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Research Article

New Fixed Point Theorem for Generalized Expansion Mappings Utilizing Banach Algebra in $\bar{\mathcal{G}}$-CMSs

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Abstract. Beg *et al.* [3, 4] were the first to propose generalized cone metric spaces as a comprehensive framework. They established the presence of fixed points in cone metrics for mappings and generalized metric spaces that adhere to specific contractive conditions. In our article, we introduce novel findings concerning fixed points within the context of $\bar{\mathcal{G}}$-*Cone Metric Spaces using Banach Algebras* (referred to as $\bar{\mathcal{G}}$-CMSBA) by utilizing generalized expansion mappings within the framework of Banach algebras.

Keywords. Generalized expansion mapping, Banach Algebras (BA) $\hat{\mathcal{B}}$, $\bar{\mathcal{G}}$-CMS

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1. Introduction

Dhage *et al.* [6] introduced the notion of generalized metric spaces, referred to as $\mathcal{D}$-metric spaces. However, this theory had been found to be incorrect. Subsequently, Mustafa *et al.* [14] developed a more suitable extension of metric spaces known as $\bar{\mathcal{G}}$-metric spaces. In the context of $\bar{\mathcal{G}}$-metric spaces, Mustafa *et al.* [13] discovered fixed point results that fulfill specific contractive conditions. Additionally, Fixed-Point (FP) ideas for noncommuting maps requiring continuity in $\bar{\mathcal{G}}$-metric spaces were obtained by Abbas and Rhoades [1].

An established *Fixed-Point* (FP) concept was created by Huang and Zhang [7], who developed the ‘idea of *Cone Metric Spaces* (CMSs) by substituting an ordered *Banach Space* (BS) for real numbers'. Rezapour and Hamlbarani [19] made a significant advancement in the study of CMSs using FP theory by eliminating the condition of normality.

Recent developments within the field of generalized metrics include the work by Beg *et al*. [5], who introduced 'generalized cone metric spaces', also referred to as $\bar{\mathfrak{G}}$-CMSs. In their definition of the generalized metric, they replaced the collection of real numbers with an ordered BS. This led to the discovery of numerous fixed point theorems that met contractive criteria, as well as an exploration of the topological characteristics of these spaces, including completeness and sequence convergence. In subsequent works of Beg *et al*. [3, 4] and Mishra *et al*. [11], they studied multiple findings related to $\bar{\mathfrak{G}}$-CMSs. Additionally, Özturk and Başarir[16] derived the concept of 'a fixed point for ϕ-maps within $\bar{\mathfrak{G}}$-CMSs'. Mustafa *et al*. [12] also presented various results concerning expansion mappings in generalized metric spaces.

Throughout this work, we propose new results for FP of generalized expansion mappings in a $\bar{\mathfrak{G}}$-CMSBA. These findings extend several previously established results by Patil and Salunke [17].

2. Preliminaries

We consider $\ddot{\mathcal{C}}$ to be a cone within $\hat{\mathcal{B}}$, where the interior of $\ddot{\mathcal{C}}$ is not empty, specifically $\operatorname{int}\ddot{\mathcal{C}} \neq \phi$ (θ, the additive identity element of $\hat{\mathcal{B}}$). Additionally, we define $\preceq$ as the partial ordering related to $\ddot{\mathcal{C}}$, with $\hat{\mathcal{B}}$ representing a Banach algebra. In this context, $\hat{\mathcal{B}}$ is characterized as a Banach space that includes a defined multiplication operation, which adheres to the properties outlined by Rudin [20] (for every $\omega, u, v \in \hat{\mathcal{B}}, \zeta \in \mathbb{R}$):

(i) $\omega(uv) = (\omega u)v$,

(ii) $v(\omega + u) = v\omega + vu$ and $(u+v)\omega = u\omega + v\omega$,

(iii) $\zeta(\omega u) = (\zeta\omega)u = \omega(\zeta u)$,

(iv) $e \in \mathcal{B}$ exists such that $e\omega = \omega = \omega e$,

(v) $\|\omega u\| \leq \|\omega\| \cdot \|u\|$,

(vi) $\|e\| = 1$.

If any $\omega^{-1} \in \hat{\mathcal{B}}$ such that $\omega^{-1}\omega = e = \omega\omega^{-1}$, then an element $\omega \in \hat{\mathcal{B}}$ is said to be invertible.

Proposition 2.1 ([9], [10])**.** The spectral radius of $\omega \in \hat{\mathcal{B}}$ is $\tau(\omega)$ if $\omega \in \hat{\mathcal{B}}$ is a BA having e as unit,

$$\tau(\omega) = \lim_{\iota\to\infty} \|\omega_\iota\|^{\frac{1}{\iota}} = \inf \|\omega^\iota\|^{\frac{1}{\iota}} < 1.$$

Moreover, $\sum\limits_{\iota=0}^{\infty} \omega^i = \frac{1}{(e-\omega)}$ and $(e-\omega)$ are invertible.

Consider the identity element e of $\hat{\mathcal{B}}$, and consider a BA $\hat{\mathcal{B}}$. $\ddot{\mathcal{C}}$ a subset of $\hat{\mathcal{B}}$ is referred to as a cone to determine if it satisfies the subsequent criteria:

(i) $\ddot{\mathcal{C}}\ddot{\mathcal{C}} = \ddot{\mathcal{C}}^2 \subset \ddot{\mathcal{C}}$;

(ii) If $\ddot{\mathcal{C}}$ is closed then $\{\Theta, e\} \subset \ddot{\mathcal{C}}$;

(iii) $\zeta\ddot{\mathcal{C}} + \xi\ddot{\mathcal{C}} \subset \ddot{\mathcal{C}}$, for any real values that are nonnegative ξ and ζ;

(iv) $\ddot{\mathcal{C}} \cap (-\ddot{\mathcal{C}}\ddot{\mathcal{C}}) = \{\Theta\}$.

