Communications in Mathematics and Applications
Vol. 16, No. 1, pp. 293–300, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i1.2908

http://www.rgnpublications.com

Research Article

Some New Oscillation Criteria for Certain Class of Quasilinear Elliptic Equations

C. Dhanalakshmi[1], G. E. Chatzarakis*[2], V. Sadhasivam[1] and S. Priyadharshini[1]

[1] *Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College, Namakkal District, Rasipuram 637401, Tamil Nadu, India*

[2] *Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), Marousi 15122, Athens, Greece*

***Corresponding author:** geaxatz@otenet.gr

Received: October 14, 2024 **Revised:** March 17, 2025 **Accepted:** April 1, 2025

Abstract. The main goals of this paper is to investigate a new oscillation for a certain class of quasilinear elliptic equations by using Riccati technique. Our main results are demonstrated with an example.

Keywords. Elliptic equations, Quasilinear, Oscillation

Mathematics Subject Classification (2020). 35B05, 35J62, 35J08

1. Introduction

The theory of differential equations oscillation has attracted a lot of attention and is an interesting area of study. This theory has numerous significant applications in the social sciences, engineering, neutral networks, etc., several authors have shown interest in studying the oscillation theory and contributed more paper on it (see Agarwal *et al*. [1], Jaros *et al*. [9], Kusano and Naito [11], and Swanson [17]).

Glazman [7] discovered the general oscillation criteria for PDEs. A set of PDEs known as elliptic PDEs is responsible for explaining both global and steady state phenomena. In fluid dynamics, heat transfer, electromagnetism, geophysics, biology, and other application domains, elliptic PDEs studied by Evans [5], Gilbarg and Trudinger [6], Pudipeddi [15], and Yoshida [20]. Many authors have worked hard in recent years to develop oscillation theory for elliptic equations with variable coefficients (see e.g., Agarwal and O'Regan [2], Headley and Swanson [8], Kreith [10], Noussair and Swanson [12], Priyadharshini and Sadhasivam [14], Santra *et al*. [16], Xu [19], and Zhuang *et al*. [22]), and oscillation criteria were obtained for quasi linear elliptic equations by Allegretto [3], Headley and Swanson [8], Wei *et al*. [18], and Yoshida [21].

Kusano and Naito [11] derived the oscillatory and non-oscillatory behavior of quasilinear differential equations.

First-order quasilinear elliptic equations have been constructed by Yoshida [21], and findings for superlinear and sublinear elliptic equations have been achieved.

The existence of radial solutions to the p-Laplace equation was examined in Pudipeddi [15],

$$\nabla(|\nabla u|^{\alpha-1}\nabla u)+|u|^{\lambda-1}u+|u|^{\mu-1}u=0,$$

where $0<\alpha<\lambda$, $0<\mu<\alpha$.

Motivated by these observations, we obtain new oscillation criteria for a certain class of quasilinear elliptic differential equations of the form

$$\nabla(|\nabla u|^{\alpha-1}\nabla u)+C(|x|)|u|^{\lambda-1}u+D(|x|)|u|^{\mu-1}u=f(x), \quad x\in\Omega, \tag{1.1}$$

where ∇ denotes gradient, i.e., $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_n}\right)$. Also,

$$|x|=r=\left(\sum_{i=1}^{n}x_i^2\right)^{\frac{1}{2}}$$

is Euclidean length of $x\in\mathbb{R}^n$. Since $x=(x_1,x_2,\dots,x_n)$ represents a point of $\mathbb{R}^n$, Ω is an exterior domain of $\mathbb{R}^n$. In other words, Ω includes the complement of some n-ball in $\mathbb{R}^n$. Let

$$E_r=\{x\in\mathbb{R}^n;|x|>r\}\subset\Omega, \quad r>r_0,$$

$$S_r=\{x\in\mathbb{R}^n;|x|=r\}, \qquad r>r_0,$$

where S_r denotes the $(n-1)$-dimensional sphere. Consider that $C(r),D(r)\in\mathbb{C}(\bar{\Omega};R_+)$, α,λ,μ are constants $\lambda>\alpha$, $0<\mu<\alpha$.

