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Research Article

Hyponormality of Toeplitz Operators on Symmetric-type Circulant Trigonometric Polynomial Symbols: An Extremal Case

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Abstract. In this paper, we give some necessary and sufficient conditions to determine the hyponormality of Toeplitz operators on symmetric-type circulant trigonometric polynomial symbols in Hardy-Hilbert spaces of analytic functions considering an extremal case.

Keywords. Toeplitz operators, Hyponormal operators, Circulant trigonometric polynomials, Symmetric-type trigonometric polynomials

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1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. For each integer n, let $e_n(e^{i\theta}) = e^{in\theta}$, regarded as a function of $\mathbb{T}$. Then it is well known that the set $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{T})$ (Martínez-Avendaño and Rosenthal [11]). The space $L^\infty(\mathbb{T})$ consists of all bounded functions defined on the unit circle $\mathbb{T}$ and the space $H^\infty(\mathbb{T})$ consists of all the functions that are analytic and bounded on $\mathbb{T}$. Thus $H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \langle f, e_n\rangle = 0 \text{ for } n < 0\}$. The Hardy-Hilbert space, to be denoted by $H^2(\mathbb{T})$, consists of all analytic functions having power series representations with square-summable complex coefficients, i.e., $H^2(\mathbb{T}) = \left\{f \in L^2(\mathbb{T}) : f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\right\}$. For a given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ, to be denoted by T_φ, is the operator on the Hardy space $H^2(\mathbb{T})$ defined by $T_\varphi(f) := P(\varphi\cdot f)$, where $f\in H^2(\mathbb{T})$ and P is the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

A polynomial $\varphi(z) = \sum_{n=-m}^{N} b_n z^n$ is said to be a trigonometric polynomial of analytic and co-analytic degree N and m, respectively, if b_{-m} and b_N are not equal to zero. Then $\varphi(z)$ having the same analytic and co-analytic degree N is called a symmetric-type trigonometric polynomial if the coefficients of it satisfy the following condition:

$$\overline{b}_N \begin{pmatrix} b_{-1} \\ b_{-2} \\ \vdots \\ b_{-N} \end{pmatrix} = b_{-N} \begin{pmatrix} \overline{b_1} \\ \overline{b_2} \\ \vdots \\ \overline{b_N} \end{pmatrix}. \tag{1.1}$$

A bounded linear operator B defined on a Hilbert space H is said to be hyponormal if its self-commutator $[B^*,B] := B^*B - BB^*$ is positive semi-definite. For arbitrary symbol $\varphi\in L^\infty(\mathbb{T})$, though Cowen [1] gave a very elegant characterization to determine the hyponormality of T_φ, but in practice, it is very complicated. Later on, Nakazi and Takahashi [12] modified this characterisation as follows:

Theorem 1.1. *For a given $\varphi\in L^\infty(\mathbb{T})$, write $\mathcal{E}(\varphi) = \{k\in H^\infty(\mathbb{T}) : \|k\|_\infty \le 1$ and $\varphi - k\overline{\varphi}\in H^\infty(\mathbb{T})\}$.*

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty. This theorem is known as Variant of Cowen's Theorem. By employing this modified version of Cowen's Theorem, several authors, e.g., Cuckovic and Curto [2], Farenick and Lee [3], Fleeman and Liaw [5], Gupta and Aggarwal [6], Kim and Lee [8], Kim *et al*. [9], Lee and Simanek [10], Phukon [13], Sadraoui *et al*. [14], Le and Simanek [10] studied the hyponormality of Toeplitz operators T_φ, especially, with the symbol φ that satisfies some certain conditions about the coefficients of φ in the Hardy as well as in the Bergman spaces. Summarily, the primary objective of these authors was to enquire about the following question: "*Which trigonometric polynomial symbols induce a hyponormal Toeplitz operator and under what conditions*?" Our particular interest here also is to explore more about the hyponormality of the Toeplitz operator T_φ where the previous literature remains silent.

