Communications in Mathematics and Applications
Vol. 15, No. 5, pp. 1431–1442, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i5.2866

http://www.rgnpublications.com

Research Article

On Pitchfork Domination Number of Corona of Some Graphs

Kris D. Barnido*[1] and Stephanie O. Espinola[2]

[1] *Kao National High School, Nabunturan East District, Schools Division of Davao de Oro, Philippines*
[2] *Department of Mathematics and Statistics, University of Southeastern Philippines, Davao City, Philippines*
***Corresponding author:** kris.barnido@deped.gov.ph

Received: August 27, 2024 **Accepted:** November 23, 2024

Abstract. A dominating set D of V is called a *pitchfork dominating set* if every vertex in it dominates at least j vertices and at most k vertices of $V-D$, for any non-negative integers j and k. The pitchfork domination number of G, denoted by $\gamma_{pf}(G)$ is the minimum cardinality over all pitchfork dominating sets in G. In this paper, the pitchfork domination when $j=1$ and $k=2$ are applied to the corona of some graphs: $G\circ C_m$, $G\circ K_{1,n}$, $G\circ K_{p,q}$, $G\circ B_{2,k}$ and $G\circ F_{2,k}$. In relation to getting new results of the corona of the mentioned graphs, we also generate results of pitchfork domination number of $P_2+\overline{K}_n$, $K_1+K_{p,q}$, $B_{2,k}$, $F_{2,k}$ and $K_1+F_{2,k}$.

Keywords. Domination, Pitchfork domination, Corona, Join

Mathematics Subject Classification (2020). 05C05, 05C38, 05C69, 05C76

1. Introduction

All graphs considered in this paper are all connected, finite, simple and undirected. A graph $G=(V,E)$ is an ordered pair, where V is a finite non-empty set and E is a set of unordered pair of distinct elements of V. The vertex-set of G whose elements are called vertices is denoted by $V=V(G)$ and $E=E(G)$ is called the edge-set of G whose elements are called edges. For definitions of the special graphs considered in this study, please refer to Aslam *et al*. [3], Asmiati [4], Chartrand and Zhang [5], and Haryanti *et al*. [7].

A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its *closed neighborhood* $N[v]$. A set S of vertices of G is a *dominating* *set* of G if every vertex of G is dominated by at least one vertex of S. Equivalently, a set S of vertices of G is a dominating set if every vertex in $V(G)-S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is the referred to as a *minimum dominating set* (Chartrand and Zhang [5]). The importance of domination in various applications, led to the appearance of different types of domination according to the purpose used (Gayathri *et al*. [6], Haynes *et al*. [8], and Venkateswari [9]). One of these is pitchfork and inverse pitchfork domination which were introduced by Abdlhusein and Al-Harere [1,2].

A dominating set D of V is called a *pitchfork dominating set* if every vertex in it dominates at least j vertices and at most k vertices of $V-D$, for any non-negative integers j and k. The pitchfork domination number of G, denoted by $\gamma_{pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G (Al-Harere and Abdlhusein [2]). In this paper, the pitchfork domination when $j=1$ and $k=2$ are applied on the graphs.

2. Results and Discussions

The Pitchfork Domination Number of $K_1+K_{p,q}$, $B_{2,k}$, $F_{2,k}$ and $K_1+F_{2,k}$

The following are the result from the paper of Abdlhusien and Al-Harere [1,2] which will be used in the proof of the graphs used in this paper.

Theorem 2.1 ([2]). Let G be a wheel graph W_n, then:

$$\gamma_{pf}(W_n)=\begin{cases}2\left\lceil\frac{n}{4}\right\rceil-1, & \text{if } n\equiv 1 \pmod 4),\\ 2\left\lceil\frac{n}{4}\right\rceil, & \text{otherwise.}\end{cases}$$

Theorem 2.2 ([1]). If G is a graph of order n, then:

(1) $\gamma_{pf}(G\circ K_2)=\gamma_{pf}(\overline{G}\circ K_2)=\gamma_{pf}(G\circ\overline{K}_2)=\gamma_{pf}(\overline{G}\circ\overline{K}_2)=n$,

(2) $\gamma_{pf}(G+K_2)=\gamma_{pf}(\overline{G}+K_2)=\gamma_{pf}(G+K_2)=\gamma_{pf}(\overline{G}+\overline{K}_2)=n$,

(3) $\gamma_{pf}(G\circ K_1)=\gamma_{pf}(\overline{G}\circ\overline{K}_1)=n$.

Lemma 2.3 below will be used in determining the $\gamma_{pf}(G\circ K_{1,n})$ in Theorem 2.9.

Lemma 2.3. Let P_2 be a path graph of order 2 and $\overline{K}_n$ be the empty graph of order $n\ge 3$. Then, the following are the only pitchfork dominating sets of $P_2+\overline{K}_n$,

$A=\{v_1,v_2,\dots,v_n\}$,

$B=(V(\overline{K}_n)\setminus\{v_i\})\cup\{u\}$, where $i\in\{1,2,\dots,n\}$,

$C=(V(\overline{K}_n)\setminus\{v_i\})\cup\{v\}$, where $i\in\{1,2,\dots,n\}$,

$D=V(\overline{K}_n)\cup\{u\}$, and

$E=V(\overline{K}_n)\cup\{v\}$.

Proof. Let $V(P_2)=\{u,v\}$ and $V(\overline{K}_n)=\{v_1,v_2,\dots,v_n\}$ as shown in Figure 1. Note by Theorem 2.2, $\gamma_{pf}(P_2+\overline{K}_n)=n$. Let S be a pitchfork dominating set in $P_2+\overline{K}_n$. Note that either $S\cap V(P_2)=\emptyset$ or $S\cap V(P_2)\neq\emptyset$. Consider the following cases:

Case 1: Suppose that $S \cap V(P_2) = \emptyset$. Then $S \subseteq V(\overline{K}_n)$. Based on Figure 1 and the definition of pitchfork dominating set, we must have $S = V(\overline{K}_n) = D = \{v_1, v_2, \dots, v_n\}$. In this case, $S = \{v_1, v_2, \dots, v_n\}$ is the only pitchfork dominating set of $P_2 + \overline{K}_n$. Otherwise if there exists v_i, where $i \in \{1, 2, \dots, n\}$ such that $v_i \notin S$, then no vertex will dominate v_i. This contradicts the assumption that S is a pitchfork dominating set of $P_2 + \overline{K}_n$. Thus, in this case, $|S| = n$.