This paper is organized as follows: We recall a few preliminaries, lemmas given in Section 2. In Section 3 new oscillation theorems developed by using Riccati transformation and Philo's type. In Section 4, we provide the suitable example illustrating our main results.

2. Preliminaries

We present the definitions and lemmas which are used in the sequel.

Definition 2.1 ([20])**.** A solution $u\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^1$ of (1.1) defined and of class $\mathbb{C}$ in a unbounded domain E_r, whose complement CE_r is compact, will be called oscillatory if the exterior of every sphere contains a zero of u and the set of the zeros of u has no interior points.

Definition 2.2 ([4])**.** If a solution to (1.1) contains arbitrarily large zeros on $[r_0,\infty)$, then it is considered oscillatory; otherwise, it is considered non-oscillatory. If all of its solutions are oscillatory, then (1.1) is considered oscillatory.

Philo's (1989) [13] introduces the function $\mathbb{H}$, as stated in the definition.

Definition 2.3 ([13])**.** The functions $\mathbb{H}(r,s)$, $h(r,s)\in\mathbb{C}'(\mathbb{D},\mathbb{R})$ in which $\mathbb{D}=\{(r,s):r\geq s\geq r_0>0\}$,
(H1): $\mathbb{H}(r,r)=0$, $r>r_0$ and $\mathbb{H}(r,s)>0$, $r>s\geq r_0$,

(H2): $\dfrac{\partial \mathbb{H}(r,s)}{\partial r} = h_1 \mathbb{H}(r,s)$,

(H3): $\dfrac{\partial \mathbb{H}(r,s)}{\partial s} = -h_2 \mathbb{H}(r,s)$,

(H4): $\mathbb{B}_1(r,s) = h_1(r,s) + \dfrac{\rho'(s)}{\rho(s)}$,

(H5): $\mathbb{B}_2(r,s) = -h_2(r,s) + \dfrac{\rho'(s)}{\rho(s)}$.

Lemma 2.1 ([20])**.** *If $u \in \mathbb{C}^2((r_0,\infty),\mathbb{R})$, then the spherical mean of $u(x)$ over S_r, i.e.,*

$$U(r) = \frac{1}{\omega_n r^{n-1}} \int_{s_r} \tilde{u}\, dS\,,$$

where the surface area of the unit sphere S_1 is indicated by ω_n and $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, if

$$\frac{1}{\omega_n r^{n-1}} \int_{s_r} \nabla(|\nabla u|^{\alpha-1}\nabla u) ds = r^{1-n}(r^{n-1}|U'(r)|^{\alpha-1}U'(r))', \quad r > r_0\,.$$

Lemma 2.2. *Let $u \in \mathbb{C}^2(\Omega,\mathbb{R})$ is a solution to* (1.1) *for some $r_1 > r_0$, $u > 0$ in (r_1,∞). In Ω, all solutions $u \in \mathbb{C}(\Omega,\mathbb{R})$ are oscillatory if the ordinary differential inequality,*

$$r^{1-n}(r^{n-1}|U'(r)|^{\alpha-1}U'(r))' + C|U(r)|^{\lambda-1}U(r) + D|U(r)|^{\mu-1}U(r) \le 0, \quad \textit{for } r > r_1 \tag{2.1}$$

has no non-negative solution.

Proof. By Lemma 2.1,

$$\begin{aligned} r^{1-n}(r^{n-1}|U'(r)|^{\alpha-1}U'(r))' &= \frac{1}{\omega_n r^{n-1}} \int_{s_r} \nabla(|\nabla u|^{\alpha-1}\nabla u) dS \\ &= -\frac{1}{\omega_n r^{n-1}} \int_{s_r} (C(|x|)|u|^{\lambda-1}u + D(|x|)|u|^{\mu-1}u) dS + \int_{s_r} f(x) dS, \\ &\qquad \text{for } r > r_0\,. \end{aligned} \tag{2.2}$$