Definition 1.1 ([4])**.** *A trigonometric polynomial $\varphi(z) = \sum_{n=-N}^{N} b_n z^n$ ($b_N \neq 0$, $b_{-N}\neq 0$) is said to be a circulant trigonometric polynomial with argument ω, if there exists $\omega\in[0,2\pi)$ such that $b_{-k} = e^{i\omega} b_{N-k+1}$, for every $1\le k\le N$.*

Thus, the coefficients of a circulant trigonometric polynomial should satisfy the following condition:

$$\begin{pmatrix} b_{-1} \\ b_{-2} \\ \vdots \\ b_{-N+1} \\ b_{-N} \end{pmatrix} = e^{i\omega} \begin{pmatrix} b_N \\ b_{N-1} \\ \vdots \\ b_2 \\ b_1 \end{pmatrix}. \tag{1.2}$$

In this paper, we determine the hyponormality of the Toeplitz operator T_φ considering a symmetric-type circulant trigonometric polynomial $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ with the extremal case where the argument $\omega = 0$. Before going to the main results, a brief description of Zhu's theorem [16] is given.

2. Zhu's Theorem

Suppose that $f(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$. If $f_0 = f$, define by induction a sequence $\{f_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, \quad |z| < 1,\ n = 0, 1, 2, \dots .$$

Note that $f_n(0)$ depends only on the values of $c_0, c_1, c_2, \dots, c_n$. We write $f_n(0) = \Phi_n\ (c_0, \dots, c_n)$ for each $n = 0, 1, 2, \dots$ as a function of $n+1$ complex variables and call them as Schur's functions. Now we state Zhu's theorem [16] briefly as follows:

Theorem 2.1. *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if*

$$\begin{pmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \vdots \\ \bar{c}_{N-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N-1} & a_N \\ a_2 & a_3 & \cdots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \overline{a}_{-1} \\ \overline{a}_{-2} \\ \vdots \\ \overline{a}_{-N} \end{pmatrix}, \tag{2.1}$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0, \dots, c_n)| \leq 1$ for each $n = 0, 1, \dots, N-1$.

3. Main Result

Before going to the main results, we need the following lemma and a formulation for Φ_{n+3}.

Lemma 3.1 ([7]). *Suppose that $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ and that $\{\Phi_n\}$ is the sequence of Schur's functions associated with $\{c_n\}$. If $c_1 = c_2 = c_3 = \dots = c_{n-1} = 0$ and $c_n \neq 0$, then*

$$\Phi_0 = c_0, \text{ and}$$

$$\Phi_1 = \Phi_2 = \Phi_3 = \dots = \Phi_{n-1} = 0,$$

$$\Phi_n = \frac{c_n}{1 - |c_0|^2},$$

$$\Phi_{n+1} = \frac{c_{n+1}}{(1 - |c_0|^2)(1 - |\Phi_n|^2)},$$

$$\Phi_{n+2} = \frac{(1 - |\Phi_n|^2) c_{n+2} c_n + |\Phi_n|^2 c_{n+1}^2}{c_n (1 - |c_0|^2)(1 - |\Phi_n|^2)^2 (1 - |\Phi_{n+1}|^2)},$$

$$\Phi_{n+3} = \frac{\begin{pmatrix} c_n c_{n+3} |c_{n+1}|^2 (1 - |\Phi_{n+1}|^2) + 2|c_n|^2 c_{n+1} c_{n+2} |\Phi_{n+1}|^2 (1 - |\Phi_n|^2) \\ + c_n \overline{c_{n+1}} c_{n+2}^2 |\Phi_{n+1}|^2 + \overline{c_n} c_{n+1}^3 |\Phi_{n+1}|^2 |\Phi_n|^2 \end{pmatrix}}{c_n |c_{n+1}|^2 (1 - |c_0|^2)(1 - |\Phi_n|^2)(1 - |\Phi_{n+1}|^2)^2 (1 - |\Phi_{n+2}|^2)}.$$