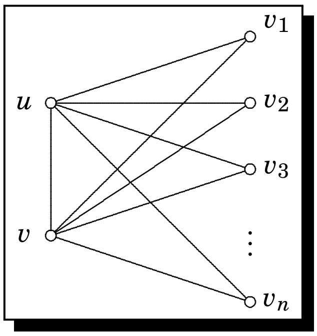

Figure 1. The join $P_2 + \overline{K}_n$

Case 2: Suppose that $S \cap V(P_2) \neq \emptyset$. Then $|S \cap V(P_2)| = 1$ or $|S \cap V(P_2)| = 2$.

Subcase 2.1: Suppose that $|S \cap V(P_2)| = 1$.

Without loss of generality, suppose that $u \in S$. Then $v \notin S$. By the definition of a pitchfork dominating set, we must include either all n vertices of $\overline{K}_n$ in S or the $n-1$ vertices of $\overline{K}_n$ in S. Note that if we include fewer than $n-1$ vertices in S, then S is not a pitchfork dominating set since $u \in S$ would dominate 3 or more vertices in $P_2 + \overline{K}_n$, which is a contradiction since S is a pitchfork dominating set. Thus, in this case, $|S| = n+1$ or $|S| = n-1+1 = n$.

Subcase 2.2: Suppose that $|S \cap V(P_2)| = 2$.

Then $u \in S$ and $v \in S$. Note that this case is not possible for $n \geq 3$. That is, S can not be a pitchfork dominating set for $n \geq 3$ because if $|S \cap V(\overline{K}_n)| = \emptyset$, then either u or v will dominate 3 or more vertices in $V(P_2 + \overline{K}_n) = V(\overline{K}_n)$, which is a contradiction since S is a pitchfork dominating set. If $|S \cap V(\overline{K}_n)| \neq \emptyset$. Then, there is $v_i \in V(\overline{K}_n)$ such that $v_i \in S$ for some $i = 1, 2, \dots, n$. By the definition of $\overline{K}_n$ and since $u, v \in S$, we have $N(v_i) \cap (V(P_2 + \overline{K}_n) \setminus S = \emptyset$, which again contradicts the pitchfork dominating set condition.

Therefore, based on the definition of $P_2 + \overline{K}_n$ and from *Case* 1 and *Case* 2, the only possible pitchfork dominating set are the following:

$A = \{v_1, v_2, \dots, v_n\}$,

$B = (V(\overline{K}_n) \setminus \{v_i\}) \cup \{u\}$, where $i \in \{1, 2, \dots, n\}$,

$C = (V(\overline{K}_n) \setminus \{v_i\}) \cup \{v\}$, where $i \in \{1, 2, \dots, n\}$,

$D = V(\overline{K}_n) \cup \{u\}$, and

$E = V(\overline{K}_n) \cup \{v\}$. □

Lemma 2.4. Let $K_{p,q}$ be a complete bipartite graph of order $p+q$, where $p, q \geq 2$ and K_1 be the empty graph of order 1. Then $\gamma_{pf}(K_1 + K_{p,q}) = (p+q) - 2$.

Proof. Let $V(K_{p,q}) = \{v_1, v_2, \dots, v_p, u_1, u_2, \dots, u_q\}$ where the partite sets are $W = \{v_1, v_2, \dots, v_p\}$ and $U = \{u_1, u_2, \dots, u_q\}$ and $V(K_1) = \{x\}$ as shown in Figure 2.

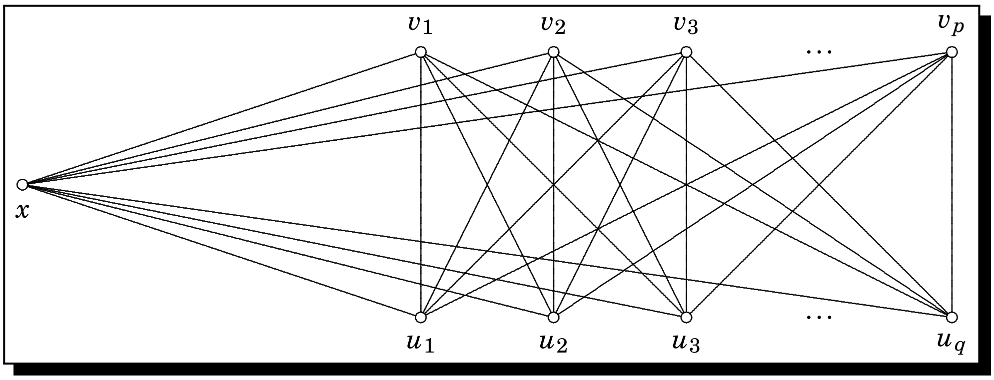

Figure 2. The join $K_1+K_{p,q}$

Let $D=\{v_1,v_2,\dots,v_{p-1}\}\cup\{u_1,u_2,\dots,u_{q-1}\}$. Then D is a pitchfork dominating set in $K_1+K_{p,q}$. Thus, the pitchfork dominating set exists in $K_1+K_{p,q}$. Consequently, $\gamma_{pf}(K_1+K_{p,q})$ exists.

Next, let us find the $\gamma_{pf}(K_1+K_{p,q})$. To do this, let S be a pitchfork dominating set in $K_1+K_{p,q}$. Note that either $S\cap V(K_1)=\emptyset$ or $S\cap V(K_1)\neq\emptyset$.

