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Research Article

Some New Results on Antimagic Labeling

C. M. Barasara* and P. J. Prajapati

Department of Mathematics, Hemchandracharya North Gujarat University, Patan 384265, Gujarat, India
***Corresponding author:** chirag.barasara@gmail.com

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Abstract. A graph with q edges is called *antimagic* if its edges can be labeled with $1,2,3,\dots,q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct. In this paper, we study antimagic labeling of one point union of cycle, book graph, path union of m copies of cycles, m-splitting of path and m-splitting of cycle.

Keywords. Antimagic labeling, Antimagic graph, Book graph, Graph operations, Splitting graph

Mathematics Subject Classification (2020). 05C78, 05C76

1. Introduction

Graph labeling has experienced rapid development during last three decades. Nearly two hundred graph labeling techniques have been studied in over two thousand research papers. For more information on graph labeling, refer dynamic survey on graph labeling by Gallian [9].

All the graphs considered here are simple, finite, connected and undirected. For all standard terminologies and notations, we follow the book by Clark and Holton [7].

Motivated by the notion of magic square, Sedláček [14] published a paper about magic labeling. A *magic labeling* is a function from the set of edges of a graph G into the non-negative real numbers, so that the sum of the edges labels around any vertex in G are all the same.

A variant of a magic labeling where all the vertex labels are distinct is called antimagic labeling and it is defined by Hartsfield and Ringel [10].

Definition 1.1. A graph with q edges is called *antimagic* if its edges can be labeled with $1,2,3,\dots,q$ without repetition such that the sums of the labels of the edges incident to each vertex are distinct.

They have also proved that paths, cycles, wheels and complete graphs admits antimagic labeling. For more interesting results on antimagic labeling and effect of graph operation on antimagic labeling (see Alon *et al*. [2], Bača *et al*. [3], Barasara and Prajapati [4,5], Cheng [6], Joseph and Kureethara [12], Sridharan and Umarani [16], Vaidya and Vyas [17,18], Wang and Hsiao [19], Wang *et al*. [20], and Zhang and Sun[21]). An excellent survey on antimagic labeling was prepared by Jin and Tu [11].

The main goal of this paper is to study the effect of graph operation on antimagicness of graph. We hope that the results reported here may have some interesting applications.

Before moving to main results, we required the following results obtained by Hartsfield and Ringel [10], Sridharan and Umarani [16], and Vaidya and Vyas [18].

Proposition 1.1 ([10])**.** *The cycle C_n is antimagic.*

Proposition 1.2 ([16])**.** *Lantern graph $G = K_2 + \overline{K_n}$ $(n \geqslant 2)$ is antimagic.*

Proposition 1.3 ([16])**.** *Friendship graph $C_3^{(t)}$ is antimagic, for all $t \geqslant 2$.*

Proposition 1.4 ([18])**.** *Splitting graph of path P_n is antimagic.*

Proposition 1.5 ([18])**.** *Splitting graph of cycle C_n is antimagic.*

2. Main Results

Definition 2.1. *One point union of cycle $C_n^{(t)}$* is a graph consists of t copies of cycle C_n sharing a common vertex.

Theorem 2.1. *The graph $C_n^{(t)}$ is an antimagic graph.*

Proof. Let $v, v_1^j, v_2^j, \dots, v_{n-1}^j$ be the vertices of jth copies of cycle C_n and v be the common vertex of $C_n^{(t)}$. Then $|V(C_n^{(t)})| = t(n-1)+1$ and $|E(C_n^{(t)})| = nt$.

We define $f : E(C_n^{(t)}) \to \{1,2,\dots,nt\}$, as per following three cases:

Case 1: For $n \geqslant 3$ and $t = 1$.
The graph $C_n^{(1)}$ is cycle C_n. By Proposition 1.1, $C_n^{(1)}$ is an antimagic graph.