Now, we proceed to formulate Φ_{n+3} as follows: From Section 2, we have

$$k_{n+2}(z) = \frac{k_{n+1}(z) - k_{n+1}(0)}{z(1 - \overline{k_{n+1}(0)} k_{n+1}(z))}$$

$$= \frac{\left(\frac{(1-|c_0|^2)\sum_{j=n+1}^{\infty} c_j z^{j-n-1} + c_n\overline{c_0}\sum_{j=1}^{\infty} c_j z^{j-1}}{(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))} - \frac{c_{n+1}(1-|c_0|^2)}{(1-|c_0|^2)^2-|c_n|^2}\right)}{z(1-\overline{k_{n+1}(0)}k_{n+1}(z))}$$

$$= \frac{A+B-C}{(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}z(1-\overline{k_{n+1}(0)}k_{n+1}(z))},$$

where

$$A = (1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+1}^{\infty} c_j z^{j-n-1}$$

$$= (1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+1} + (1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}z\sum_{j=n+2}^{\infty} c_j z^{j-n-2},$$

$$B = \{(1-|c_0|^2)^2-|c_n|^2\}c_n\overline{c_0}\sum_{j=1}^{\infty} c_j z^{j-1}$$

$$= \{(1-|c_0|^2)^2-|c_n|^2\}c_n\overline{c_0}z\sum_{j=2}^{\infty} c_j z^{j-2},$$

$$C = c_{n+1}(1-|c_0|^2)^2(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))$$

$$= c_{n+1}(1-|c_0|^2)^3 - \overline{c_0}c_{n+1}(1-|c_0|^2)^2 z\sum_{j=1}^{\infty} c_j z^{j-1} - c_{n+1}(1-|c_0|^2)|c_n|^2$$

$$-\overline{c_n}c_{n+1}(1-|c_0|^2)z\sum_{j=n+1}^{\infty} c_j z^{j-n-1}.$$

Now, by putting the values of A, B and C and by further simplifications, we have

$$k_{n+2}(z) = \frac{\left(\begin{array}{l}(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+2}^{\infty} c_j z^{j-n-2} + \{(1-|c_0|^2)^2-|c_n|^2\}c_n\overline{c_0}\sum_{j=2}^{\infty} c_j z^{j-2}\\ +\overline{c_0}c_{n+1}(1-|c_0|^2)^2\sum_{j=1}^{\infty} c_j z^{j-1} + \overline{c_n}c_{n+1}(1-|c_0|^2)\sum_{j=n+1}^{\infty} c_j z^{j-n-1}\end{array}\right)}{(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}(1-\overline{k_{n+1}(0)}k_{n+1}(z))}$$

$$= \frac{R}{S},$$

where

$$R = (1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+2}^{\infty} c_j z^{j-n-2} + \{(1-|c_0|^2)^2-|c_n|^2\}c_n\overline{c_0}\sum_{j=2}^{\infty} c_j z^{j-2}$$

$$+\overline{c_0}c_{n+1}(1-|c_0|^2)^2\sum_{j=1}^{\infty} c_j z^{j-1} + \overline{c_n}c_{n+1}(1-|c_0|^2)\sum_{j=n+1}^{\infty} c_j z^{j-n-1},$$

$$S = (1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}(1-\overline{k_{n+1}(0)}k_{n+1}(z)).$$

Thus,

$$k_{n+2}(0) = \frac{(1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2} + \overline{c_n}c_{n+1}^2\}}{(1-|c_0|^2)^2(1-|k_n(0)|^2)(1-|k_{n+1}(0)|^2)\{(1-|c_0|^2)^2-|c_n|^2\}} = \frac{U}{V},$$

where

$$U = (1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2} + \overline{c_n}c_{n+1}^2\},$$

$$V=(1-|c_0|^2)^2(1-|k_n(0)|^2)(1-|k_{n+1}(0)|^2)\{(1-|c_0|^2)^2-|c_n|^2\}.$$

Hence,

$$k_{n+3}(z)=\frac{k_{n+2}(z)-k_{n+2}(0)}{z(1-\overline{k_{n+2}(0)}k_{n+2}(z))}=\frac{\frac{R}{S}-\frac{U}{V}}{z(1-\overline{k_{n+2}(0)}k_{n+2}(z))}.$$