Case 1: Suppose that $S\cap V(K_1)=\emptyset$.
This implies that $x\notin S$ and therefore $S\subseteq V(K_{p,q})$. From Figure 2 and the definition of pitchfork dominating set, the only possible pitchfork dominating sets of $K_1+K_{p,q}$ are:

$$S=(W\setminus\{v_i\})\cup(U\setminus\{u_j\}),\ \text{where } i\in\{1,2,\dots,p\} \text{ and } j\in\{1,2,\dots,q\}$$

or

$$S=(W\setminus\{v_k\})\cup U,\ \text{where } k\in\{1,2,\dots,p\}$$

or

$$S=(U\setminus\{u_t\})\cup W,\ \text{where } t\in\{1,2,\dots,q\}.$$

Case 2: Suppose that $S\cap V(K_1)\neq\emptyset$.
Then $|S\cap V(K_1)|=1$ which implies that $x\in S$. By definition of a pitchfork dominating set, the only possible pitchfork dominating set of $K_1+K_{p,q}$ in this case is $S=(W\setminus\{v_i\})\cup(U\setminus\{u_j\})$ where $i\in\{1,2,\dots,p\}$ and $j\in\{1,2,\dots,q\}$.

Therefore, by *Case* 1 and *Case* 2,

$$\gamma_{pf}(K_1+K_{p,q})=\min\{p+q-2,p+q-1\}=p+q-2.$$ □

Theorem 2.5. Let $B_{2,k}$ be a banana tree graph where $k\geq 5$, then $\gamma_{pf}(B_{2,k})=2k-3$.

Proof. Let $B_{2,k}$ be a banana tree graph with $|V(B_{2,k})|=2k+1$ and $|E(B_{2,k})|=2k$. Assume that $A=\{u_1,u_2,\dots,u_k\}$, $B=\{v_1,v_2,\dots,v_k\}$. Then $V(B_{2,k})=A\cup B\cup\{w\}$. Based on the definition of $B_{2,k}$, and structure shown in Figure 3, the only possible pitchfork dominating sets are the following:

Case 1: Let $D_1=\{u_2,u_3,\dots,u_{k-1}\}$, $D_2=\{v_2,v_3,\dots,v_{k-1}\}$ and $D_3=\{w\}$ so that $D=D_1\cup D_2\cup D_3$ and $D_1\cap D_2\cap D_3=\emptyset$. Observe that for every $u_k\in D_1\subseteq D$, where $k\in\{2,3,\dots,k-1\}$, u_k dominates u_1 and for every $v_k\in D_2\subseteq D$, where $k\in\{2,3,\dots,k-1\}$, v_k dominates v_1 and $w\in D_3\subseteq D$ dominates u_k and v_k. Hence, D is a pitchfork dominating set with $|D|=k-2+k-2+1=2(k-2)+1=2k-3$.

Case 2: Let $C_1 = \{u_2, u_3, \ldots, u_k\}$, $C_2 = \{v_2, v_3, \ldots, v_k\}$ so that $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. Then every $u_k \in C_1 \subseteq C$ where $k = 2, 3, \ldots, k-1$ dominates u_1 while u_k dominates u_1 and w. Similarly, every $v_k \in C_2 \subseteq C$ such that $k \in \{2, 3, \ldots, k-1\}$ dominates u_1 while v_k dominates v_1 ad w. So, C is a pitchfork dominating set with $|C| = 2(k-1) = 2k-2$.

Case 3: Let $E_1 = \{u_2, u_3, \ldots, u_k\}$, $E_2 = \{v_2, v_3, \ldots, v_{k-1}\}$, $E_3 = w\}$ then $E = E_1 \cup E_2 \cup E_3$ and $E_1 \cap E_2 \cap E_3 = \emptyset$. Observe that every vertex in $E_1 \subseteq E$ dominates $u_1 \in V(B_{2,k}) - E$. Similarly, every vertex in $E_2 \subseteq E$ dominates $v_1 \in V(B_{2,k}) - E$ and $w \in E_3 \subseteq E$ dominates v_k. Thus, E is a pitchfork dominating set with $|E| = k-1+k-2+1 = 2k-2$.

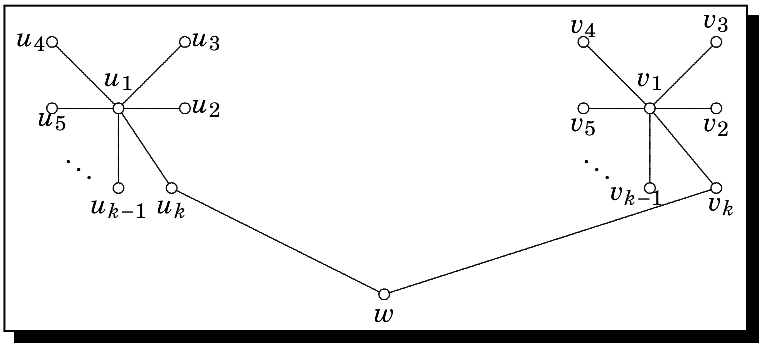

Figure 3. $B_{2,k}$

Case 4: Let $F_1 = \{u_2, u_3, \ldots, u_{k-1}\}$, $F_2 = \{v_2, v_3, \ldots, v_k\}$, $F_3 = \{w\}$ then $F = F_1 \cup F_2 \cup F_3$ and $F_1 \cap F_2 \cap F_3 = \emptyset$. Observe that every vertex in $F_1 \subseteq F$ dominates $u_1 \in V(B_{2,k}) - F$. Similarly, every vertex in $F_2 \subseteq F$ dominates $v_1 \in V(B_{2,k}) - F$ and $w \in F_3 \subseteq F$ dominates u_k. Thus, F is a pitchfork dominating set with $|E| = k-2+k-1+1 = 2k-2$.

Observe that among the cases listed above, the minimum pitchfork dominating set is D with $|D| = 2k-3$. Therefore, $\gamma_{pf}(B_{2,k}) = 2k-3$. □

Theorem 2.6. Let $F_{2,k}$ be a firecracker graph where $k \geq 4$, then $\gamma_{pf}(F_{2,k}) = 2k-3$.