$u \succeq \omega$ iff $(u-\omega) \in \ddot{\mathcal{C}}$, $u \succ \omega$ if $u \succeq \omega$ and $u \neq \omega$, $u \gg \omega$ signify $(u-\omega) \in \mathrm{int}\,\ddot{\mathcal{C}}$ is the definition of a partial ordering denoted by $\succeq$ in relation to a cone $\ddot{\mathcal{C}}$.

$\ddot{\mathcal{C}}$ referred to as a cone of $\hat{\mathcal{B}}$ (Banach algebra), $\ddot{\mathcal{C}} \subset \hat{\mathcal{B}}$, and all of these requirements hold.

Remark 2.2 ([21]). $\|\omega_\iota\| \to 0$ as $\iota \to \infty$, if $\tau(\omega) < 1$.

Definition 2.3 ([9], [10]). While Y having a set, $\hat{\mathcal{B}}$ being a BA, $\ddot{\mathcal{C}} \subseteq \hat{\mathcal{B}}$ representing a cone is considered. Let us assume the following for any ω, u, v are element of Y the maps $\bar{d} : Y^2 \to \hat{\mathcal{B}}$ satisfies:

(i) $\bar{d}(u,\omega) = \Theta$ iff $\omega = u$, and $\Theta \preceq \bar{d}(u,\omega)$,

(ii) $\bar{d}(u,\omega) = \bar{d}(\omega,u)$,

(iii) $\bar{d}(u,v) + \bar{d}(v,\omega) \succeq \bar{d}(u,t)$, every $u,v,\omega \in Y$.

In this case, $(Y,\bar{d})$ represents the CMSBA with $\hat{\mathcal{B}}$, and d is known as a cone metric (CSMBA in short). Keep in mind that for any $\omega, v \in Y$, $\bar{d}(\omega,v) \in \ddot{\mathcal{C}}$.

Definition 2.4 ([9]). Consider a CMSBA $(Y,\bar{d})$. $\{\omega_n\}$ a sequence in Y. We claim that

(i) If $c \gg \Theta$ for each $c \in \hat{\mathcal{B}}$ then $c \gg \bar{d}(\omega_n,t)$ for everyone $n \geq N.\{\omega_n\}$ has sequence of convergent.

(ii) A sequence $\{\omega_n\}$ is Cauchy. If appears a N so that $c \gg \bar{d}(\omega_n,\omega_m)$ for every $N \leq m,n$, and every c element of $\hat{\mathcal{B}}$ by $\Theta \gg c$.

(iii) For any $Y,(Y,\bar{d})$ has complete CMSs so that all Cauchy sequences have convergence.

Lemma 2.5 ([21]). *Suppose κ is a vector within $\hat{\mathcal{B}}$. When given that $1 > r(\kappa) \geq 0$, there is*

$$\frac{1}{(1-\tau(\kappa))} > \tau\frac{1}{(e-\kappa)}.$$

Lemma 2.6 ([21]). *Consider ω, u are vectors in Banach algebra $\hat{\mathcal{B}}$. When ω and u commute, those that follow is accurate:*

(i) $\tau(u\omega) \leq \tau(u).\ \tau(\omega)$;

(ii) $\tau(\omega+u) \leq \tau(\omega) + \tau(u)$;

(iii) $|\tau(\omega) - \tau(u)|\tau(\omega-u) \leq \tau(\omega-u)$.

Lemma 2.7 ([18]). *Given a real Banach algebra $\hat{\mathcal{B}}$, a sequence $\{\omega_\iota\}$ in $\hat{\mathcal{B}}$, and a solid cone $\ddot{\mathcal{C}}$. Assume that for any $\Theta \ll c$, $\|\omega_\iota| \to 0$ $(\iota \to \infty)$. For every N^1 is element of N and less than n, $\omega_\iota \ll c$.*

Lemma 2.8 ([21]). *$N \in \mathbb{N}$ is present so that, for all $\iota > N$, it has $\omega_\iota \ll c$ for every $\Theta \ll c$,, that $\|\omega_\iota\| \to 0$ as $\iota \to \infty$ and that $\hat{\mathcal{B}}$ has a Banach space real within cone $\ddot{\mathcal{C}}$ solid are assumed.*

Lemma 2.9 ([22]). *With a cone form the limit of a convergent sequence is not the same in a metric space.*

Definition 2.10 ([2], [10])**.** A function $\bar{\mathcal{G}}: Y^3 \to \hat{\mathcal{B}}$ fulfills the condition listed here, and let Y is a nonempty set of $\hat{\mathcal{B}}$:

(i) $\Theta = \bar{\mathcal{G}}(v,\omega,u)$ iff $u = \omega = v$;

(ii) $\Theta \prec \bar{\mathcal{G}}(v,t,u)$, for all $v,\omega \in Y$, using $v \neq \omega$,

(iii) $\bar{\mathcal{G}}(v,\omega,u) \succeq \bar{\mathcal{G}}(v,v,u)$ for all $v,\omega,u \in Y$, with $v \neq \omega$;

(iv) Symmetry $\bar{\mathcal{G}}(u,v,\omega) = \bar{\mathcal{G}}(v,u,\omega) = \bar{\mathcal{G}}(v,\omega,u)$,

(v) Rectangle inequality $\bar{\mathcal{G}}(u,\omega,v) \preceq \bar{\mathcal{G}}(v,b,b) + \bar{\mathcal{G}}(b,\omega,v)$ for every $b,u,\omega,v \in Y$.