Now,

$$\begin{aligned}
RV=&\Bigg[(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+2}^{\infty}c_jz^{j-n-2}+\{(1-|c_0|^2)^2-|c_n|^2\}c_n\overline{c_0}\sum_{j=2}^{\infty}c_jz^{j-2}\\
&+\overline{c_0}c_{n+1}(1-|c_0|^2)^2\sum_{j=1}^{\infty}c_jz^{j-1}+\overline{c_n}c_{n+1}(1-|c_0|^2)\sum_{j=n+1}^{\infty}c_jz^{j-n-1}\Bigg]\\
&\times[(1-|c_0|^2)^2(1-|k_n(0)|^2)(1-|k_{n+1}(0)|^2)\{(1-|c_0|^2)^2-|c_n|^2\}]\\
=&\Bigg[\{(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2}\}+\left\{(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}z\sum_{j=n+3}^{\infty}c_jz^{j-n-3}\right\}\\
&+\left\{\overline{c_0}c_n\{(1-|c_0|^2)^2-|c_n|^2\}z\sum_{j=3}^{\infty}c_jz^{j-3}\right\}+\left\{\overline{c_0}c_{n+1}(1-|c_0|^2)^2z\sum_{j=2}^{\infty}c_jz^{j-2}\right\}\\
&+\{\overline{c_n}c_{n+1}^2(1-|c_0|^2)\}+\left\{\overline{c_n}c_{n+1}(1-|c_0|^2)z\sum_{j=n+2}^{\infty}c_jz^{j-n-2}\right\}\Bigg]\\
&\times[\{(1-|c_0|^2)^2-|c_n|^2\}^2-|c_{n+1}|^2(1-|c_0|^2)^2],
\end{aligned}$$

$$\begin{aligned}
SU=&[(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}\{1-\overline{k_{n+1}(0)}k_{n+1}(z)\}]\\
&\times[(1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2}+\overline{c_n}c_{n+1}^2\}]\\
=&\Bigg[(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}\\
&\times\left\{1-\left(\overline{\frac{c_{n+1}(1-|c_0|^2)}{\{(1-|c_0|^2)^2-|c_n|^2\}}}\right)\frac{(1-|c_0|^2)\sum_{j=n+1}^{\infty}c_jz^{j-n-1}+c_n\overline{c_0}\sum_{j=1}^{\infty}c_jz^{j-1}}{(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))}\right\}\Bigg]\\
&\times[(1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2}+\overline{c_n}c_{n+1}^2\}]\\
=&\Bigg[\{(1-|c_0|^2)^2-|c_n|^2\}^2-\overline{c_0}(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}z\sum_{j=1}^{\infty}c_jz^{j-1}\\
&-\overline{c_n}\{(1-|c_0|^2)^2-|c_n|^2\}z\sum_{j=n+1}^{\infty}c_jz^{j-n-1}\\
&-|c_{n+1}|^2(1-|c_0|^2)^2-\overline{c_{n+1}}(1-|c_0|^2)^2z\sum_{j=n+2}^{\infty}c_jz^{j-n-2}-\overline{c_0}c_n\overline{c_{n+1}}(1-|c_0|^2)z\sum_{j=2}^{\infty}c_jz^{j-2}\Bigg]\\
&\times[(1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2}+\overline{c_n}c_{n+1}^2\}].
\end{aligned}$$

Thus,

$$RV-SU=z\Bigg[\Bigg\{(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+3}^{\infty}c_jz^{j-n-3}+c_n\overline{c_0}\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=3}^{\infty}c_jz^{j-3}$$

$$+\overline{c_0}c_{n+1}(1-|c_0|^2)^2\sum_{j=2}^{\infty}c_jz^{j-2}+\overline{c_n}c_{n+1}(1-|c_0|^2)\sum_{j=n+2}^{\infty}c_jz^{j-n-2}\Bigg\}$$
$$\times\{\{(1-|c_0|^2)^2-|c_n|^2\}^2-|c_{n+1}|^2(1-|c_0|^2)^2\}$$
$$-\{(1-|c_0|^2)\{\{(1-|c_0|^2)^2-|c_n|^2\}c_{n+2}+\overline{c_n}c_{n+1}^2\}\}$$
$$\times\Bigg\{-\overline{c_0}(1-|c_0|^2)\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=1}^{\infty}c_jz^{j-1}-\overline{c_n}\{(1-|c_0|^2)^2-|c_n|^2\}\sum_{j=n+1}^{\infty}c_jz^{j-n-1}$$
$$-\overline{c_{n+1}}(1-|c_0|^2)^2\sum_{j=n+2}^{\infty}c_jz^{j-n-2}-\overline{c_0}c_n\overline{c_{n+1}}(1-|c_0|^2)\sum_{j=2}^{\infty}c_jz^{j-2}\Bigg\}\Bigg],$$