Proof. Let $F_{2,k}$ be a firecracker graph with $|V(F_{2,k})| = 2k$ and $|E(F_{2,k})| = 2k-1$. Assume that $A = \{u_1, u_2, \ldots, u_k\}$ and $B = \{v_1, v_2, \ldots, v_k\}$. Then $V(F_{2,k}) = A \cup B$. Note that from the definition of $F_{2,k}$, and as shown in Figure 4, the only possible pitchfork dominating sets are the following:

Case 1: Let $C_1 = \{u_2, u_3, \ldots, u_k\}$, $C_2 = \{v_2, v_3, \ldots, v_k\}$ so that $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. Observe that for every $u_i \in C_1 \subseteq C$, where $i \in \{2, 3, \ldots, k\}$, u_i dominates u_1. For every $v_i \in C_2 \subseteq C$, where $i \in \{2, 3, \ldots, k\}$, v_i dominates v_1. Hence, C is a pitchfork dominating set in $F_{2,k}$ with $|C| = k-1+k-1 = 2k-2$.

Case 2: Let $D_1 = \{u_2, u_3, \ldots, u_{k-1}\}$, $D_2 = \{v_2, v_3, \ldots, v_k\}$ so that $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$. Observe that for every $u_i \in D_1 \subseteq D$, where $i \in \{2, 3, \ldots, k-1\}$, u_i dominates u_1 while for every $v_i \in D_2 \subseteq D$, where $i \in \{2, 3, \ldots, k-1\}$, v_i dominates v_1. Meanwhile, v_k dominates two vertices which are v_1 and u_k. Thus, D is a pitchfork dominating set in $F_{2,k}$ with $|D| = k-2+k-1 = 2k-3$.

Case 3: Let $E_1 = \{u_2, u_3, \ldots, u_k\}$, $E_2 = \{v_2, v_3, \ldots, v_{k-1}\}$ so that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Observe that for every $u_i \in E_1 \subseteq E$, where $i \in \{2, 3, \ldots, k-1\}$, u_i dominates u_1 while for every $v_i \in E_2 \subseteq E$, where $i \in \{2, 3, \ldots, k-1\}$, v_i dominates v_1. Meanwhile, u_k dominates two vertices which are u_1 and v_k. Thus, E is a pitchfork dominating set in $F_{2,k}$ with $|E| = k-1+k-2 = 2k-3$.

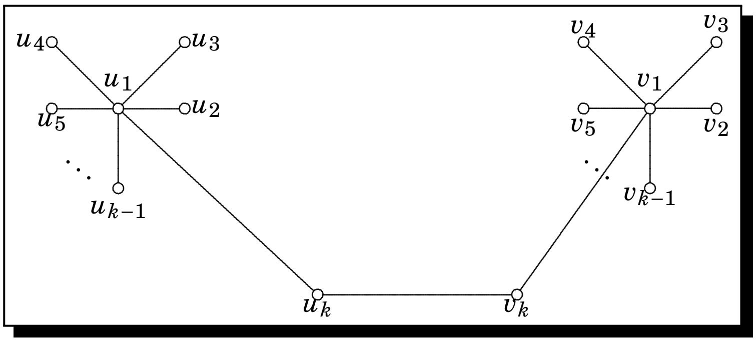

Figure 4. $F_{2,k}$

Observe that among the cases listed above, the minimum pitchfork dominating sets of $F_{2,k}$ are D and E. Therefore, $\gamma_{pf}(F_{2,k}) = 2k-3$. □

Lemma 2.7. Let K_1 be an empty graph of order 1 and $F_{2,k}$, where $k \geq 4$ be a firecracker graph of order $2k$. Then $\gamma_{pf}(K_1 + F_{2,k}) = 2k-2 = 2(k-1)$.

Proof. Let $V(K_1) = \{w\}$ and $V(F_{2,k}) = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k\}$ as shown in Figure 5. The following are the only pitchfork dominating sets of $K_1 + F_{2,k}$.

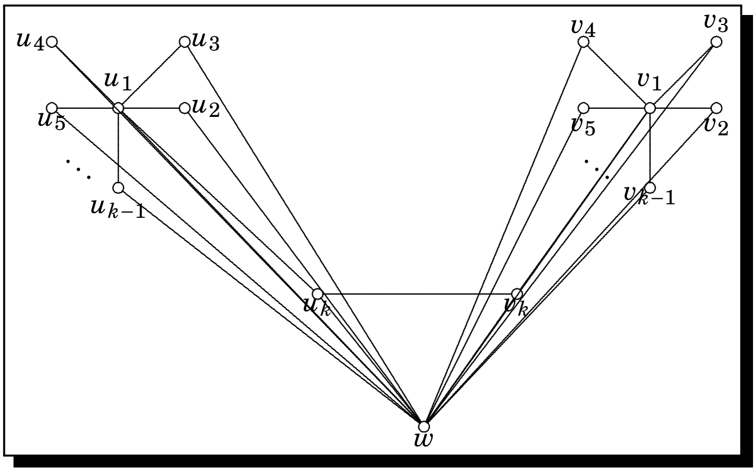

Figure 5. $K_1 + F_{2,k}$

Case 1: Let $D = \{w, u_2, u_3, \dots, u_k, v_2, v_3, \dots, v_k\}$. Observe that every u_i, where $i \in \{2,3,\dots,k\}$ dominates $u_1 \in V(K_1 + F_{2,k}) - D$ while every v_i, where $i \in \{2,3,\dots,k\}$ dominates $v_1 \in V(K_1 + F_{2,k}) - D$ and w dominates u_1 and $v_1 \in V(K_1 + F_{2,k}) - D$. It follows that D is a pitchfork dominating set of $K_1 + F_{2,k}$ with $|D| = 1 + 2k - 2 = 2k - 1$.

Case 2: Let $E = V(F_{2,k}) \setminus \{u_1, v_1\}$. Observe that every u_i where $i \in \{2,3,\dots,k\}$ dominates u_1 and $w \in V(K_1 + F_{2,k}) - E$. In addition, every v_j, where $j \in \{2,3,\dots,k\}$ dominates v_1 and $w \in V(K_1 + F_{2,k}) - E$. Hence, E is a pitchfork dominating set of $K_1 + F_{2,k}$ with $|E| = 2k - 2$.

Let $F = V(F_{2,k}) \setminus \{u_1\}$, then every u_i where $i \in \{2,3,\dots,k\}$ dominates u_1 and $w \in V(K_1 + F_{2,k}) - E$ while every v_j where $j \in \{1,2,\dots,k\}$ dominates $w \in V(K_1 + F_{2,k}) - E$. Thus, F is a pitchfork dominating set of $K_1 + F_{2,k}$ with $|F| = 2k - 1$.