Using the condition $-\frac{1}{\omega_n r^{n-1}} \int_{S_r} f(x) dS \le 0$ in equation (2.2), we have

$$\begin{aligned} r^{1-n}(r^{n-1}|U'(r)|^{\alpha-1}U'(r))' &\le -\frac{C(r)}{\omega_n r^{n-1}} \int_{s_1} |u|^{\lambda} r^{n-1} d\omega - \frac{D(r)}{\omega_n r^{n-1}} \int_{s_1} |u|^{\mu} r^{n-1} d\omega \\ &\le -(C|u|^{\lambda}(r) + D|u|^{\mu}(r)). \end{aligned}$$

The proof of the lemma is complete. □

3. Main Results

Our goal is to minimize the multidimensional problems to one dimensional problem of oscillation of an ordinary differential equation. In the following theorem, we establish some new oscillation by using the Riccati techniques and Philo's type.

Theorem 3.1. *If there exists a function $\rho(s) \in \mathbb{C}((r_0,\infty),\mathbb{R})$ such that*

$$\limsup_{r\to\infty} \int_{r_0}^{r} \rho(s)\mathbb{H}(r,s)\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - k|\mathbb{B}_2(r,s)|^{\alpha+1}\right) ds = \infty, \tag{3.1}$$

then all solution of (1.1) *are oscillatory.*

Proof. Let $u(x) > 0$ is non-oscillatory. We define the Riccati transformation

$$W(r) = \rho(r)\left(\frac{r^{n-1}|U'(r)|^{\alpha-1}U'(r)}{|U(r)|^{\alpha-1}U(r)}\right),$$

$$W'(r) \le \frac{\rho'(r)}{\rho(r)}W(r) - \frac{\rho(r)}{r^{1-n}}\left(\frac{C|U(r)|^{\lambda-1}U(r)}{|U(r)|^{\alpha-1}U(r)}\right) - \frac{\rho(r)}{r^{1-n}}\left(\frac{D|U(r)|^{\mu-1}U(r)}{|U(r)|^{\alpha-1}U(r)}\right) - \alpha\rho(r)r^{n-1}\left(\frac{|U'(r)|}{|U(r)|}\right)^{\alpha+1}. \tag{3.2}$$

Using the inequality,

$$C(r)|v|^{\lambda-\alpha} + \frac{D(r)}{|u|^{\alpha-\mu}} \ge \frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(r)^{\frac{\alpha-\lambda}{\lambda-\mu}} D(r)^{\frac{\lambda-\alpha}{\lambda-\mu}}. \tag{3.3}$$

In equation (3.3) substituted in (3.2),

$$W'(r) \le W(r)\frac{\rho'(r)}{\rho(r)} - r^{n-1}\rho(r)\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right) C(r)^{\frac{\alpha-\mu}{\lambda-\mu}} D(r)^{\frac{\lambda-\alpha}{\lambda-\mu}} - \alpha r^{n-1}\rho(r)\left(\frac{W(r)}{r^{n-1}\rho(r)}\right)^{\frac{\alpha+1}{\alpha}}. \tag{3.4}$$

Integrating on both sides from r_0 to r,

$$\begin{aligned}\int_{r_0}^{r} \mathbb{H}(r,s)W'(s)ds \le& \int_{r_0}^{r} \mathbb{H}(r,s)\frac{\rho'(s)}{\rho(s)}W(s)ds \\ &- \int_{r_0}^{r} \mathbb{H}(r,s)\rho(s)s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right) C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} ds \\ &- \alpha\int_{r_0}^{r} \mathbb{H}(r,s)s^{n-1}\rho(s)\left(\frac{W(r)}{s^{n-1}\rho(s)}\right)^{\frac{\alpha+1}{\alpha}} ds \\ \le& \int_{r_0}^{r} \mathbb{H}(r,s)W(s)B_2(r,s)ds - \alpha\int_{r_0}^{r} \mathbb{H}(r,s)\rho(s)s^{n-1}\left(\frac{W(r)}{s^{n-1}\rho(s)}\right)^{\frac{\alpha+1}{\alpha}} ds \\ &- \int_{r_0}^{r} \mathbb{H}(r,s)\rho(s)s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right) C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} ds.\end{aligned}$$