$$\begin{aligned}zSV(1-\overline{k_{n+2}(0)}k_{n+2}(z))&=z[(1-|c_0|^2)(1-\overline{c_0}k_0(z))(1-\overline{k_n(0)}k_n(z))\{(1-|c_0|^2)^2-|c_n|^2\}(1-\overline{k_{n+1}(0)}k_{n+1}(z))]\\&\cdot[(1-|c_0|^2)^2(1-|k_n(0)|^2)(1-|k_{n+1}(0)|^2)\{(1-|c_0|^2)^2-|c_n|^2\}]\{1-\overline{k_{n+2}(0)}k_{n+2}(z)\}.\end{aligned}$$

Hence,

$$\begin{aligned}\Phi_{n+3}=k_{n+3}(0)&=\frac{\begin{pmatrix}c_{n+3}\{(1-|c_0|^2)^2-|c_n|^2\}^2+\overline{c_n}c_{n+1}c_{n+2}\{(1-|c_0|^2)^2-|c_n|^2\}\\-c_{n+3}|c_{n+1}|^2(1-|c_0|^2)^2+\overline{c_n}c_{n+1}c_{n+2}\{(1-|c_0|^2)^2-|c_n|^2\}\\+(\overline{c_n}^2)c_{n+1}^3+\overline{c_{n+1}}c_{n+2}^2(1-|c_0|^2)^2\end{pmatrix}}{(1-|c_0|^2)^3(1-|k_n(0)|^2)^2(1-|k_{n+1}(0)|^2)^2\{(1-|c_0|^2)^2-|c_n|^2\}\{1-|k_{n+2}(0)|^2\}}\\&=\frac{\begin{pmatrix}c_{n+3}[\{(1-|c_0|^2)^2-|c_n|^2\}^2-|c_{n+1}|^2(1-|c_0|^2)^2]\\+2\overline{c_n}c_{n+1}c_{n+2}\{(1-|c_0|^2)^2-|c_n|^2\}\\+\overline{c_{n+1}}c_{n+2}^2(1-|c_0|^2)^2+(\overline{c_n}^2)c_{n+1}^3\end{pmatrix}}{(1-|c_0|^2)^3(1-|\phi_n|^2)^2(1-|\phi_{n+1}|^2)^2\{(1-|c_0|^2)^2-|c_n|^2\}\{1-|\phi_{n+2}|^2\}}\\&=\frac{\begin{pmatrix}c_nc_{n+3}|c_{n+1}|^2(1-|\Phi_{n+1}|^2)+2|c_n|^2c_{n+1}c_{n+2}|\Phi_{n+1}|^2(1-|\Phi_n|^2)\\+c_n\overline{c_{n+1}}c_{n+2}^2|\Phi_{n+1}|^2+\overline{c_n}c_{n+1}^3|\Phi_{n+1}|^2|\Phi_n|^2\end{pmatrix}}{c_n|c_{n+1}|^2(1-|c_0|^2)(1-|\Phi_n|^2)(1-|\Phi_{n+1}|^2)^2(1-|\Phi_{n+2}|^2)}.\end{aligned}$$

Now, we proceed to our main results.

Theorem 3.1. *Let $\varphi(z)=\sum_{n=-N}^{N}a_nz^n$ (with $a_N\neq 0, a_{-N}\neq 0$) be a circulant trigonometric polynomial with argument 0 and let the coefficients of φ satisfy the following condition:*

$$\overline{a}_N\begin{pmatrix}a_{-2}\\a_{-4}\\\vdots\\a_{-N}\end{pmatrix}=a_{-N}\begin{pmatrix}\overline{a_2}\\\overline{a_4}\\\vdots\\\overline{a_N}\end{pmatrix}. \tag{3.1}$$

If $\alpha=\frac{a_2\overline{a_3}}{|a_3|^2-|a_1|^2}$, then T_φ is hyponormal if and only if

(i) $|a_3|\le|a_1|\le|a_N|$,

(ii) $|\alpha|\le 1$,

(iii) $\left|1+\alpha^2\overline{\left(\frac{a_1}{a_3}\right)}\right|\le 1-|\alpha|^2$.