Let $H = V(F_{2,k})\setminus\{v_1\}$, then every v_j where $j \in \{2,3,\dots,k\}$ dominates v_1 and $w \in V(K_1+F_{2,k})-E$ while every u_i where $i \in \{1,2,\dots,k\}$ dominates $w \in V(K_1+F_{2,k})-E$. Thus, H is a pitchfork dominating set of $K_1+F_{2,k}$ with $|H| = 2k-1$.

Assume that $I = V(F_{2,k})$, then every u_i where $i \in \{1,2,3,\dots,k\}$ and every v_j where $j \in \{1,2,3,\dots,k\}$ dominates $w \in V(K_1+F_{2,k})-I$. Therefore, I is a pitchfork dominating set of $K_1+F_{2,k}$ with $|I| = 2k$.

Observe that among all pitchfork dominating sets listed above, the minimum pitchfork dominating set is E. Therefore, $\gamma_{pf}(K_1+F_{2,k}) = 2k-2 = 2(k-1)$. □

The Pitchfork Domination Number of Corona of Graphs

From the paper of Al-Harere and Abdlhusein [2], they obtained results on (1,2)-pitchfork domination of wheel graph and in their study, the minimum pitchfork dominating set is a subset of C_m where $W_m = K_1 + C_m$. Thus, we obtained the result on $G \circ C_m$ below:

Theorem 2.8. Let G be a connected graph of order $n \geq 2$ and a cycle graph C_m of order $m \geq 6$, then

$$\gamma_{pf}(G \circ C_m) = \begin{cases} n\left(2\left\lceil \frac{m}{4}\right\rceil - 1\right), & \text{if } m \equiv 1 \pmod 4, \\ n\left(2\left\lceil \frac{m}{4}\right\rceil\right), & \text{otherwise.} \end{cases}$$

Proof. For every $u \in V(G)$, denote by C_m^u, the copy of C_m whose vertices are attached one by one to the vertex u. Denote by $u + C_m^u$ the subgraph of the corona $G \circ C_m$, corresponding to the join $\langle\{u\}\rangle + C_m^u$. Note that for every $u \in V(G)$, the copy $u + C_m^u$ is isomorphic to the wheel graph W_m. By Theorem 2.1,

$$\gamma_{pf}(W_m) = \begin{cases} 2\left\lceil \frac{m}{4}\right\rceil - 1, & \text{if } m \equiv 1 \pmod 4, \\ 2\left\lceil \frac{m}{4}\right\rceil, & \text{otherwise.} \end{cases}$$

Let S be a pitchfork dominating set in $G \circ C_m$ and let $S_u = S \cap V(C_m^u)$. Then $S_u \subseteq S$. Let us show first that every vertex in $V(G)$ should not be an element of S. Suppose that there is a vertex $u \in V(G)$ such that $u \in S$. Now since $u \in S$, it follows that by the definition of pitchfork dominating set and the corona $G \circ C_m$, we must have

$$|S_u| = m-2 \quad \text{or} \quad |S_u| = m-1,$$

otherwise if $|S_u| \leq m-3$, u will dominate at least three vertices. A contradiction since S is a pitchfork dominating set of H. Since $m \geq 6$, it follows that in either case, there exists $v \in S_u \subseteq S$ that is not adjacent to any vertex in $V(G \circ C_m)\setminus S$. This is a contradiction since S is a pitchfork dominating set of $G \circ C_m$. Since $u \in V(G)$ is arbitrary, it follows that if S is a pitchfork dominating set of $G \circ C_m$, then $u \notin S$ for every $u \in V(G)$.

Thus, to have a minimum pitchfork dominating set in $G \circ C_m$, we should not include any vertex of G in the set. Hence, we will not consider any $N_G(u)$ where $u \in V(G)$. By Theorem 2.1, the minimum pitchfork dominating set is a subset of C_m, where $W_m = K_1 + C_m$. Therefore, if S is a minimum pitchfork dominating set of $G \circ C_m$, $S \subseteq \bigcup_{u \in V(G)} V(C_m^u)$ where $S_u = S \cap V(C_m^u)$, $S_u \cap S_v = \emptyset$ for $u \neq v$ where $u,v \in V(G)$ and

$$|S_u| = \gamma_{pf}(u + C_m^u) = \begin{cases} 2\left(\left\lceil \frac{m}{4}\right\rceil - 1\right), & \text{if } m \equiv 1 \pmod 4, \\ 2\left\lceil \frac{m}{4}\right\rceil, & \text{otherwise.} \end{cases}$$

Then $S = \bigcup_{u \in V(G)} S_u$ is a minimum pitchfork dominating set in $G \circ C_m$ and

$$|S| = \gamma_{pf}(G \circ C_m) = \begin{cases} n\left(2\left\lceil \frac{m}{4} \right\rceil - 1\right), & \text{if } m \equiv 1 \pmod 4, \\ n\left(2\left\lceil \frac{m}{4} \right\rceil\right), & \text{otherwise.} \end{cases}$$ □

Theorem 2.9. Let G be a connected graph with order $m \geq 3$ and a complete bipartite graph $K_{1,n}$ of order $n+1$ where $n \geq 3$. Then $\gamma_{pf}(G \circ K_{1,n}) = nm$.

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2, \ldots, v_n\}$ with partite sets $U = \{v\}$ and $W = \{v_1, v_2, \ldots, v_n\}$. For every $u \in V(G)$, denote by $K_{1,n}^u$ the copy of $K_{1,n}$ whose vertices are attached one by one to the vertex u. Denote by $u + K_{1,n}^u$ the subgraph of the corona $G \circ K_{1,n}$, corresponding to the join $\langle\{u\}\rangle + K_{1,n}^u$. Note that for every $u \in V(G)$, the copy $u + K_{1,n}^u$ is isomorphic to $P_2 + \overline{K}_n$ and by Lemma 2.3, $\gamma_{pf}(P_2 + \overline{K}_n) = n$.

Let S be a pitchfork dominating set in $G \circ K_{1,n}$ and let $S_u = S \cap V(K_{1,n}^u)$. Let $u \in V(G)$ and consider the following cases:

Case 1: $u \in S$.