Case 2: For $n = 3$ and $t \geqslant 2$.
The graph $C_3^{(t)}$ is also known as friendship graph. By Proposition 1.3, $C_3^{(t)}$ is an antimagic graph.

Case 3: For $n \geqslant 4$ and $t \geqslant 2$.

$$f(vv_1^j) = nj - (n-1); \quad \text{for } 1 \leqslant j \leqslant t,$$
$$f(vv_{n-1}^j) = nj; \qquad \text{for } 1 \leqslant j \leqslant t,$$

$$f(v_i^j v_{i+1}^j) = (n-1)(j-1) + i + j; \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-2, \\ 1 \leqslant j \leqslant t. \end{cases}$$

Due to this edge labeling, all generated vertex labels are different. Hence, the theorem is proved. □

Illustration 2.1. *The graph $C_6^{(5)}$ and its antimagic labeling is shown in Figure* 1.

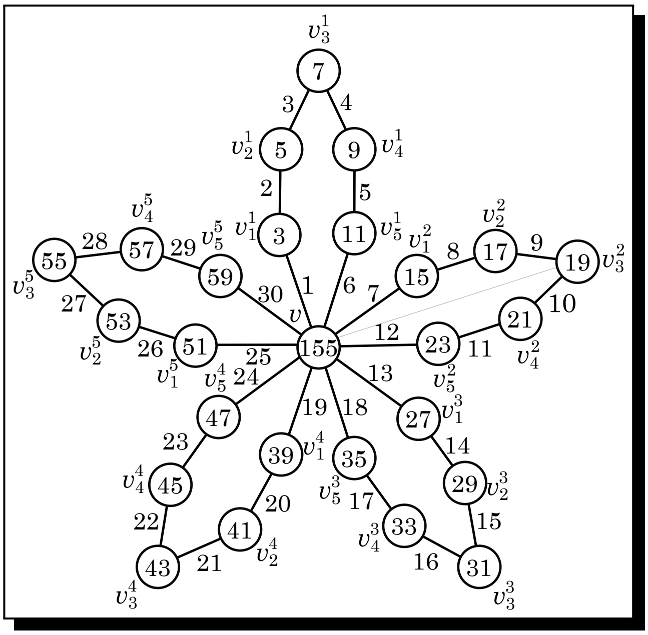

Figure 1. $C_6^{(5)}$ and its antimagic labeling

Definition 2.2. An m page *book* with n-polygonal pages is the graph made from m copies of C_n that share an edge. It is denoted by $\Theta(C_n)^m$.

Theorem 2.2. *The graph $\Theta(C_n)^m$ is an antimagic graph.*

Proof. Let $u, v, v_1^j, v_2^j, \dots, v_{n-2}^j$ be the vertices of jth copies of cycle C_n and u and v be the end vertices of shared edge of book graph $\Theta(C_n)^m$. Then $|V(\Theta(C_n)^m)| = (n-2)m + 2$ and $|E(\Theta(C_n)^m)| = (n-1)m + 1$.

We define $f : E(\Theta(C_n)^m) \to \{1, 2, \dots, (n-1)m + 1\}$, as per following four cases:

Case 1: For $n \geqslant 3$ and $m = 1$.
The graph $\Theta(C_n)^1$ is cycle C_n. By Proposition 1.1, $\Theta(C_n)^1$ is an antimagic graph.

Case 2: For $n = 3$ and $m \geqslant 2$.
The graph $\Theta(C_3)^m$ is also known as lantern graph $K_2 + \overline{K_n}$. By Proposition 1.2, $\Theta(C_3)^m$ is an antimagic graph.