The combination $(Y,\bar{\mathcal{G}})$ indicates a $\bar{\mathcal{G}}$-CMSBA, and $\bar{\mathcal{G}}$ is referred to as a $\bar{\mathcal{G}}$-cone metric in $\hat{\mathcal{B}}$.

Definition 2.11 ([2])**.** The metric space over Banach algebra and $(Y,\bar{\mathcal{G}})$ has G-CMSBA. Now we say that $\bar{\mathcal{G}}$ is symmetric when

$$\bar{\mathcal{G}}(v,u,u) = \bar{\mathcal{G}}(v,v,u),$$

for any $v,u \in Y$.

Here following lists both nonsymmetric and symmetric $\bar{\mathcal{G}}$-CMSBA as examples.

Definition 2.12 ([2])**.** Suppose $(Y,\bar{\mathcal{G}})$ is $\bar{\mathcal{G}}$-CMSBA. Exists $\kappa > \Theta$ so that for every $\omega,v,u \in Y$, then $\hat{\mathcal{B}}$ is $\bar{\mathcal{G}}$-bounded,

$$\|\bar{\mathcal{G}}(u,v,\omega)\| \leq \kappa.$$

Definition 2.13 ([2])**.** Suppose $\{\omega_n\}$ is a sequence, $\Theta \ll c$ and $c \in \hat{\mathcal{B}}$, let $(Y,\bar{\mathcal{G}})$ be a $\bar{\mathcal{G}}$-CMSBA.

(i) If $\bar{\mathcal{G}}(\omega_m,\omega_n,\omega) \ll c$, for all $N^1 < j$, ι element of N, then $\{\omega_i\}$ converges for $\omega \in Y$.

(ii) If $\bar{\mathcal{G}}(\omega_\iota,\omega_j,\omega_p) \ll c$, for all $\iota, j > p$, then $\{\omega_\iota\}$ to be Cauchy, $N^1 \in N$.

(iii) If all Cauchy sequences converge, then $(Y,\bar{\mathcal{G}})$ has completed of $\bar{\mathcal{G}}$-CMSBA.

Definition 2.14 ([12], [10])**.** Let f is a self-mapping of Y, and let $(Y,\bar{\mathcal{G}})$ has $\bar{\mathcal{G}}$-CMSBA. If

$$\bar{\mathcal{G}}(\urcorner u,\urcorner\omega,\urcorner v) \geq a\bar{\mathcal{G}}(u,\omega,v)$$

with a has constant and $a > 1$ for every $\omega,u,v \in Y$, then $\urcorner$ is referred to as an expansion mapping.

Definition 2.15 ([8])**.** Consider $\ddot{\mathcal{C}}$ is solid cone in $\hat{\mathcal{B}}$. $\iota_0 \in \mathbb{N}$ exist such that $\omega_\iota \ll c$ for $n \geq i_0$, then a sequence $\{\omega_\iota\} \subset \ddot{\mathcal{C}}$ to be c-sequence.

Proposition 2.16 ([8])**.** Consider $\ddot{\mathcal{C}}$ is solid cone in $\hat{\mathcal{B}}$. Two sequences in $\ddot{\mathcal{C}}$ are $\{\omega_\iota\}$ and $\{u_\iota\}$. $\{\zeta\omega_\iota + \xi u_\iota\}$ is a c-sequence if $\{\omega_\iota\}$ and $\{u_\iota\}$ are c-sequences and $\zeta,\xi > 0$.

Proposition 2.17 ([8])**.** Consider $\ddot{\mathcal{C}}$ is solid cone in $\hat{\mathcal{B}}$. If sequence $\{\omega_\iota\}$ exists with $\ddot{\mathcal{C}}$, then the subsequent statements hold equal meaning:

(i) A c-sequence is $\{\omega_\iota\}$.

(ii) For any of the $\Theta \ll c$, it exists $i_0 \in \mathbb{N}$ it gives $\omega_n \prec c$ over $n_0 \leq \iota$.

(iii) During every $\Theta \ll c$, $\omega_\iota \prec c$ for $\iota_1 \leq n$, an exists $\iota_1 \in \mathbb{N}$.

Proposition 2.18 ([21]). Consider $\ddot{\mathcal{C}}$ is solid cone in $\hat{\mathcal{B}}$. $\ddot{\mathcal{C}}$ contains a series $\{\omega_\iota\}$. Assume that $\{\omega_\iota\}$ is c-sequence for $\ddot{\mathcal{C}}$ and $\kappa \in \ddot{\mathcal{C}}$ is an vector that is supplied at arbitrarily. If so, $\{\kappa\omega_\iota\}$ is c-sequence.

Proposition 2.19 ([21]). Suppose $(Y, \bar{d})$ is complete CMSBA. Let $\ddot{\mathcal{C}}$ be a solid cone. Consider a sequence $\{\omega_\iota\}$ in Y, assuming that converges to $\omega \in Y$.

(i) The expression $\{\bar{d}(\omega_\iota, \omega)\}$ is a c-sequence.

(ii) A c-sequence is $\{\bar{d}(\omega_\iota, \omega_{\iota+p})\}$ for any $p \in \mathbb{N}$.

3. Main Result

Definition 3.1. Consider $\urcorner : Y \to Y$ serve as maps and let $(Y, \bar{\mathfrak{G}})$ and $\bar{\mathfrak{G}}$-CMSBA $\hat{\mathcal{B}}$. If there exists $\mu \in \ddot{\mathcal{C}}$ with $\tau(\mu) > 1$ so that

$$\bar{\mathfrak{G}}(\urcorner u, \urcorner v, \urcorner \omega) \succeq \mu \bar{\mathfrak{G}}(u, v, \omega),$$

then $\urcorner$ is a generalized expanding mapping.