Subcase 1.1. $N_G(u) \cap (V(G) \setminus S) = \emptyset$.
Then by Lemma 2.3 (*Subcase* 2.1), $|S_u| = n-1$, where S_u consists of $n-1$ vertices from W.

Subcase 1.2. $N_G(u) \cap (V(G) \setminus S) \neq \emptyset$.
Note that by definition of pitchfork dominating set,

$$|N_G(u) \cap (V(G) \setminus S)| = 1 \text{ or } |N_G(u) \cap (V(G) \setminus S)| = 2.$$

Suppose that $|N_G(u) \cap (V(G) \setminus S)| = 1$. Then $|S_u| = n$, where S_u is consists of n vertices from W only. If $|N_G(u) \cap (V(G) \setminus S)| = 2$, then it is not possible since $n \geq 3$ and by the definition of $u + K_{1,n}^u$ and the pitchfork dominating set, we must include all vertices of $K_{1,n}^u$ to S and S is not a pitchfork dominating set since for every $x \in V(K_{1,n}^u)$, x will not dominate at least 1 vertex or at most 2 vertices in $V(G \circ K_{1,n}) \setminus S$. This is a contradiction since S is a pitchfork dominating set of $G \circ K_{1,n}$. Thus, the case where $|N_G(u) \cap (V(G) \setminus S)| = 2$ is not possible.

Case 2: $u \notin S$.

Subcase 2.1. $N_G(u) \cap (V(G) \setminus S) = \emptyset$.

Then by Lemma 2.3 (*Case* 1 and *Subcase* 2.1), either $S_u = W$ or $S_u = (W \setminus \{v_i\}) \cup \{v\}$ where $i \in \{1, 2, \ldots, n\}$.

Subcase 2.2. $N_G(u) \cap (V(G) \setminus S) \neq \emptyset$.
Since $u \notin S$, by the definition of $G \circ K_{1,n}$ and pitchfork dominating set and by Lemma 2.3 (*Case* 1 and *Subcase* 2.1), either $S_u = W$ or $S_u = (W \setminus \{v_i\}) \cup \{v\}$ where $i \in \{1, 2, \ldots, n\}$.

Note that in determining S_u in *Case* 1 and *Case* 2, we consider the vertices as minimum as possible. Suppose that $|V(G) \cap S| = t$, where $t \leq m$. Then by *Case* 1 and *Case* 2, we have

$$\begin{aligned} \gamma_{pf}(G \circ K_{1,n}) &= \min\{t(n-1) + (m-t)(n) + t, t(n) + (m-t)(n) + t\} \\ &= \min\{tn - t + mn - tn + t, tn + mn - tn + t\} \\ &= \min\{mn, mn + t\} \\ &= mn. \end{aligned}$$ □

Theorem 2.10. Let G be a connected graph with order $m \geq 1$ and a complete bipartite graph $K_{p,q}$ of order $p+q$ where $p \geq 2, q \geq 2$. Then $\gamma_{pf}(G \circ K_{p,q}) = m[(p+q)-2]$.

Proof. Let $V(K_{p,q}) = \{v_1, v_2, \dots, v_p, u_1, u_2, \dots, u_q\}$ where the partite sets are $W = \{v_1, v_2, \dots, v_p\}$ and $U = \{u_1, u_2, \dots, u_q\}$. For every $x \in V(G)$, denote by $K^x_{p,q}$ the copy of $K_{p,q}$ where the vertices are attached one by one to the vertex x. Denote by $x + K^x_{p,q}$ the subgraph of corona $G \circ K_{p,q}$ corresponding to the join $\langle\{x\}\rangle + K^x_{p,q}$. Note that for every $x \in V(G)$, the copy $x + K^x_{p,q}$ is isomorphic to $K_1 + K_{p,q}$ and by Lemma 2.4, $\gamma_{pf}(K_1 + K_{p,q}) = (p+q)-2$.

Let S be a pitchfork dominating set in $G \circ K_{p,q}$ and let $S_x = S \cap K^x_{p,q}$, where $x \in V(G)$. Consider the following cases:

Case 1: $x \in S$

Subcase 1.1. $N_G(x) \cap (V(G) \setminus S) = \emptyset$.
Then by Lemma 2.4 (*Case* 2), $S_x = (W \setminus \{v_i\}) \cup (U \setminus \{u_j\})$ where $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,q\}$. Thus, $|S_x| = p+q-2$.

Subcase 1.2. $N_G(x) \cap (V(G) \setminus S) \neq \emptyset$.
Then by the definition of a pitchfork dominating set,

$$|N_G(x) \cap (V(G) \setminus S)| = 1 \text{ or } |N_G(x) \cap (V(G) \setminus S)| = 2.$$

Note that the case where $|N_G(x) \cap (V(G) \setminus S)| = 2$ is not possible since $x \in S$ and so $|S_x| = |V(K^x_{p,q})|$ implying that S is not a pitchfork dominating set since there exists $w \in V(K^x_{p,q}) \setminus S$ such that w is not adjacent to any vertex not in S.

Now suppose that $|N_G(x) \cap (V(G) \setminus S)| = 1$, where $x \in S$. Then by definition of pitchfork dominating set, $|S_x| = |V(K^x_{p,q})| - 1$. Without loss of generality, suppose that we include p vertices in W and $q-1$ vertices in U in the set S_x. Then by definition of $K_{p,q}$ and corona, there exists vertex $u \in U \cap S_x$, where u is not adjacent to any vertex not in S. This is a contradiction since S is a pitchfork dominating set.

Thus, from *Subcase* 1.1 and *Subcase* 1.2, if $x \in S$, it follows that $N_G(x) \cap (V(G) \setminus S) = \emptyset$.