Case 3: For odd $n \geqslant 5$ and $m = 2$.

$$f(uv) = mn - (m-1),$$

$$f(vv_1^j) = (n-1)j - 1; \qquad \text{for } 1 \leqslant j \leqslant m,$$
$$f(uv_{n-2}^j) = (n-1)j; \qquad \text{for } 1 \leqslant j \leqslant m,$$
$$f(v_i^j v_{i+1}^j) = (n-1)j - 2i - 1; \qquad \text{for } \begin{cases} 1 \leqslant i \leqslant \frac{n-3}{2}, \\ 1 \leqslant j \leqslant m, \end{cases}$$
$$f\left(v^j_{\frac{n-3}{2}+i} v^j_{\frac{n-1}{2}+i}\right) = (n-1)(j-1) + 2i; \quad \text{for } \begin{cases} 1 \leqslant i \leqslant \frac{n-3}{2}, \\ 1 \leqslant j \leqslant m. \end{cases}$$

Case 4: For all remaining n and m.

$$f(uv) = mn - (m-1),$$
$$f(vv_1^j) = (n-1)(j-1) + 1; \qquad \text{for } 1 \leqslant j \leqslant m,$$
$$f(uv_{n-2}^j) = (n-1)j; \qquad \text{for } 1 \leqslant j \leqslant m,$$
$$f(v_i^j v_{i+1}^j) = (n-1)(j-1) + i + 1; \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-3, \\ 1 \leqslant j \leqslant m. \end{cases}$$

Due to this edge labeling, all generated vertex labels are different.
Hence, the theorem is proved. □

Illustration 2.2. *The graph* $\Theta(C_7)^3$ *and its antimagic labeling is shown in Figure* 2.

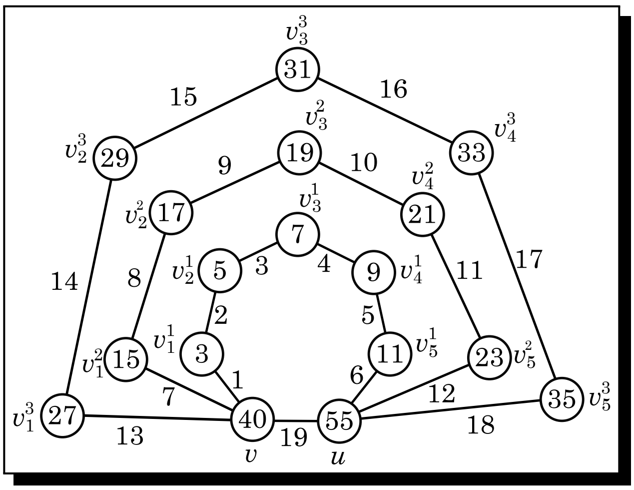

Figure 2. $\Theta(C_7)^3$ and its antimagic labeling

Definition 2.3. Let $G_1, G_2, G_3, \dots, G_m$ $(m \geqslant 2)$ be m-copies of graph G. The *path union of* G is the graph obtained by adding an edge between corresponding vertices of G_j and G_{j+1}, $1 \leqslant j \leqslant n-1$. It is denoted by $P(mC_n)$.

Theorem 2.3. *The graph* $P(mC_n)$ *is an antimagic graph.*

Proof. Let $v_1^j, v_2^j, \dots, v_n^j$ be the vertices of jth copies of cycle C_n. To construct path union of m copies of cycle C_n, add an edge between vertices v_1^j and v_1^{j+1} for $1 \leqslant j \leqslant m-1$. Then $|V(P(mC_n))| = mn$ and $|E(P(mC_n))| = mn + (m-1)$.

We define $f : E(P(mC_n)) \to \{1, 2, \ldots, mn + (m-1)\}$, as per following two cases.

