n & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & l & m & n \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \vdots \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 3ke^x \\ \vdots \\ 3ke^x \end{bmatrix}$$

for $j = 0, 1, 2, \dots, N_t$, where $l = 1 + \frac{3}{4}r^2$, $m = 4 - \frac{3}{2}r^2$ and $n = 1 + \frac{3}{4}r^2$.

From boundary conditions

$$u_x(0,t) = 2\sinh(t/2) + 1 \Longrightarrow \frac{u_{i+1}^j - u_{i-1}^j}{2h} = 2\sinh(t^j/2) + 1, \quad j \ge 0$$
$$\Longrightarrow u_1^j = u_{-1}^j + 2h(2\sinh(t^j/2) + 1), \quad j \ge 0$$

and

$$u_x(1,t) = 2e\sinh(t/2) + 1 \Longrightarrow \frac{u_{i+1}^j - u_{i-1}^j}{2h} = 2e\sinh(t^j/2) + 1, \quad j \ge 0$$
$$\Longrightarrow u_{N_x+1}^j = u_{N_x-1}^j + 2h(2e\sinh(t^j/2) + 1), \quad j \ge 0$$

The above matrix reduces to

$$\begin{bmatrix} 4 & 2(1) & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2(1) & 4 \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} + \begin{bmatrix} 2h(2h\sinh(t/2)+1) \\ 0 \\ \vdots \\ 0 \\ 2h(2h\sinh(t/2)+1) \end{bmatrix}$$

$$= \begin{bmatrix} m & 2n & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & l & m & n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & l & m & n & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & l & m & n \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2l & m \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \vdots \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 2h(2h\sinh(t/2)+1) + 3ke^x \\ 3ke^x \\ \vdots \\ 3ke^x \\ 2h(2h\sinh(t/2)+1) + 3ke^x \end{bmatrix}.$$

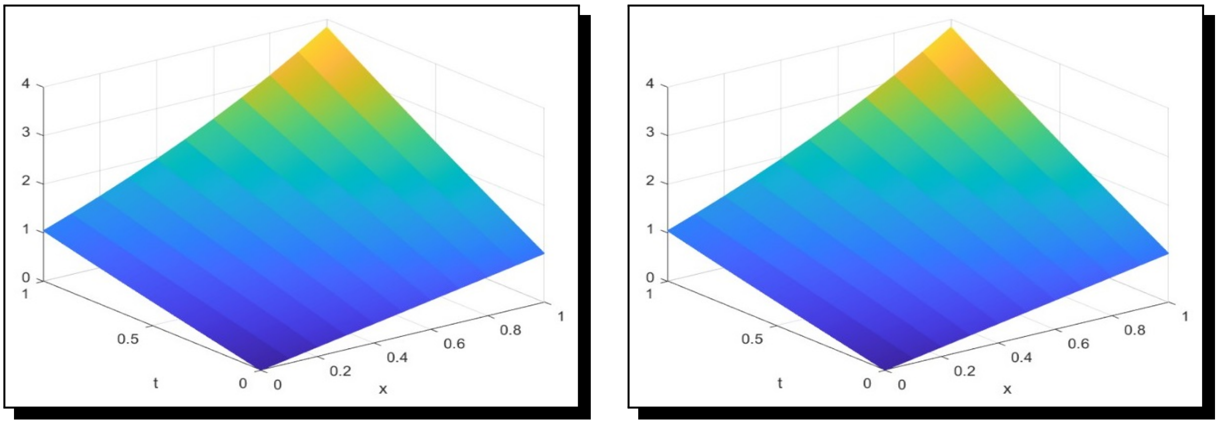
In short form

$$M_L X^{j+1} + C_L = M_R X^j + C_R\,.$$

Thus, the solution can be obtained from

$$X^{j+1} = (M_L)^{-1}[M_R X^j + C_R - C_L].$$

The accuracy of the NCS explicit with analytical solution for Example 3.3 is shown in Figure 5.



(a) (b)

Figure 5. Solution of Example 3.3 using (a) NCS explicit method and (b) analytical solution

Implicit Method

By NCS implicit method, eq. (3.17) becomes

$$(4-3r^2)u_{i-1}^{j+1}+(16+6r^2)u_i^{j+1}+(4-3r^2)u_{i+1}^{j+1} \tag{3.20}$$

$$=(3r^2-4)u_{i+1}^{j-1}-(6r^2+4)u_i^{j-1}+(3r^2-4)u_{i+1}^{j-1}+8(u_{i+1}^{j}+4u_i^j+u_{i-1}^j), \tag{3.21}$$

where $r^2=\frac{k^2}{h^2}$. From the given initial condition,

$$u_t(x,0)=e^x \Rightarrow \frac{u_i^{j+1}-u_i^{j-1}}{2k}=e^x$$

Eq. (3.20) reduces to

$$\left(1-\frac{3}{4}r^2\right)u_{i-1}^{j+1}+\left(4+\frac{3}{2}r^2\right)u_i^{j+1}+\left(1-\frac{3}{4}r^2\right)u_{i+1}^{j+1}=u_{i+1}^j+4u_i^j+u_{i-1}^j+3ke^{x_i^j}. \tag{3.22}$$

Matrix form of (3.22) becomes

$$\begin{bmatrix} a & b & c & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & a & b & c & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & a & b & c & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & a & b & c\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a & b \end{bmatrix} \begin{bmatrix} u_0^{j+1}\\ u_1^{j+1}\\ \vdots\\ u_{N_x-1}^{j+1}\\ u_{N_x}^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} u_0^{j}\\ u_1^{j}\\ \vdots\\ u_{N_x-1}^{j}\\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 3ke^x\\ \vdots\\ 3ke^x \end{bmatrix},$$

where $a=1-\frac{3}{4}r^2$, $b=4+\frac{3}{2}r^2$ and $c=1-\frac{3}{4}r^2$.