Lemma 3.2. *If ω is invertible, then $\mu \in \tau(\omega)$ iff $\mu^{-1} \in \tau(\omega^{-1})$.*

Lemma 3.3. *Consider $\hat{\mathcal{B}}$ has a Banach algebra using unity. In $\hat{\mathcal{B}}$, suppose that $\ddot{\mathcal{C}}$ has a cone. $\tau(\omega^{-1}) < 1$ if has invertible and $1 < \tau(\omega)$.*

We will now demonstrate certain new fixed point results with generalized expanding maps within a structure of Banach algebras for $\bar{\mathfrak{G}}$-CMSBA.

Theorem 3.4. *Consider the cone $\ddot{\mathcal{C}}$ within the Banach algebra $\hat{\mathcal{B}}$ and the complete of $\bar{\mathfrak{G}}$-CMSBA $(Y, \bar{\mathfrak{G}})$. Assume that $\urcorner : Y \to Y$ is surjective mapping and fulfilled the condition*

$$\bar{\mathfrak{G}}(\urcorner u, \urcorner v, \urcorner \omega) \succeq \mu \bar{\mathfrak{G}}(u, v, \omega). \tag{3.1}$$

$\urcorner$ has a fixed point in Y if μ is invertible and $\tau(\mu) > 1$ for every $\omega, u, v \in Y$ and $\mu \in \ddot{\mathcal{C}}$.

Proof. Assuming that $\urcorner : Y \to Y$ has surjective maps, it exists ω_1 an element of Y so that $\urcorner(\omega_1) = \omega_0$ for every $\omega_0 \in Y$. Similarly, there exists $\omega_\iota \in Y$ so that $\urcorner(\omega_\iota) = \omega_{\iota-1}$ for every $\iota \geq 1$. A fixed point of $\urcorner$ is ω_ι if $\urcorner(\omega_\iota) = \omega_{\iota-1}$. Assume that for $\iota \geq 1$, $\urcorner(\omega_\iota) \neq \omega_{\iota-1}$. Consequently,

$$\bar{\mathfrak{G}}(\omega_{\iota-1}, \omega_{\iota-2}, \omega_{\iota-2}) = \bar{\mathfrak{G}}(\urcorner\omega_\iota, \urcorner\omega_{\iota-1}, \urcorner\omega_{\iota-1}) \succeq \mu \bar{\mathfrak{G}}(\omega_\iota, \omega_{\iota-1}, \omega_{\iota-1}),$$
$$\bar{\mathfrak{G}}(\omega_{\iota-1}, \omega_\iota, \omega_\iota) \preceq \mu^{-1} \bar{\mathfrak{G}}(\omega_{\iota-2}, \omega_{\iota-1}, \omega_{\iota-1}) = h \bar{\mathfrak{G}}(\omega_{\iota-2}, \omega_{\iota-1}, \omega_{\iota-1}),$$

where μ^{-1} replace by h. Lemma 3.3 implies that $\tau(h) = \tau(\mu^{-1}) < 1$ since $\tau(\mu) > 1$. It has got

$$\bar{\mathfrak{G}}(\omega_{\iota-1}, \omega_\iota, \omega_\iota) \preceq h \bar{\mathfrak{G}}(\omega_{\iota-2}, \omega_{\iota-1}, \omega_{\iota-1}) \preceq h^2 \bar{\mathfrak{G}}(\omega_{\iota-3}, \omega_{\iota-2}, \omega_{\iota-2}).$$

Usually,

$$\bar{\mathfrak{G}}(\omega_{\iota-1}, \omega_\iota, \omega_\iota) \preceq h^\iota \bar{\mathfrak{G}}(\omega_0, \omega_1, \omega_1).$$

Assuming that $j > \iota$ and every ι, j elements of $\mathbb{N}$, we obtain

$$\begin{aligned}\bar{\mathfrak{G}}(\omega_{\iota+1}, \omega_j, \omega_j) &\preceq \bar{\mathfrak{G}}(\omega_{\iota+1}, \omega_{\iota+2}, \omega_{\iota+2}) + \bar{\mathfrak{G}}(\omega_{\iota+2}, \omega_{\iota+3}, \omega_{\iota+3}) \ldots \bar{\mathfrak{G}}(\omega_{j-1}, \omega_j, \omega_j)\\ &\preceq h^\iota \bar{\mathfrak{G}}(\omega_0, \omega_1, \omega_1) + h^{\iota+1} \bar{\mathfrak{G}}(\omega_0, \omega_1, \omega_1) + \ldots + h^{j-1} \bar{\mathfrak{G}}(\omega_0, \omega_1, \omega_1)\\ &\preceq h^\iota (e + h + h^2 + \ldots + h^{j-\iota-1}) \bar{\mathfrak{G}}(\omega_0, \omega_1, \omega_1)\end{aligned}$$

$$\preceq h^{\iota}(e-h)^{-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1).$$

Using Remark 2.2 and Lemma 2.8, we can infer that, for every $N < \iota < j$, and $N \in \mathbb{N}$ so that $\urcorner\omega_{\iota+1} = \omega_\iota \to q$ as $(\iota \to \infty)$. This is because for any $\mu \in \hat{\mathcal{B}}$ with $\mu \gg \Theta$.

The fact that $\{\omega_\iota\}$ is a Cauchy sequence is implied by the notion that $\bar{\mathcal{G}}(\omega_\iota,\omega_j,\omega_j) \preceq h^{\iota}(e-h)^{-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1) \ll \mu$. Since Y is complete, $\urcorner\omega_{\iota+1} = \omega_\iota \to q$ as $(\iota \to \infty)$ occurs for all $q \in Y$.