Using the assumption,

$$G\left(\frac{W(s)}{s^{n-1}\rho(s)}\right) = \mathbb{B}_2(r,s)s^{n-1}\left(\frac{W(s)}{s^{n-1}\rho(s)}\right) - \alpha s^{n-1}\left(\frac{W(s)}{s^{n-1}\rho(s)}\right)^{\frac{\alpha+1}{\alpha}},$$

$$G\left(\frac{W(s)}{s^{n-1}\rho(s)}\right) \le G_{\max} = K|\mathbb{B}(r,s)|^{\alpha+1},$$

$$\frac{1}{\mathbb{H}(r,r_0)}\int_{r_0}^{r} \rho(s)\mathbb{H}(r,s)\left(s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right) C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - k|\mathbb{B}_2(r,s)|^{\alpha+1}\right) ds \le W(r_0). \tag{3.5}$$

Taking the limit supremum on both sides,

$$\limsup_{r\to\infty} \frac{1}{\mathbb{H}(r,r_0)}\int_{r_0}^{r} \rho(s)\mathbb{H}(r,s)\left(s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right) C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - k|\mathbb{B}_2(r,s)|^{\alpha+1}\right) ds \le W(r_0),$$

which contracts (3.1). The proof of the theorem is complete. □

Theorem 3.2. *Assume that, each $R_0 \ge r_0$ for sufficiently large, there exist $a_1, a_2, a_3 \in \mathbb{R}$ with $R_0 \le a_1 < a_3 < a_2$ such that*

$$\frac{1}{\mathbb{H}(a_3,a_1)}\int_{a_1}^{a_3} \left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - K|\mathbb{B}_1(s,a_1)|^{\alpha+1}\right)\rho(s)\mathbb{H}(s,a_1)ds$$

$$+\frac{1}{\mathbb{H}(a_2,a_3)}\int_{a_3}^{a_2}\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}C(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}-K|\mathbb{B}_2(a_2,s)|^{\alpha+1}\right)\rho(s)\mathbb{H}(a_2,s)ds>0. \tag{3.6}$$

Then all solution of (1.1) *are oscillatory.*

Proof. Assume the contradiction. Define $W(r)$ as in Theorem 3.1, $r_0 \le a_1 < a_3 < a_2$. We can obtain (3.1) with r_0, $\mathbb{H}(r,s)$ replaced by a_3, $h_2(r,s)$, respectively. It follows that

$$\int_{a_3}^{r}\rho(s)\mathbb{H}(r,s)\left(s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right)C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}-k|\mathbb{B}_2(r,s)|^{\alpha+1}\right)ds\le\mathbb{H}(r,a_3)W(a_3), \tag{3.7}$$

where $r\in[a_3,a_2)$. Letting $r\to a_2^-$ in equation (3.8) and dividing both sides by $\mathbb{H}(a_2,a_3)$. Then, we have

$$\frac{1}{\mathbb{H}(a_2,a_3)}\int_{a_3}^{a_2}\rho(s)\mathbb{H}(a_2,s)\left(s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right)C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}-k|\mathbb{B}_2(a_2,s)|^{\alpha+1}\right)ds\le W(a_3). \tag{3.8}$$

Corresponding to the work of Theorem 3.1, multiplying $\mathbb{H}(s,r)$ with r replaced by s, integrating for $r\in(a_1,a_3]$,

$$\begin{aligned}\int_r^{a_3}\mathbb{H}(s,r)W'(s)ds\le&\int_r^{a_3}\frac{\rho'}{\rho}\mathbb{H}(s,r)W(s)ds\\&-\int_r^{a_3}\rho(s)\mathbb{H}(s,r)s^{n-1}\left(\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}\right)C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}ds\\&-\int_r^{a_3}\rho(s)\mathbb{H}(s,r)s^{n-1}\left(\frac{W(s)}{s^{n-1}\rho(s)}\right)ds.\end{aligned}$$