Proof. By the definition of a circulant polynomial with argument 0 we have:

$$\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-N} \end{pmatrix} = \begin{pmatrix} a_N \\ a_{N-1} \\ \vdots \\ a_1 \end{pmatrix}.$$

Now, the equation (3.1) together with this condition yields that

$$\overline{a_N}\begin{pmatrix} a_{N-1} \\ a_{N-3} \\ \vdots \\ a_1 \end{pmatrix} = a_1\begin{pmatrix} \overline{a_2} \\ \overline{a_4} \\ \vdots \\ \overline{a_N} \end{pmatrix}.$$

Now, by Theorem 1.1, T_φ will be hyponormal if and only if we can find a unique analytic polynomial $c(z) = \sum_{j=0}^{\infty} c_j z^j$ in $H^\infty(\mathbb{T})$ such that $\varphi - c\bar{\varphi} \in H^\infty(\mathbb{T})$. Because c satisfies the property $\varphi - c\bar{\varphi} \in H^\infty(\mathbb{T})$, then c necessarily satisfies the property that

$$c\sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{N} a_{-n} z^{-n} \in H^\infty(\mathbb{T}). \tag{3.2}$$

From (3.2), one can compute the Fourier's coefficients $\widehat{c}(0), \widehat{c}(1), \dots, \widehat{c}(N-1)$ of c to be $\widehat{c}(n) = c_n$ for each $n = 0, 1, \dots, N-1$ uniquely as follows:

$$c_0 = \frac{a_1}{\overline{a_N}}, \quad c_1 = \dots = c_{N-4} = 0,$$

$$c_{N-3} = (\overline{a_N})^{-1}\left(a_{-3} - \sum_{j=0}^{N-4} c_j \overline{a_{j+3}}\right) = (\overline{a_N})^{-1}(a_{-3} - c_0\overline{a_3} - \dots) = \frac{a_1\overline{a_3}d_1}{|a_1|^2(\overline{a_N})^2},$$

$$c_{N-2} = (\overline{a_N})^{-1}\left(a_{-2} - \sum_{j=0}^{N-3} c_j \overline{a_{j+2}}\right) = (\overline{a_N})^{-1}(a_{-2} - c_0\overline{a_2} - \dots - c_{N-3}\overline{a_{N-1}}) = -\frac{a_2\overline{a_3}d_1}{|a_1|^2(\overline{a_N})^2},$$

$$c_{N-1} = (\overline{a_N})^{-1}\left(a_{-1} - \sum_{j=0}^{N-2} c_j \overline{a_{j+1}}\right) = (\overline{a_N})^{-1}(a_{-1} - c_0\overline{a_1} - \dots - c_{N-2}\overline{a_{N-1}}) = \frac{\{-a_1d_2 + a_2^2\overline{a_3}\}d_1}{a_1|a_1|^2(\overline{a_N})^2},$$

where $d_1 = |a_N|^2 - |a_1|^2$ and $d_2 = |a_3|^2 - |a_1|^2$. Hence $c(z) = c_0 + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ is the unique analytic polynomial of degree less than N which satisfies the condition $\varphi - c\bar{\varphi} \in H^\infty(\mathbb{T})$. Now, if $\{\Phi_n\}$ is a sequence of Schur's functions associated with $\{c_n\}$, where $n = 0, 1, \dots, N-1$, then by Lemma 3.1, one can compute Schur's functions Φ_n's as follows:

$$\Phi_0 = c_0 = \frac{a_1}{\overline{a_N}},$$

$$\Phi_{N-3} = \frac{c_{N-3}}{1 - |c_0|^2} = \frac{a_1\overline{a_3}d_1}{|a_1|^2(\overline{a_N})^2(1 - |c_0|^2)} = \frac{\overline{a_3}a_N}{(\overline{a_1 a_N})},$$

$$\Phi_{N-2} = \frac{c_{N-2}}{(1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)} = \frac{-\frac{a_2\overline{a_3}d_1}{|a_1|^2(\overline{a_N})^2}}{(1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)} = \frac{a_2\overline{a_3}a_N}{\overline{a_N}d_2},$$

$$\Phi_{N-1} = \frac{(1 - |\Phi_{N-3}|^2)c_{N-1}c_{N-3} + |\Phi_{N-3}|^2 c_{N-2}^2}{c_{N-3}(1 - |c_0|^2)(1 - |\Phi_{N-3}|^2)^2(1 - |\Phi_{N-2}|^2)} = \frac{(d_2^2 + \overline{a_1}a_2^2\overline{a_3})a_N}{(d_2^2 - |a_2|^2|a_3|^2)\overline{a_N}}.$$