As a result, $\urcorner\tau = q$ may be found for every $\tau \in Y$. We now show $\tau = q$.

In (3.1), if we replace $\tau = \omega, v = \omega_\iota$, we obtain

$$\begin{aligned}\bar{\mathcal{G}}(q,\omega_{\iota-1},\omega_{\iota-1}) &= \bar{\mathcal{G}}(\urcorner\tau,\urcorner\omega_\iota,\omega_\iota)\\ &\succeq \mu\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota).\end{aligned}$$

Through triangular inequality,

$$\bar{\mathcal{G}}(q,\omega_{\iota-1},\omega_{\iota-1}) \preceq \bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota) + \bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}).$$

When we substitute the previous inequality, we get

$$\begin{aligned}&\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota) + \bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}) \succeq \mu\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota),\\ &\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota) \preceq \mu^{-1}(\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota) + \bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})),\\ &\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota) \preceq h(\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota) + \bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})).\end{aligned}$$

We obtain $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota) \preceq u_{\iota-1}$ from Proposition 2.17, Proposition 2.18, and 2.19, where $u_{\iota-1}$ is a c-sequence in cone $\mathcal{C}$.

Therefore, $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota) \ll \mu$ for each $\mu \gg \Theta$. This means that $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota) \to \Theta$ as $\iota \to \infty$. As $\iota \to \infty$, $\omega_\iota \to \tau$.

Lemma 2.9 gives us $q = \tau$. As a result, function $\urcorner$ possesses a distinct fixed point. □

Theorem 3.5. *Consider the cone $\ddot{\mathcal{C}}$ within the Banach algebra $\hat{\mathcal{B}}$ and the complete of $\bar{\mathcal{G}}$-CMSBA $(Y,\bar{\mathcal{G}})$. Assume that $\urcorner : Y \to Y$ is surjective mapping and fulfilled the condition*

$$\mathcal{G}(\urcorner u,\urcorner v,\urcorner\omega) \succeq \mu_1\bar{\mathcal{G}}(\urcorner u,v,\omega) + \mu_2\bar{\mathcal{G}}(\urcorner v,v,v). \tag{3.2}$$

There is unique fixed point for $\urcorner$ in Y if $(\mu_1+\mu_2)$ is invertible and $\tau(\mu_1+\mu_2) > 1$, for every $u,v,\omega \in Y$ and $\mu_1,\mu_2 \in \ddot{\mathcal{C}}$.

Proof. Considering that $\urcorner : Y \to Y$ is a surjective mapping, there exists $\omega_1 \in Y$ so that $\urcorner(\omega_1) = \omega_0$ for any $\omega_0 \in Y$. Furthermore, there exists $\omega_{\iota+1} \in Y$ so that $\urcorner(\omega_\iota) = \omega_{\iota-1}$ for every $n \geq 1$. ω_ι has fixed point of $\urcorner$ if $\urcorner(\omega_{\iota+1}) = \omega_i$. Let us now assume that for $\iota \geq 1$, $\omega_{\iota-1} \neq \urcorner(\omega_\iota)$. Consequently,

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{v-2},\omega_{\iota-2}) &= \bar{\mathcal{G}}(\urcorner\omega_\iota,\urcorner\omega_{\iota-1},\urcorner\omega_{\iota-1})\\ &\succeq \mu_1\bar{\mathcal{G}}(\urcorner\omega_\iota,\omega_\iota,\omega_\iota) + \mu_2\bar{\mathcal{G}}(\urcorner\omega_{\iota-1},\omega_{\iota-1},\omega_{\iota-1})\\ &= \mu_1\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota) + \mu_2\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1}),\end{aligned}$$

$$(e-\mu_2)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-2},\omega_{\iota-2}) \succeq \mu_1\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota). \tag{3.3}$$

Also,

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1}) &= \bar{\mathcal{G}}(\urcorner\omega_{\iota-1},\urcorner\omega_\iota,\urcorner\omega_\iota)\\ &\succeq \mu_1\bar{\mathcal{G}}(\urcorner\omega_{\iota-1},\omega_{\iota-1},\omega_{\iota-1}) + \mu_2\bar{\mathcal{G}}(\urcorner\omega_\iota,\omega_\iota,\omega_\iota)\\ &= \mu_1\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1}) + \mu_2\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota),\end{aligned}$$

$$(e-\mu_1)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-2},\omega_{\iota-2}) \succeq \mu_2\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota). \tag{3.4}$$

When we add equations (3.3) and (3.4) together, we obtain

$$(2e-\mu_1-\mu_2)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-2},\omega_{\iota-2}) \succeq (\mu_1+\mu_2)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota),$$
$$(\mu_1+\mu_2)^{-1}(2e-(\mu_1+\mu_2))\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1}) \succeq \bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota),$$
$$(2\mu^{-1}-e)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-1},\omega_{\iota-1}) \succeq \bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota),$$

where $\mu=\mu_1+\mu_2$. Since $r(\mu)>1$ by Lemma 3.3, $r(\mu^{-1})<1$ so $(2\mu^{-1}-e)<1, \bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota) \preceq h\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1})$, where $h=(2\mu^{-1}-e)$.