It follows that,

$$\begin{aligned}\mathbb{H}(a_3,r)W(a_3)\le&\int_r^{a_3}\mathbb{H}(s,r)\rho(s)\left(\mathbb{B}_1(s,r)s^{n-1}\left(\frac{W(s)}{s^{n-1}\rho(s)}\right)-\alpha s^{n-1}\left(\frac{W(s)}{s^{n-1}\rho(s)}\right)^{\frac{\alpha+1}{\alpha}}\right)ds\\&-\int_r^{a_3}s^{n-1}\rho(s)\mathbb{H}(s,r)\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}ds.\end{aligned}$$

Since

$$G\left(\frac{W(s)}{s^{n-1}\rho(s)}\right)\le G_{\max}=K|\mathbb{B}(r,s)|^{\alpha+1}.$$

Thus, we obtain

$$\begin{aligned}\mathbb{H}(a_3,r)W(a_3)\le&\int_r^{a_3}\mathbb{H}(s,r)\rho(s)k|\mathbb{B}_1(s,r)|^{\alpha+1}ds\\&-\int_r^{a_3}s^{n-1}\rho(s)\mathbb{H}(s,r)\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}ds.\end{aligned}$$

Therefore,

$$\int_r^{a_3}\rho(s)\mathbb{H}(s,r)\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}}C(s)^{\frac{\alpha-\mu}{\lambda-\mu}}D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}-K|\mathbb{B}_1(s,r)|^{\alpha+1}\right)ds\le-\mathbb{H}(a_3,r)W(a_3),$$

where $r \in (a_1, a_3]$. Letting $r \to a_1^+$, dividing both sides by $\mathbb{H}(a_3, a_1)$. Then, we have

$$\frac{1}{\mathbb{H}(a_3,a_1)}\int_{a_1}^{a_3} \rho(s)\mathbb{H}(s,a_1)\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - K|\mathbb{B}_1(s,a_1)|^{\alpha+1}\right)ds \le -W(a_3). \tag{3.9}$$

Adding (3.8) and (3.9),

$$\frac{1}{\mathbb{H}(a_3,a_1)}\int_{a_1}^{a_3}\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\beta}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - K|\mathbb{B}_1(s,a_1)|^{\alpha+1}\right)\rho(s)\mathbb{H}(s,a_1)ds$$
$$+\frac{1}{\mathbb{H}(a_2,a_3)}\int_{a_3}^{a_2}\left(s^{n-1}\frac{\lambda-mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - K|\mathbb{B}_2(a_2,s)|^{\alpha+1}\right)\rho(s)\mathbb{H}(a_2,s)ds \le 0$$

which contradicts (3.1). The proof of the theorem is complete. □

Corollary 3.1. *Let* $\mathbb{H}, h_1, h_2,, \rho$ *be the same in Theorem* 3.1. *Moreover, if*

$$\limsup_{r\to\infty}\frac{1}{\mathbb{H}(r,r_0)}\int_{r_0}^{r}\rho(s)\mathbb{H}(r,s)s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}ds = \infty, \tag{3.10}$$

$$\limsup_{n\to\infty}\frac{1}{\mathbb{H}(r,r_0)}\int_{r_0}^{r}\rho(s)\mathbb{H}(r,s)K|\mathbb{B}_2(r,s)|^{\alpha+1}ds \le \infty. \tag{3.11}$$