Case 2: $x \notin S$

Subcase 2.1. $N_G(x) \cap (V(G) \setminus S) = \emptyset$.
Then by Lemma 2.4 (*Case* 1),

$$S_x = (W \setminus \{v_i\}) \cup (U \setminus \{u_j\}), \text{ where } i \in \{1,2,\dots,p\} \text{ and } j \in \{1,2,\dots,q\} \text{ or}$$

$$S_x = (W \setminus \{v_i\}) \cup U, \text{ where } i \in \{1,2,\dots,p\} \text{ or}$$

$$S_x = (U \setminus \{u_j\}) \cup W, \text{ where } j \in \{1,2,\dots,q\}.$$

Thus, $|S_x| = p+q-2$ or $|S_x| = p+q-1$.

Subcase 2.2. $N_G(x) \cap (V(G) \setminus S) \neq \emptyset$.
Then as in *Subcase* 2.1, $|S_x| = p+q-2$ or $|S_x| = p+q-1$.
Therefore, by *Case* 1 and *Case* 2 we have

$$\gamma_{pf}(G \circ K_{p,q}) = \min\{m(p+q-2), m(p+q-2), m(p+q-1)\} = m(p+q-2). \qquad \square$$

Theorem 2.11. Let G be corona of a connected graph of order $n \geq 2$ and a banana tree graph $B_{2,k}$ where $k \geq 5$. Then $\gamma_{pf}(G \circ B_{2,k}) = n(2k-2) = 2n(k-1)$.

Proof. For every $u \in V(G)$, denote by $B^u_{2,k}$, the copy of $B_{2,k}$ whose vertices are attached one by one to the vertex u. Denote by $u+B^u_{2,k}$, the subgraph of the corona $G \circ B^u_{2,k}$, corresponding to the join $\langle\{u\}\rangle + B^u_{2,k}$. Let S be a pitchfork dominating set $G \circ B^u_{2,k}$ and let $S_u = S \cap V(B_{2,k})$. Consider the following cases:

Case 1: $u \in S$
Suppose that there is a vertex $u \in V(G)$ such that $u \in S$. Then, by the definition of pitchfork domination and the corona $G \circ B^u_{2,k}$, then $|S_u| = 2k$ or $|S_u| = 2k+1$. But observe that there is at least one vertex in $B_{2,k}$ that does not dominate a vertex in $V(B^u_{2,k}) - S$. Hence, when $u \in S$, we cannot form a pitchfork dominating set.

Case 2: $u \notin S$
Assume that $V(u+B^u_{2,k}) = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k, w, u\}$. Without loss of generality, let v_k and u_k connect w and let $u \in V(G)$, where all of the vertices of $B^u_{2,k}$ are adjacent. Note that the following are the only pitchfork dominating sets in $u+B^u_{2,k}$.

Let $A_1 = \{u_2, u_3, \dots, u_k\}$, $A_2 = \{v_2, v_3, \dots, v_{k-1}\}$, $A_3 = \{w\}$ and $A = A_1 \cup A_2 \cup A_3$ with $A_1 \cap A_2 \cap A_3 = \emptyset$. Observe that every vertices in $A_1 \subseteq A$ dominates u_1 and $u \in V(u+B^u_{2,k}) - A$. Moreover, every vertices in $A_2 \subseteq A$ dominates v_1 and $u \in V(u+B^u_{2,k}) - A$, while $w \in A_3 \subseteq A$ dominates v_k and $u \in V(u+B^u_{2,k}) - A$. Thus, A is a pitchfork dominating set of $u+B^u_{2,k}$ with $|A| = k-1+k-2+1 = 2k-2 = 2(k-1)$.

Let $D_1 = \{u_2, u_3, \dots, u_{k-1}\}$, $D_2 = \{v_2, v_3, \dots, v_k\}$, $D_3 = \{w\}$ and $D = D_1 \cup D_2 \cup D_3$ with $D_1 \cap D_2 \cap D_3 = \emptyset$. Observe that every vertices in $D_1 \subseteq D$ dominates u_1 and $u \in V(u+B^u_{2,k}) - D$. Moreover, every vertices in $D_2 \subseteq D$ dominates v_1 and $u \in V(u+B^u_{2,k}) - D$, while $w \in D_3 \subseteq D$ dominates u_k and $u \in V(u+B^u_{2,k}) - D$. Thus, D is a pitchfork dominating set of $u+B^u_{2,k}$ with $|A| = k-1+k-2+1 = 2k-2 = 2(k-1)$.

Let $B_1 = \{u_1, u_2, u_3, \dots, u_k\}$, $B_2 = \{v_1, v_2, v_3, \dots, v_k\}$ and $B = B_1 \cup B_2$ with $B_1 \cap B_2 = \emptyset$. Observe that every vertex in $B_1 \setminus \{u_k\} \subset B$ dominates u while $u_k \in B_1$ dominates $w, u \in V(u+B^u_{2,k}) - B$. Further observe that every vertex in $B_2 \setminus \{v_k\} \subset B$ dominates u while $v_k \in B_2$ dominates $w, u \in V(u+B^u_{2,k}) - B$. Hence, B is a pitchfork dominating set of $u+B^u_{2,k}$ with $|B| = k+k = 2k$.

Note that if $u \notin S$, the sets A, D and B are the only pitchfork dominating sets of $u+B^u_{2,k}$. Moreover, sets A and D are the only pitchfork dominating sets of $u+B^u_{2,k}$ of cardinality $2k-2$. Thus, among all the pitchfork dominating sets of $u+B^u_{2,k}$ above, the minimum pitchfork dominating sets of $u+B^u_{2,k}$ are A and D.

Henceforth, by *Case* 1 and *Case* 2, to have a minimum pitchfork dominating set in $G \circ B_{2,k}$, we should not include any vertex in G in the set. Therefore, if S is a minimum pitchfork dominating set of $G \circ B_{2,k}$, $S \subseteq \bigcup_{u \in V(G)} V(B^u_{2,k})$. For each $u \in V(G)$, let $S_u \cong A$. Then $S = \bigcup_{u \in V(G)} S_u$ is a minimum pitchfork dominating set in $G \circ B_{2,k}$ and $|S| = \gamma_{pf}(G \circ B_{2,k}) = 2n(k-1)$. □

Theorem 2.12. Let G be a connected graph with order $n \geq 1$ and a firecracker graph $F_{2,k}$ of order $2k$. Then $\gamma_{pf}(G \circ F_{2,k}) = 2n(k-1)$.