From this it has, $G(\omega_{\iota-1},\omega_\iota,\omega_\iota) \preceq h^\iota G(\omega_0,\omega_1,\omega_1)$

With $n<m$ for every $m,n\in\mathbb{N}$, it has

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-1},\omega_j,\omega_j) &\preceq \bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota+1},\omega_{\iota+1})\dots\bar{\mathcal{G}}(\omega_{j-1},\omega_j,\omega_j)\\ &\preceq h^\iota\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)+h^{\iota+1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)+\dots+h^{j-1}G(\omega_0,\omega_1,\omega_1)\\ &\preceq h^\iota(e+h+h^2+\dots+h^{j-\iota-1})\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)\\ &\preceq h^i(e-h)^{-1}G(\omega_0,\omega_1,\omega_1).\end{aligned}$$

Lemma 2.8 and Remark 2.2 allow us to deduce that, for every $N<\iota<j$, it exists $N\in\mathbb{N}$ so that, for all $\mu\in\hat{\mathcal{B}}$ for $\Theta\ll\mu$.

It has

$$\bar{\mathcal{G}}(\omega_{\iota-1},\omega_j,\omega_j) \preceq h^\iota(e-h)^{-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1) \ll \mu.$$

This shows that the sequence $\{\omega_\iota\}$ is Cauchy. Given that Y is complete, it exists a $q\in Y$ so that $i\to\infty$ and $\urcorner(\omega_\iota)=\omega_{\iota-1}\to q$.

Obtaining $\tau\in Y$ so that $\urcorner\tau=q$ is therefore possible. We will now show that $\tau=q$.

When we replace $\omega=\tau, v=\omega_\iota$ in (3.2), we obtain

$$\begin{aligned}\bar{\mathcal{G}}(q,\omega_{\iota-1},\omega_{\iota-1}) &= \bar{\mathcal{G}}(\urcorner\tau,\urcorner\omega_\iota,\omega_\iota)\\ &\succeq \mu_1\bar{\mathcal{G}}(\urcorner\tau,\tau,\tau)+\mu_2\bar{\mathcal{G}}(\urcorner\omega_\iota,\omega_\iota,\omega_\iota)\\ &= \mu_1\bar{\mathcal{G}}(q,\tau,\tau)+\mu_2\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota).\end{aligned} \tag{3.5}$$

Using the triangle inequality,

$$\bar{\mathcal{G}}(q,\omega_{\iota-1},\omega_{\iota-1}) \preceq \bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}). \tag{3.6}$$

Also,

$$\bar{\mathcal{G}}(\omega_\iota,\tau,\tau) \preceq \bar{\mathcal{G}}(\omega_\iota,q,q)+\bar{\mathcal{G}}(q,\tau,\tau),$$
$$\bar{\mathcal{G}}(\omega_\iota,\tau,\tau)-\bar{\mathcal{G}}(\omega_\iota,q,q) \preceq \mathcal{G}(q,\tau,\tau). \tag{3.7}$$

Using these in (3.5),

$$\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1} \succeq \mu_1\bar{\mathcal{G}}(\omega_\iota,\tau,\tau)-\mu_1\bar{\mathcal{G}}(\omega_\iota,q,q)+\mu_2\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}),$$
$$(e+\mu_1)\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+(e-\mu_2)G(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}) \succeq \mu_1\bar{\mathcal{G}}(t_\iota,\tau,\tau). \tag{3.8}$$

Substituting $\omega=\omega_\iota, v=\tau$ in (3.2), we get

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-1},q,q) &= \bar{\mathcal{G}}(\urcorner\omega_\iota,\urcorner\tau,\urcorner\tau)\\ &\succeq \mu_1\bar{\mathcal{G}}(\urcorner\omega_\iota,\omega_\iota,\omega_\iota)+\mu_2\bar{\mathcal{G}}(\urcorner\tau,\tau,\tau)\end{aligned}$$

$$\succeq \mu_1\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)+\mu_2\bar{\mathcal{G}}(q,\tau,\tau).$$

Using equations (3.6) and (3.7), in above equation, we get

$$\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})\succeq \mu_1\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})+\mu_2\bar{\mathcal{G}}(\omega_\iota,\tau,\tau)-\mu_2\bar{\mathcal{G}}(\omega_\iota,q,q),$$

$$(e+\mu_2)\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+(e-\mu_1)\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})\succeq \mu_2\bar{\mathcal{G}}(\omega_\iota,\tau,\tau). \tag{3.9}$$

When we add equations (3.8) and (3.9) together, we get

$$(2e+\mu_1+\mu_2)\bar{\mathcal{G}}(q,\omega_i,\omega_\iota)+(2e-(\mu_1+\mu_2))\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})\succeq(\mu_1+\mu_2)\bar{\mathcal{G}}(\omega_\iota,\tau,\tau),$$

$$\bar{\mathcal{G}}(\omega_\iota,\tau,\tau)\preceq(2e+\mu)\mu^{-1}\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+(2e-\mu)\mu^{-1}\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})$$

$$\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau)\preceq(2\mu^{-1}+e)\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+(2\mu^{-1}-e)\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1}).$$

Since $\tau(\mu)>1$ so by Lemma 3.3, $\tau(\mu^{-1})<1$. Hence $\tau(2\mu^{-1}-e)=\tau(h)<1$

$$\bar{\mathcal{G}}(x_\iota,\tau,\tau)\preceq(2\mu^{-1}+e)\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota-1},\omega_{\iota-1})$$

We obtain $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota)\preceq u_\iota$ from Proposition 2.17, Proposition 2.18 and 2.19, where u_i is a c-sequence in cone $\mathcal{C}$.

Therefore, $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota)\ll\mu$ for each $\mu\gg\Theta$. This means that $\bar{\mathcal{G}}(\tau,\omega_\iota,\omega_\iota)\to\Theta$ as $\iota\to\infty$. As $\iota\to\infty$, $\omega_\iota\to\tau$.