Hence, with an application of Theorem 2.1 and a straightforward calculation, the results can be computed easily. □

Theorem 3.2. *Let* $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ *(with* $a_N \neq 0, a_{-N} \neq 0$*) be a circulant trigonometric polynomial with argument* 0 *whose coefficients satisfy the following condition:*

$$\overline{a_N}\begin{pmatrix} a_{-2} \\ a_{-3} \\ a_{-5} \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_2} \\ \overline{a_3} \\ \overline{a_5} \\ \vdots \\ \overline{a_N} \end{pmatrix}.$$

If $d_1 = |a_N|^2 - |a_1|^2$, $d_3 = |a_1|^2 - |a_4|^2$, $\alpha = (a_2^2 - a_1 a_3)$ *and* $\beta = \{2a_1 a_3 - a_2^2\}$, *then* T_φ *is hyponormal if and only if*

(i) $|a_4| \le |a_1| \le |a_N|$,

(ii) $|a_2 a_4| \le d_3$,

(iii) $|a_4|\,|\overline{a_1} a_2^2 - d_3 a_3| \le d_3^2 - |a_2 a_4|^2$,

(iv) $d_3|(a_1^2 d_3 + a_2 \overline{a_4}\beta)(d_3^2 - |a_2 a_4|^2) - \overline{a_4}|a_4|^2(2a_2 \alpha d_3 + \overline{a_2}\alpha^2 + a_2^3|a_4|^2)|$

$$\le \big|\,|a_1|^2(d_3^2 - |a_2 a_4|^2)^2 - |a_4|^2|d_3\alpha + a_2^2|a_4|^2|^2\big|.$$

Proof. Applying the technique used in the proof of Theorem 3.1, we can get a unique analytic polynomial $c(z) = c_0 + c_{N-4}z^{N-4} + c_{N-3}z^{N-3} + c_{N-2}z^{N-2} + c_{N-1}z^{N-1}$ of degree less than N which satisfies the condition $\varphi - c\bar{\varphi} \in H^\infty(\mathbb{T})$. Now, if $\{\Phi_n\}$ is a sequence of Schur's functions associated with $\{c_n\}$, where $n = 0, 1, \dots, N-1$, then by Lemma 3.1, Schur's functions Φ_n's can be computed as follows:

$$\Phi_0 = c_0 = \frac{a_1}{\overline{a_N}}, \quad \Phi_{N-4} = \frac{c_{N-4}}{1-|c_0|^2} = \frac{\overline{a_4} a_N}{\overline{a_1} a_N},$$

$$\Phi_{N-3} = \frac{c_{N-3}}{(1-|c_0|^2)(1-|\Phi_{N-4}|^2)} = \frac{-a_2\overline{a_4}a_N}{(\overline{a_N} d_3)},$$

$$\Phi_{N-2} = \frac{(1-|\Phi_{N-4}|^2)c_{N-2}c_{N-4} + |\Phi_{N-4}|^2 c_{N-3}^2}{c_{N-4}(1-|c_0|^2)(1-|\Phi_{N-4}|^2)^2(1-|\Phi_{N-3}|^2)} = \frac{A+B}{C},$$

where

$$A = (1-|\Phi_{N-4}|^2)c_{N-2}c_{N-4} = \frac{(\overline{a_4})^2 d_1^2 d_3 \alpha}{|a_1|^6(\overline{a_N})^4},$$

$$B = |\Phi_{N-4}|^2(c_{N-3})^2 = \frac{a_2^2(\overline{a_4})^2|a_4|^2 d_1^2}{|a_1|^6(\overline{a_N})^4},$$

$$C = c_{N-4}(1-|c_0|^2)(1-|\Phi_{N-4}|^2)^2(1-|\Phi_{N-3}|^2) = \frac{\overline{a_4} d_1^2(|d_3|^2 - |a_2|^2|a_4|^2)}{\overline{a_1}(\overline{a_N})^2|a_N|^2|a_1|^4}.$$