From boundary conditions,

$$u(0,t)=2\sinh(t/2)+1 \Longrightarrow \frac{u_{i+1}^j-u_{i-1}^j}{2h}=2\sinh(t^j/2)+1, \quad j\ge 0$$

$$\implies u_1^j = u_{-1}^j + 2h(2\sinh(t^j/2)+1), \quad j \ge 0$$

another condition:

$$u_x(1,t) = 2e\sinh(t/2)+1 \implies \frac{u_{i+1}^j - u_{i-1}^j}{2h} = 2e\sinh(t^j/2)+1, \quad j \ge 0$$

$$\implies u_{N_x+1}^j = u_{N_x-1}^j + 2h(2e\sinh(t^j/2)+1), \quad j \ge 0$$

Therefore, matrix form reduces to

$$\begin{bmatrix} m & 2n & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & l & m & n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & l & m & n & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & l & m & n \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2l & m \end{bmatrix} \begin{bmatrix} u_0^{j+1} \\ u_1^{j+1} \\ \vdots \\ u_{N_x-1}^{j+1} \\ u_{N_x}^{j+1} \end{bmatrix} + \begin{bmatrix} 2h(2h\sinh(t/2)+1) \\ 0 \\ \vdots \\ 0 \\ 2h(2h\sinh(t/2)+1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2(1) & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2(1) & 4 \end{bmatrix} \begin{bmatrix} u_0^{j} \\ u_1^{j} \\ \vdots \\ u_{N_x-1}^{j} \\ u_{N_x}^{j} \end{bmatrix} + \begin{bmatrix} 2h(2h\sinh(t/2)+1)+3ke^x \\ 3ke^x \\ \vdots \\ 3ke^x \\ 2h(2h\sinh(t/2)+1)+3ke^x \end{bmatrix}$$

$$\implies \quad M_L X^{j+1} + C_L = M_R X^j + C_R$$

$$\implies \quad X^{j+1} = (M_L)^{-1}[M_R X^j + C_R - C_L]$$

The accuracy of the NCS explicit with analytical solution for Example 3.3 is shown in Figure 6.

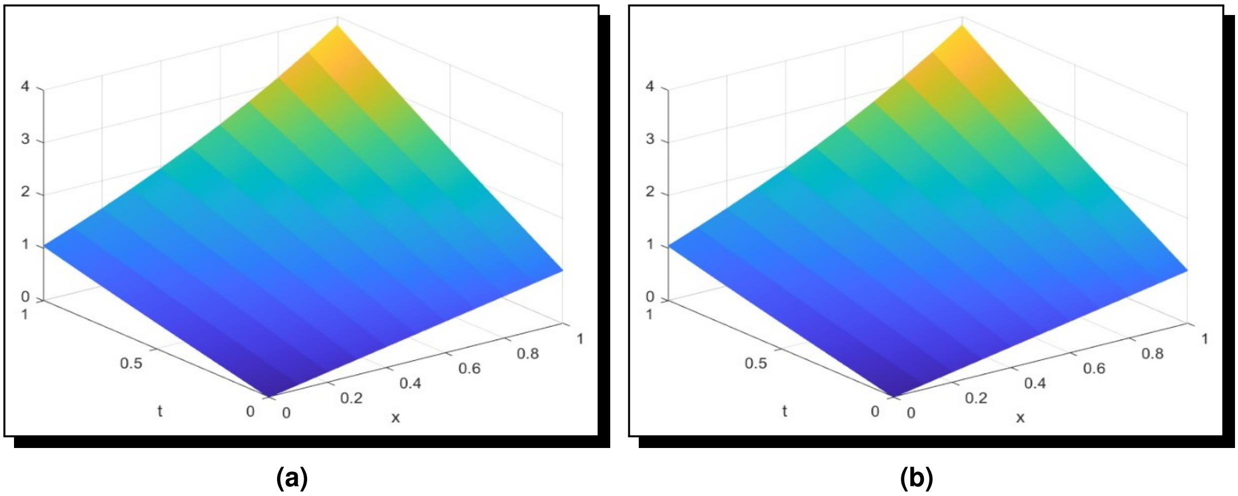


Figure 6. Solution of Example 3.3 using (a) NCS implicit method and (b) analytical solution

4. Conclusion

NCS method is employed to solve hyperbolic PDE in this paper. The detailed procedure is explained for NCS method for two different types such as explicit and implicit. The time and space derivatives are replaced by forward finite difference operator and NCS derivatives.

A tri-diagonal system of equations is obtained for different index values and converted into matrix form. With the given initial condition, matrix form is revised and incorporated the boundary conditions. The resultant matrix is solved using inverse method. We took different examples to check the developed NCS method. We demonstrated the NCS method for different PDEs with different examples and presented their results graphically.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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