Lemma 2.9 gives us $\tau=q$. As a result, function $\urcorner$ possesses a distinct fixed point. □

Theorem 3.6. *Consider the cone $\ddot{\mathcal{C}}$ within the Banach algebra $\hat{\mathcal{B}}$ and the complete of $\bar{\mathcal{G}}$-CMSBA $(Y,\bar{\mathcal{G}})$. Assume that $\urcorner:Y\to Y$ is surjective mapping and fulfilled the condition*

$$\bar{\mathcal{G}}(\urcorner u,\urcorner v,\urcorner\omega)\succeq\mu_1\bar{\mathcal{G}}(u,v,\omega)+\mu_2\bar{\mathcal{G}}(u,u,\urcorner u)+\mu_3\bar{\mathcal{G}}(v,v,\urcorner v)+\mu_4\bar{\mathcal{G}}(\omega,\omega,\urcorner\omega). \tag{3.10}$$

There are $u,v,\omega\in Y$, and all $\mu\in\ddot{\mathcal{C}}$, if $\sum_{\iota=1}^{4}\mu_\iota$ is invertible, $\tau(\mu_1+\mu_2+\mu_3+\mu_4)>1$ then $\urcorner$ has only one fixed point in Y.

Proof. Given that $\urcorner:Y\to Y$ is a surjective mapping, for every $\omega_0\in Y$, it exists $\omega_1\in Y$ so that $\urcorner(\omega_1)=\omega_0$. Moreover, for each $\iota\geq 1$, it exists $\omega_\iota\in Y$ so that $\urcorner(\omega_\iota)=\omega_{\iota-1}$. If $\urcorner\omega_\iota)=\omega_{\iota-1}$, then ω_ι is a fixed point of $\urcorner$. Now, let's consider that $\urcorner(\omega_\iota)\neq\omega_{\iota-1}$ for $\iota\geq 1$. From equation (3.10), we have

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1})&=\bar{\mathcal{G}}(\urcorner\omega_{\iota-1},\urcorner\omega_\iota,\urcorner\omega_\iota)\\&\succeq\mu_1\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)+\mu_2\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-1},\urcorner\omega_{\iota-1})\\&\quad+\mu_3\bar{\mathcal{G}}(\omega_\iota,\omega_\iota,\urcorner\omega_\iota)+\mu_4\bar{\mathcal{G}}(\omega_\iota,\omega_\iota,\urcorner\omega_\iota)\\&\succeq\mu_1\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)+\mu_2\bar{\mathcal{G}}(\omega_{\iota-1},\omega_{\iota-1},\urcorner\omega_{\iota-1})\\&\quad+\mu_3\bar{\mathcal{G}}(\omega_\iota,\omega_\iota,\omega_{\iota-1})+\mu_4\bar{\mathcal{G}}(\omega_\iota,\omega_\iota,\omega_{\iota-1}).\end{aligned}$$

Hence

$$(e-\mu_2)\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1})=(\mu_1+\mu_3+\mu_4)\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota),$$

$$\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)\preceq\frac{(e-\mu_2)}{(\mu_1+\mu_3+\mu_4)}\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1})\preceq h\bar{\mathcal{G}}(\omega_{\iota-2},\omega_{\iota-1},\omega_{\iota-1}), \tag{3.11}$$

where $h=\frac{e-\mu_2}{\mu_1+\mu_3+\mu_4}$, since $(\mu_1+\mu_2+\mu_3+\mu_4)>1$. It suggests $h<1$. Employing (3.11) frequently, we receive

$$\bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)\preceq h^\iota\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1).$$

Now for each and every ι, j elements of $\mathbb{N}$ about $\iota < j$, it has

$$\begin{aligned}\bar{\mathcal{G}}(\omega_{\iota-1},\omega_j,\omega_j) &\preceq \bar{\mathcal{G}}(\omega_{\iota-1},\omega_\iota,\omega_\iota)+\bar{\mathcal{G}}(\omega_\iota,\omega_{\iota+1},\omega_{\iota+1})\dots\bar{\mathcal{G}}(\omega_{j-1},\omega_j,\omega_j)\\ &\preceq h^\iota\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)+h^{\iota+1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)+\dots+h^{j-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)\\ &\preceq h^\iota(e+h+h^2+\dots+h^{j-\iota-1})\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1)\\ &\preceq h^\iota(e-h)^{-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1).\end{aligned}$$

By using Remark 2.2 and Lemma 2.8, we can infer that for $\mu \in \hat{\mathcal{B}}$ with $\mu \gg \Theta$, it exists an N belong in $\mathbb{N}$ so that for any $j > \iota > N$, we have

$$\bar{\mathcal{G}}(\omega_\iota,\omega_j,\omega_j) \preceq h^\iota(e-h)^{-1}\bar{\mathcal{G}}(\omega_0,\omega_1,\omega_1) \ll \mu.$$

$\{\omega_\iota\}$ is therefore a Cauchy sequence. The existence of $q \in Y$ implies that $\urcorner(\omega_{\iota+1}) = \omega_\iota \to q$ as $\iota \to \infty$. This is because Y is complete.

Thus, $\tau \in Y$ such that $\urcorner\tau = q$ can be found. We will now demonstrate that $\tau = q$.