Then all solution of (1.1) *are oscillatory.*

Corollary 3.2. *Let* $\mathbb{H}, h_1, h_2, \rho$*,* B_1 *and* B_2 *be the same as in Theorem* 3.1. *Moreover, suppose that* (3.2) *is replaced by*

$$\frac{1}{\mathbb{H}(a_3,a_1)}\int_{a_1}^{a_3} S^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}\rho(s)\mathbb{H}(s,a_1)ds$$
$$+\frac{1}{\mathbb{H}(a_2,a_3)}\int_{a_3}^{a_2} S^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}}\rho(s)\mathbb{H}(a_2,s)ds$$
$$> \frac{1}{\mathbb{H}(a_3,a_1)}\int_{a_1}^{a_3} K|\mathbb{B}_1(s,a_1)|^{\alpha+1}\rho(s)\mathbb{H}(s,a_1)ds + \frac{1}{\mathbb{H}(a_2,a_3)}\int_{a_3}^{a_2} K|\mathbb{B}_2(a_2,s)|^{\alpha+1}\rho(s)\mathbb{H}(a_2,s)ds. \tag{3.12}$$

Then all solution of (1.1) *are oscillatory.*

4. Example

We provide an example to highlight the findings from Section 3.

Example 4.1. We assume the quasilinear elliptic equations

$$\nabla(|\nabla u|^{\alpha-1}\nabla u) + c(r)|u|^{\beta-1}u + D(r)|u|^{\gamma-1}u = f(r), \quad x \in \Omega, \tag{4.1}$$

where $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{3}{4}$ and $\mu = \frac{1}{4}$, $\rho(s) = s$, $C(r) = \frac{(\sin r)^{\frac{1}{4}}}{2(\cos r)^{\frac{1}{2}}}$ and $D(r) = \frac{(\sin r)^{\frac{1}{4}}}{r}$, $\mathbb{H}(r,s) = (r-s)$,

$$\limsup_{r\to\infty}\int_{r_0}^{r}\rho(s)\mathbb{H}(r,s)\left(s^{n-1}\frac{\lambda-\mu}{\alpha-\mu}\left(\frac{\lambda-\alpha}{\alpha-\mu}\right)^{\frac{\alpha-\lambda}{\lambda-\mu}} C(s)^{\frac{\alpha-\mu}{\lambda-\mu}} D(s)^{\frac{\lambda-\alpha}{\lambda-\mu}} - k|\mathbb{B}_2(r,s)|^{\alpha+1}\right)ds$$
$$= \limsup_{r\to\infty}\int_{r_0}^{r} s(r-s)s^{3-1}2(1)^{\frac{-1}{2}}\left(\frac{(\sin s)^{\frac{1}{4}}}{2(\cos s)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\frac{(\sin s)^{\frac{1}{4}}}{s} + \text{constant} = \infty.$$

Therefore, all requirements given in Theorem 3.1 are fulfilled. As a result, all of (4.1) solutions oscillate. One such solution is $u(x) = \sin(r)$.

5. Conclusion

In this work, some new oscillation criteria for a certain class of the quasi linear elliptic equations is discussed. By using new sufficient conditions for the quasilinear elliptic equation is established. We have also present an example to illustrate our new results.

Acknowledgment

The second author was supported by the Special Account for Research of ASPETE through the funding program "Strengthening research of ASPETE faculty members".

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R. P. Agarwal, S. R. Grace and D. O'Regan, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, 1st edition, Springer, Dordrecht, xiv + 672 pages (2002), DOI: 10.1007/978-94-017-2515-6.

[2] R. P. Agarwal and D. O'Regan, *An Introduction to Ordinary Differential Equations*, Springer, New York, xii + 321 pages (2008).

[3] W. Allegretto, Oscillation criteria for quasilinear equations, *Canadian Journal of Mathematics* **26**(4) (1974), 931 – 947.

[4] G. E. Chatzarakis, M. Deepa, N. Nagajothi and V. Sadhasivam, Some new oscillation criteria for second-order hybrid differential equations, *Hacettepe Journal of Mathematics and Statistics* **49** (2020), 1334 – 1345, DOI: 10.15672/hujms.541773.

[5] L. C. Evans, *Partial Differential Equations*, 2nd edition, Graduate Studies in Mathematics series, Vol. 19, American Mathematical Society, (1998), DOI: 10.1090/gsm/019.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 1st edition, Springer, Berlin — Heidelberg, x + 401 pages (1977), DOI: 10.1007/978-3-642-96379-7.