Hence,

$$\Phi_{N-2} = \frac{\overline{a_4} a_N\{d_3\alpha + a_2^2|a_4|^2\}}{a_1\overline{a_N}\{|d_3|^2 - |a_2 a_4|^2\}},$$

$$\Phi_{N-1} = \frac{\begin{pmatrix} c_{N-4}|c_{N-3}|^2 c_{N-1}(1-|\Phi_{N-3}|^2) + 2|c_{N-4}|^2 c_{N-3}c_{N-2}|\Phi_{N-3}|^2(1-|\Phi_{N-4}|^2) \\ +c_{N-4}\overline{c_{N-3}}c_{N-2}^2|\Phi_{N-3}|^2 + \overline{c_{N-4}}c_{N-3}^3|\Phi_{N-3}|^2|\Phi_{N-4}|^2 \end{pmatrix}}{c_{N-4}|c_{N-3}|^2(1-|c_0|^2)(1-|\Phi_{N-4}|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)}$$
$$= \frac{E+F+G+H}{K},$$

where

$$E = c_{N-4}|c_{N-3}|^2 c_{N-1}(1-|\Phi_{N-3}|^2) = \frac{\overline{a_4}|a_2a_4|^2 d_1^4(a_1^2 d_3 + a_2\overline{a_4}\beta)(|d_3|^2 - |a_2a_4|^2)}{a_1|a_1|^8(\overline{a_N})^4|a_N|^4|d_3|^2},$$

$$F = 2|c_{N-4}|^2 c_{N-3}c_{N-2}|\Phi_{N-3}|^2(1-|\Phi_{N-4}|^2) = -2\frac{\overline{a_4}|a_2a_4|^2 d_1^4 a_2\overline{a_4}|a_4|^2\alpha d_3}{a_1|a_1|^8(\overline{a_N})^4|a_N|^4|d_3|^2},$$

$$G = c_{N-4}\overline{c_{N-3}}c_{N-2}^2|\Phi_{N-3}|^2 = -\frac{\overline{a_4}|a_2a_4|^2 d_1^4\overline{a_2a_4}|a_4|^2\alpha^2}{a_1|a_1|^8(\overline{a_N})^4|a_N|^4|d_3|^2},$$

$$H = \overline{c_{N-4}}c_{N-3}^3|\Phi_{N-3}|^2|\Phi_{N-4}|^2 = -\frac{\overline{a_4}|a_2a_4|^2 d_1^4 a_2^3|a_4|^4\overline{a_4}}{a_1|a_1|^8(\overline{a_N})^4|a_N|^4|d_3|^2},$$

$$K = c_{N-4}|c_{N-3}|^2(1-|c_0|^2)(1-|\Phi_{N-4}|^2)(1-|\Phi_{N-3}|^2)^2(1-|\Phi_{N-2}|^2)$$
$$= \frac{\overline{a_4}\,|a_2a_4|^2 d_1^4 d_3}{\overline{a_1}\,|a_1|^8(\overline{a_N})^2|a_N|^6|d_3|^4}(|a_1\{|d_3|^2 - |a_2a_4|^2\}|^2 - |\overline{a_4}\{d_3\alpha + a_2^2|a_4|^2\}|^2).$$

Now by putting the above values of E, F, G, H and K and simplifying, we get

$$\Phi_{N-1} = \frac{\overline{a_1}a_N d_3}{a_1\overline{a_N}}\,\frac{[(a_1^2 d_3 + a_2\overline{a_4}\beta)(|d_3|^2 - |a_2|^2|a_4|^2) - \overline{a_4}|a_4|^2(2a_2\alpha d_3 + \overline{a_2}\alpha^2 + a_2^3|a_4|^2)]}{[|a_1(|d_3|^2) - |a_2|^2|a_4|^2)|^2 - |\overline{a_4}(d_3\alpha + a_2^2|a_4|^2)|^2]}.$$

Hence, by an application of Theorem 2.1, the results can be computed easily. □

4. Conclusion

Theorems 3.1 and 3.2 will give the researchers a new insight in exploring the hyponormality of Toeplitz operators on symmetric and circulant-type [4] trigonometric polynomial symbols in the Hardy-Hilbert spaces.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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