Changing $\omega = \tau, v = \omega_{\iota+1}$ in (3.10), we get

$$\begin{aligned}\bar{\mathcal{G}}(\omega_\iota,q,q) &= \bar{\mathcal{G}}(\urcorner\omega_{\iota+1},\urcorner\tau,\urcorner\tau)\\ &\succeq \mu_1\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau)+\mu_2\bar{\mathcal{G}}(\omega_{\iota+1},\omega_{\iota+1},\urcorner\omega_{\iota+1})+\mu_3\bar{\mathcal{G}}(\tau,\tau,\urcorner\tau)+\mu_4\bar{\mathcal{G}}(\tau,\tau,\urcorner\tau)\\ &= \mu_1\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau)+\mu_2\bar{\mathcal{G}}(\omega_{\iota+1},\omega_{\iota+1},\omega_\iota)+(\mu_3+\mu_4)\bar{\mathcal{G}}(\tau,\tau,q).\end{aligned}\tag{3.12}$$

Using equations (3.6), (3.7) and $\bar{\mathcal{G}}(\omega_\iota,\tau,\tau) \succeq \bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau)-\bar{\mathcal{G}}(\omega_{\iota+1},\omega_\iota,\omega_\iota)$ in above, we have

$$(e+\mu_2-\mu_4)\bar{\mathcal{G}}(\omega_{\iota+1},q,q)+(e-\mu_3)\bar{\mathcal{G}}(\omega_{\iota+1},\omega_\iota,\omega_\iota) \succeq (\mu_1+\mu_2)\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau).\tag{3.13}$$

Substituting $\omega = \omega_{\iota+1}, v = \tau$ in (3.10), we get

$$\begin{aligned}\bar{\mathcal{G}}(q,\omega_\iota,\omega_\iota) &= \bar{\mathcal{G}}(\urcorner\tau,\urcorner\omega_{\iota+1},\urcorner\omega_{\iota+1})\\ &\succeq \mu_1\bar{\mathcal{G}}(\tau,\omega_{\iota+1},\omega_{\iota+1})+\mu_2\bar{\mathcal{G}}(t_{\iota+1},\omega_{\iota+1},\urcorner\omega_{v+1})\\ &\quad +\mu_3\bar{\mathcal{G}}(\omega_{\iota+1},\omega_{\iota+1},\urcorner\omega_{\iota+1})+\mu_4\bar{\mathcal{G}}(\omega_{\iota+1},\omega_{\iota+1},\urcorner\omega_{\iota+1})\\ &= \mu_1\bar{\mathcal{G}}(\tau,\omega_{\iota+1},\omega_{\iota+1})+\mu_2\bar{\mathcal{G}}(\tau,\tau,\omega_\iota)+(\mu_3+\mu_4)\bar{\mathcal{G}}(\omega_{\iota+1},\omega_{\iota+1},q).\end{aligned}\tag{3.14}$$

Using equations (3.6), (3.7) and $\bar{\mathcal{G}}(\omega_\iota,\tau,\tau) \succeq \bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau)-\bar{\mathcal{G}}(\omega_{\iota+1},\omega_\iota,\omega_u)$ in above, we have

$$(e+\mu_3)\bar{\mathcal{G}}(q,\omega_{\iota+1},\omega_{\iota+1})+(e-\mu_2+\mu_4)\bar{\mathcal{G}}(\omega_{\iota+1},\omega_\iota,\omega_\iota) \succeq (\mu_1+\mu_3+\mu_4)\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau).\tag{3.15}$$

Add up equations (3.13) and (3.15), we get

$$\bar{\mathcal{G}}(\omega_{\iota+1},\tau,\tau) \preceq h_1\bar{\mathcal{G}}(q,\omega_{\iota+1},\omega_{\iota+1})+h_2\bar{\mathcal{G}}(\omega_{\iota+1},\omega_\iota,\omega_\iota),$$

where $h_1 = \frac{(2e+\mu_2+\mu_3-\mu_4)}{(2\mu_1+\mu_2+\mu_3+\mu_4)}$ and $h_2 = \frac{(2e-\mu_2-\mu_3+\mu_4)}{(2\mu_1+\mu_2+\mu_3+\mu_4)}$.

From Proposition 2.17, Proposition 2.18 and 2.19. We have $\bar{\mathcal{G}}(\tau,\omega_{\iota+1},\omega_{\iota+1}) \preceq u_n$, where u_n has a c-sequence of cone $\ddot{\mathcal{C}}$.

Therefore, for every $\mu \gg \Theta$, we have $\bar{\mathcal{G}}(\tau,\omega_{\iota+1},\omega_{\iota+1}) \ll \mu$, so $\bar{\mathcal{G}}(\tau,\omega_{\iota+1},\omega_{\iota+1}) \to \Theta$ as $n \to \infty$. So $\omega_{\iota+1} \to \tau$ as $\iota \to \infty$.

Lemma 2.9 gives us $\tau = q$. As a result, $\urcorner$ has only one fixed point. □

Corollary 3.7. Consider the cone $\ddot{\mathcal{C}}$ within the Banach algebra $\hat{\mathcal{B}}$ and the complete of $\bar{\mathcal{G}}$-CMSBA $(Y,\bar{\mathcal{G}})$. Assume that $\urcorner : Y \to Y$ is surjective mapping and fulfilled the condition

$$\bar{\mathcal{G}}(\urcorner u,\urcorner v,\urcorner\omega) \succeq \mu_1\bar{\mathcal{G}}(u,v,\omega)+\mu_2(\bar{\mathcal{G}}(u,u,\urcorner u)+\bar{\mathcal{G}}(v,v,\urcorner v)+\bar{\mathcal{G}}(\omega,\omega,\urcorner\omega)),\tag{3.16}$$

where $\tau(\mu_1+3\mu_2) > 1$ and $\mu_2 < \frac{1}{2}$. Then $\urcorner$ possesses a distinct fixed point in Y.

Proof. There is proof originates from result (3.6) since it reduces the equation (3.10) to equation (3.16) in the previous theorem $\mu_2 = \mu_3 = \mu_4$. □

4. Conclusion

The goal of this work is to find FP findings for generalized expansive maps within the context of $\bar{\mathcal{G}}$-CMSBA. Compared to existing literature, our results provide a substantial improvement. These findings, in our opinion, will advance the theory of fixed points and, with the right circumstances, may also apply to other spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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