[7] I. M. Glazman, On the negative part of the spectrum of one-dimensional and multidimensional differential operators over vector-functions, *Doklady Akademii Nauk SSSR* **119**(3) (1958), 421 – 424, URL: https://www.mathnet.ru/eng/dan/v119/i3/p421.

[8] V. B. Headley and C. A. Swanson, Oscillation criteria for elliptic equations, *Pacific Journal of Mathematics* **27**(3) (1968), 501 – 506, URL: https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-27/issue-3/Oscillation-criteria-for-elliptic-equations/pjm/1102983771.pdf.

[9] J. Jaros, T. Kusano and N. Yoshida, Force superlinear oscillation via Picone's identity, *Acta Mathematica Universitatis Comenianae* **69**(1) (2000), 107 – 113, URL: http://www.iam.fmph.uniba.sk/amuc/_vol-69/_no_1/_jaros/jaros.html.

[10] K. Kreith, Oscillation theorems for elliptic equations, *Proceedings of the American Mathematical Society* **15** (1964), 341 – 344, DOI: 10.2307/2034499.

[11] T. Kusano and M. Naito, Oscillation and nonoscillation criteria for second order quasilinear differential equations, *Acta Mathematica Hungarica* **76**(1-2) (1997), 81 – 99, DOI: 10.1007/BF02907054.

[12] E. S. Noussair and C. A. Swanson, Oscillation of semilinear elliptic inequalities by Riccati transformation, *Canadian Journal of Mathematics* **32**(4) (1980), 908 – 923, DOI: 10.4153/CJM-1980-069-8.

[13] Ch. G. Philos, Oscillation theorems for linear differential equations of second order, *Archiv der Mathematik* **53** (1989), 482 – 492, DOI: 10.1007/BF01324723.

[14] S. Priyadharshini and V. Sadhasivam, Forced oscillation of solutions of conformable hybrid elliptic partial differential equations, *Journal Computational Mathematics* **6**(1) (2022), 396 – 412, DOI: 10.26524/cm140.

[15] S. Pudipeddi, Localized radial solutions for nonlinear p-Laplacian equation in R^n, *Electronic Journal of Qualitative Theory of Differential Equations* **2008**(20) (2008), 1 – 23, DOI: 10.14232/ejqtde.2008.1.20.

[16] S. S. Santra, S. Priyadharshini, V. Sadhasivam, J. Kavitha, U. Fernandez-Gamiz, S. Noeiaghdam and K. M. Khedher, On the oscillation of certain class of conformable Emden-Fowler type elliptic partial differential equations, *AIMS Mathematics* **8**(6) (2023), 12622 – 12636, DOI: 10.3934/math.2023634.

[17] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Mathematics in Science and Engineering Series, Vol. 48, Academic Press, New York, (1968).

[18] D. Wei, T. Ji and X. Zhao, Oscillation and non-oscillation criteria for quasi-linear second order differential equations, *WSEAS Transactions on Mathematics* **11**(7) (2008), 627 – 636.

[19] Z. T. Xu, Oscillation and nonoscillation of solutions of second order damped elliptic differential equations, *Acta Mathematica Sincia, Chinese Series* **47**(6) (2004), 1099 – 1106, DOI: 10.12386/A2004sxxb0136. (in Chinese)

[20] N. Yoshida, *Oscillation Theory of Partial Differential Equations*, World Scientific, (2008), DOI: 10.1142/7046.

[21] N. Yoshida, Forced oscillation criteria superlinear-sublinear elliptic equations via Picone-type inequality, *Journal of Mathematical Analysis and Applications* **363**(2) (2010), 711 – 717, DOI: 10.1016/j.jmaa.2009.10.008.

[22] R. K. Zhuang, Q. R. Wang and S. M. Zhu, Domain oscillation criteria for second order nonlinear elliptic differential equations, *Acta Mathematica Sincia, Chinese Series* **48**(3) (2005), 527 – 534, DOI: 10.12386/A2005sxxb0063. (in Chinese)