Case 1: For $m \equiv 1, 2, 3 \pmod 4$.

$$f(v_i^j v_{i+1}^j) = (n+1)(j-1) + i; \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_n^j v_1^j) = n + (n+1)(j-1); \quad \text{for } 1 \leqslant j \leqslant m,$$

$$f(v_1^j v_1^{j+1}) = j(n+1); \quad \text{for } 1 \leqslant j \leqslant m-1.$$

Case 2: For $m \equiv 0 \pmod 4$.
Label the edges $v_1^1 v_2^1, v_2^1 v_3^1, \ldots, v_n^1 v_1^1$ by $2, 4, 6, \ldots, 2n$, edges $v_1^2 v_2^2, v_2^2 v_3^2, \ldots, v_n^2 v_1^2$ by $2n+2, 2n+4, 2n+6, \ldots, 4n$, continue in this way till we reach up to label $mn + (m-2)$. If the label $mn + (m-2)$ is given to edge $v_i^j v_{i+1}^j$ then label the edges $v_{i+1}^j v_{i+2}^j, v_{i+2}^j v_{i+3}^j, \ldots, v_n^j v_1^j, v_1^{j+1} v_2^{j+1}, v_2^{j+1} v_3^{j+1}, \ldots, v_n^{j+1} v_1^{j+1}, v_1^{j+2} v_2^{j+2}, v_2^{j+2} v_3^{j+2}, \ldots, v_n^{j+2} v_1^{j+2}, \ldots, v_1^m v_2^m, v_2^m v_3^m, \ldots, v_n^m v_1^m$ by $1, 3, 5, \ldots, mn - m + 1$. Now label the edges $v_1^1 v_1^2, v_1^2 v_1^3, \ldots, v_1^{m-1} v_1^m$ by $mn - m + 3, mn - m + 5, mn - m + 7, \ldots, mn + (m-1)$.
Due to this edge labeling, all generated vertex labels are different.
Hence, the theorem is proved. □

Illustration 2.3. *The graph $P(4C_5)$ and its antimagic labeling is shown in Figure* 3.

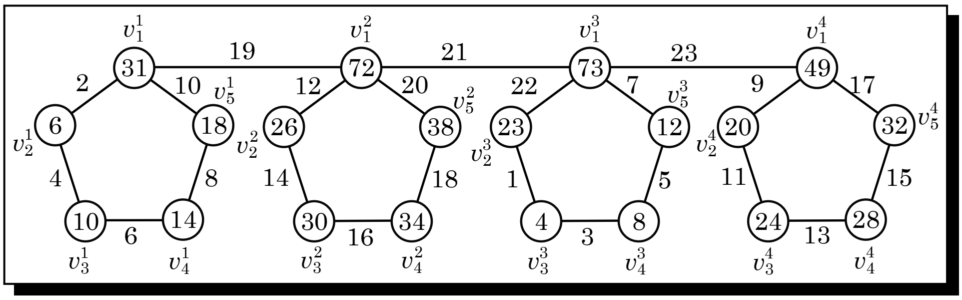

Figure 3. $P(4C_5)$ and its antimagic labeling

Definition 2.4. The *splitting graph* $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to each vertex that is adjacent to v in G.

Definition 2.5. The *m-splitting graph* $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \ldots, v_m$ such that v_i, $1 \leqslant i \leqslant m$ is adjacent to every vertex that is adjacent to v in G.

Theorem 2.4. *The graph $Spl_m(P_n)$ is an antimagic graph.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of path P_n. To construct $Spl_m(P_n)$, add the vertices $v_i^1, v_i^2, v_i^3, \ldots, v_i^m$ corresponding to vertex v_i of path P_n and join the vertex v_i^j to all the neighbours of v_i for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then $|V(Spl_m(P_n))| = n(m+1)$ and $|E(Spl_m(P_n))| = 3(n-1) + 2(n-1)(m-1)$.

We define $f : E(Spl_m(P_n)) \to \{1, 2, \ldots, 3(n-1) + 2(n-1)(m-1)\}$, as per following three cases:

Case 1: For $n \geqslant 2$ and $m = 1$.
The graph $Spl_1(P_n)$ is $S'(P_n)$. By Proposition 1.4, $Spl_1(P_n)$ is an antimagic graph.