Proof. Let $V(F_{2,k}) = \{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k\}$. For every $w \in V(G)$, denote by $F^w_{2,k}$ the copy of $F_{2,k}$ where the vertices are attached one by one to the vertex w. Denote by $w+F^w_{2,k}$ the subgraph of corona $G \circ F_{2,k}$ corresponding to the join $\langle\{w\}\rangle + F^w_{2,k}$. Note that for every $w \in V(G)$, the copy $w+F^w_{2,k}$ is isomorphic to $K_1 + F_{2,k}$ and by Lemma 2.7, $\gamma_{pf}(K_1 + F_{2,k}) = 2(k-1)$.

Let S be a pitchfork dominating set in $G \circ F_{2,k}$ and let $S_w = S \cap F^w_{2,k}$, where $w \in V(G)$. Consider the following cases:

Case 1: $w \in S$

Subcase 1.1. $N_G(w) \cap (V(G) \setminus S) = \emptyset$.
Then by Lemma 2.7 (*Case* 1), $D = S_w = \{w, u_2, u_3, \dots, u_k, v_2, v_3, \dots, v_k\}$. Thus, $|S_w| = 2k-1$.

Subcase 1.2. $N_G(w) \cap (V(G) \setminus S) \neq \emptyset$.
Then by the definition of a pitchfork dominating set,

$$|N_G(w) \cap (V(G) \setminus S)| = 1 \text{ or } |N_G(w) \cap (V(G) \setminus S)| = 2.$$

Note that the case where $|N_G(w) \cap (V(G) \setminus S)| = 2$ is not possible since $w \in S$ and so $|S_w| = |V(F^w_{2,k})|$ implying that S is not a pitchfork dominating set since there exists $x \in V(K^w_{p,q}) \setminus S$ such that x is not adjacent to any vertex not in S.

Now suppose that $|N_G(w) \cap (V(G) \setminus S)| = 1$, where $w \in S$. Then by definition of pitchfork dominating set, $|S_w| = |V(F^w_{2,k})| - 1$. Without loss of generality, suppose that we include $2k-1$ vertices of $F_{2,k}$ in the set S_w. Then by definition of $F_{2,k}$ and corona, there exists vertex $u_i \in S_w$ where $i \in \{1, 2, \dots, k\}$ and u_i is not adjacent to any vertex not in S. This is a contradiction since S is a pitchfork dominating set.

Thus, from *Subcase* 1.1 and *Subcase* 1.2, if $w \in S$, it follows that $N_G(w) \cap (V(G) \setminus S) = \emptyset$.

Case 2: $w \notin S$

Subcase 2.1. $N_G(w) \cap (V(G) \setminus S) = \emptyset$.
Then by Lemma 2.7 (*Case* 2), S_w is one of the following sets:

$S_w = E = V(F_{2,k}) \setminus \{u_1, v_1\}$ with $|S_w| = 2k-2$ or

$S_w = F = V(F_{2,k}) \setminus \{u_1\}$ with $|S_w| = 2k-1$ or

$S_w = H = V(F_{2,k}) \setminus \{v_1\}$ with $|S_w| = 2k-1$ or

$S_w = I = V(F_{2,k})$ with $|S_w| = 2k$.

Thus, $|S_w| = 2k-2$ or $|S_w| = 2k-1$ or $|S_w| = 2k$.

Subcase 2.2. $N_G(w) \cap (V(G) \setminus S) \neq \emptyset$.
Then as in *Subcase* 2.1, $|S_w| = 2k-2$ or $|S_w| = 2k-1$ or $|S_w| = 2k$.
Therefore, by *Case* 1 and *Case* 2 we have

$$\gamma_{pf}(G \circ F_{2,k}) = \min\{n(2k-1), n(2k-2), n(2k)\} = n(2k-2) = 2n(k-1). \qquad \square$$

3. Conclusion

In this study, the researchers determined the pitchfork domination numbers for various types of graphs, including banana trees, firecracker graphs, join graphs such as $P_2 + \overline{K}_n$, $K_1 + K_{p,q}$, $K_1 + F_{2,k}$ and corona graphs $G \circ C_m$, $G \circ K_{1,n}$, $G \circ K_{p,q}$, $G \circ B_{2,k}$ and $G \circ F_{2,k}$. These results demonstrate how pitchfork domination applies across different graph families and provide a foundation for future exploration.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. A. Abdlhusein and M. N. Al-Harere, Pitchfork domination and its inverse for corona and join operations in graphs, *Proceedings of International Mathematical Sciences* **1**(2) (2019), 51 – 55.

[2] M. N. Al-Harere and M. A. Abdlhusein, Pitchfork domination in graphs, *Discrete Mathematics, Algorithms and Applications* **12**(02) (2020), 2050025, DOI: 10.1142/S1793830920500251.

[3] A. Aslam, J. L. G. Guirao, S. Ahmad and W. Gao, Topological indices of the line graph of subdivision graph of complete bipartite graphs, *Applied Mathematics & Information Sciences* **11** (2017), 1631 – 1636, DOI: 10.18576/amis/110610.

[4] Asmiati, Locating chromatic number of banana tree, *International Mathematical Forum* **12**(1) (2017), 39 – 45, DOI: 10.12988/imf.2017.610138.

[5] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, 1st edition, Chapman and Hall/CRC, New York, 504 pages (2008), DOI: 10.1201/9781584888017.

[6] A. Gayathri, A. Muneera, T. N. Rao and T. S. Rao, Study of various dominations in graph theory and its applications, *International Journal of Scientific & Technology Research* **9**(2) (2020), 3426 – 3429.

[7] A. Haryanti, D. Indriati and T. S. Martini, On the total vertex irregularity strength of firecracker graph, *Journal of Physics: Conference Series* **1211** (2019), 012011, DOI: 10.1088/1742-6596/1211/1/012011.

[8] T. W. Haynes, S. T. Hedetneime and P. J. Slater, *Domination in Graphs*, Volume 2: Advanced Topics, 1st Edition, Routledge, 520 pages (1998), DOI: 10.1201/9781315141428.

[9] R. Venkateswari, Applications of domination graphs in real life, *Journal of Composition Theory* **12** (2019), 22 – 28.