Case 2: For ($n \equiv 2 \pmod 4$ and $m \geqslant 2$) or ($n \equiv 0 \pmod 4$ and $m > 2$).

$$f(v_i v_{i+1}^j) = 2i + 2n(j-1) - 2(j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i^j v_{i+1}) = 2i + 2n(j-1) - (2j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i v_{i+1}) = 3(n-1) + 2(n-1)(m-1) + 1 - i; \quad \text{for } 1 \leqslant i \leqslant n-1.$$

Case 3: For ($n \equiv 1, 3 \pmod 4$ and $m \geqslant 2$) or ($n \equiv 0 \pmod 4$ and $m = 2$).

$$f(v_i v_{i+1}^j) = 2i + 2n(j-1) - (2j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i^j v_{i+1}) = 2i + 2n(j-1) - 2(j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i v_{i+1}) = 3(n-1) + 2(n-1)(m-1) + 1 - i; \quad \text{for } 1 \leqslant i \leqslant n-1.$$

Due to this edge labeling, all generated vertex labels are different. Hence, the theorem is proved. □

Illustration 2.4. *The graph $Spl_2(P_7)$ and its antimagic labeling is shown in Figure* 4.

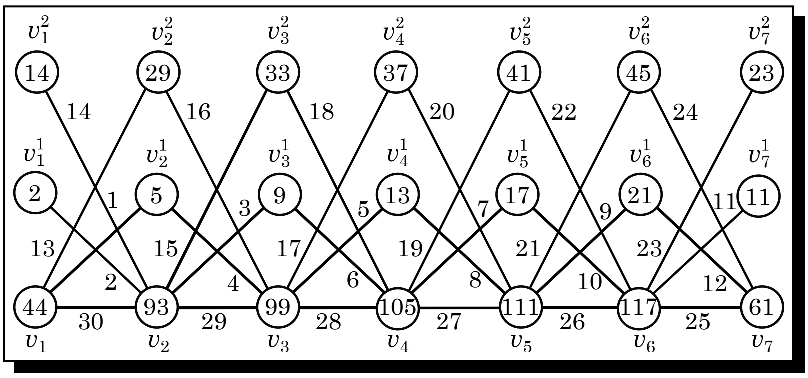

Figure 4. $Spl_2(P_7)$ and its antimagic labeling

Theorem 2.5. *The graph $Spl_m(C_n)$ is an antimagic graph.*

Proof. Let $v_1, v_2, v_3, \dots, v_n$ be the vertices of cycle C_n. To construct $Spl_m(C_n)$, add the vertices $v_i^1, v_i^2, v_i^3, \dots, v_i^m$ corresponding to vertex v_i of cycle C_n and join the vertex v_i^j to all the neighbours of v_i, for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Then $|V(Spl_m(C_n))| = n(m+1)$ and $|E(Spl_m(C_n))| = 3n + 2n(m-1)$.

We define $f : E(Spl_m(C_n)) \to \{1, 2, \dots, 3n + 2n(m-1)\}$, as per following two cases:

Case 1: For $n \geqslant 3$ and $m = 1$.
The graph $Spl_1(C_n)$ is $S'(C_n)$. Thus, by Proposition 1.5, $Spl_1(C_n)$ is an antimagic graph.

Case 2: For $n \geqslant 3$ and $m \geqslant 2$.

$$f(v_i v_{i+1}^j) = 2i + 2n(j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i^j v_{i+1}) = 2i - 1 + 2n(j-1); \quad \text{for } \begin{cases} 1 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant m, \end{cases}$$

$$f(v_i v_{i+1}) = 3n + 2n(m-1) + 1 - i; \quad \text{for } 1 \leqslant i \leqslant n-1,$$

$$f(v_n v_1) = 2nm + 1,$$

$$f(v_n^j v_1) = 2nj; \quad \text{for } 1 \leqslant j \leqslant m,$$

$$f(v_1^j v_n) = 2nj - 1; \quad \text{for } 1 \leqslant j \leqslant m.$$

Due to this edge labeling, all generated vertex labels are different.
Hence, the theorem is proved. □

Illustration 2.5. *The graph $Spl_2(C_6)$ and its antimagic labeling is shown in Figure* 5.

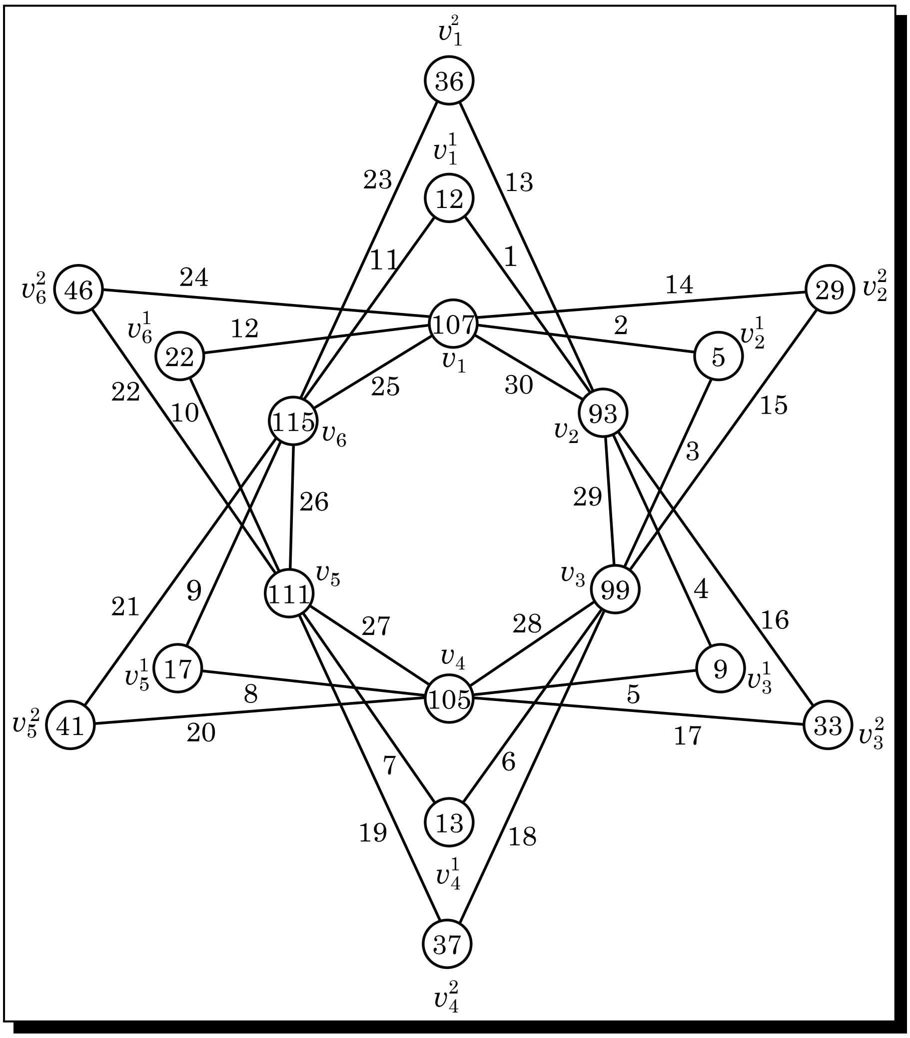

Figure 5. $Spl_2(C_6)$ and its antimagic labeling

3. Applications of Antimagic Labeling

Afzal *et al*. [1] have shown that antimagic labeling is useful for saving data from hackers attacks, channel assignment problem as well as routing problem. Encryption and decryption algorithm using antimagic labeling were developed by Krishnaa [13], Femina and Xavier [8], and Selvakumar and Gupta [15].

4. Concluding Remark

Hartsfield and Ringel [10] conjectured that every graph other than K_2 is antimagic. The conjecture is still open even in the case of trees. In this paper, we have verified the conjecture for one point union of cycle, book graph, path union of m copies of cycles, m splitting of path and m splitting of cycle.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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