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Research Article

Path Fuzzy Bitopological Spaces on Fuzzy Directed Graphs

P. S. Gholap[1], V. E. Nikumbh*[2], P. G. Andhare[3] and S. B. Gaikwad[4]

[1] *Department of Mathematics, Arts, Science and Commerce College (affiliated to Savitribai Phule Pune University), Rahuri 414705, Maharashtra, India*
[2] *Department of Mathematics, P. V. P. College of Arts, Science and Commerce (affiliated to Savitribai Phule Pune University), Pravaranagar, Ahmednagar 413713, Maharashtra, India*
[3] *Department of Mathematics, R. B. N. B. College (affiliated to Savitribai Phule Pune University), Shrirampur, Ahmednagar 413709, Maharashtra, India*
[4] *Department of Mathematics, New Arts, Commerce and Science College (affiliated to Savitribai Phule Pune University), Ahmednagar 414001, Maharashtra, India*
***Corresponding author:** vinayaknikumbh@yahoo.co.in

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Abstract. In this paper, we introduce a new concept of fuzzy topological spaces on fuzzy directed graphs by utilizing path relations between vertices and discuss several key results related to fuzzy topological spaces on various fuzzy digraphs. We define strongly connected, symmetric, and asymmetric fuzzy directed graphs, demonstrating that the associated path topological spaces are indiscrete. Furthermore, by employing path fuzzy topological spaces, we define a path fuzzy bitopological space associated with a fuzzy digraph. Various separation axioms, such as fuzzy pairwise T_0, T_1, T_2 and fuzzy pairwise weakly T_2, are studied for path fuzzy bitopological spaces, and the interrelations between path topological spaces and path fuzzy bitopological spaces are examined. This paper also initiates the concept of fuzzy bitopological spaces for fuzzy directed graphs.

Keywords. Path fuzzy topological space, Path fuzzy bitopological space, Fuzzy directed graph

Mathematics Subject Classification (2020). 05C72, 54A40

1. Introduction

The concept of bitopological spaces was first introduced by Kelly in 1963 [8]. Later, in 1965, Zadeh [14] described the notion of fuzzy sets, and in 1968, Chang [4] extended this idea by defining fuzzy topological spaces on fuzzy sets. In 1975, Rosenfeld [12] introduced fuzzy graphs to capture the fuzziness inherent in crisp graphs. Kandil *et al.* [7] were among the first to explore fuzzy bitopological spaces, studying various separation axioms within this framework.

Girija and Pilakkat [6] discussed bitopological spaces in the context of crisp directed graphs. Abdu and Kilicman [1] extended the idea by associating bitopological spaces with finite undirected graphs. More recently, Gholap and Nikumbh [5] defined various fuzzy topological spaces on fuzzy undirected graphs and investigated related results.

In this paper, we define two types of fuzzy topological spaces up path fuzzy topological space and down path fuzzy topological space on fuzzy directed graphs by utilizing path relations between vertices. We introduce the concepts of strongly connected, symmetric and asymmetric fuzzy directed graphs and we explore the relationships between up path and down path fuzzy topological spaces within these graphs. Additionally, we establish important results related to path fuzzy topological spaces for path, cycle and complete fuzzy directed graphs.

Furthermore, we define path fuzzy bitopological spaces by combining up and down path fuzzy topological spaces and we study the corresponding separation axioms. Our work aims to establish a connection between fuzzy directed graphs and fuzzy bitopological spaces. Since both concepts are rooted in fuzzy set theory, they provide valuable tools for modeling and analyzing systems with inherent uncertainty. Advancing the study of these areas is essential for both theoretical mathematics and the development of practical tools for real world applications.

2. Preliminaries

In this section, we mentioned some terminology related to fuzzy directed graph, fuzzy topology and fuzzy bitopological spaces.

Definition 2.1 ([9]). A fuzzy directed graph is a order triplet $D=(V,\sigma,\mu)$, where V be a finite nonempty vertex set, $\sigma:V\to[0,1]$ is a membership function of vertex set V and $\mu:V\times V\to[0,1]$ is a membership function of directed edge set E of D such that $\mu(u,v)\le\sigma(u)\wedge\sigma(v)$, for all $u,v\in V$. A fuzzy directed graph is complete if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$, for all $u,v\in V$. In degree of a vertex $u\in V(D)$ is the sum of the μ values of the edges incident to u, denoted by $d^-(u)$. Similarly, out degree of a vertex $u\in V(D)$ is sum of the μ values of the edges incident from the u to the all other vertices, denoted by $d^+(u)$. Complement of fuzzy digraph $D=(V,\sigma,\mu)$ is defined as $D^c=(V,\sigma,\mu^c)$, where $\mu^c(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v)$, for all $u,v\in V$. A sequence of vertices $u_1,u_2,\dots,u_r$ such that $\mu(u_{j-1},u_j)>0$, $j=1,2,\dots,r$ is a path P in a fuzzy digraph graph D. A path P is a circuit if $u_1=u_r$ and a cycle if $u_1=u_r$ with $r\ge3$. A fuzzy digraph is connected if $\mu(u,v)>0$ or $\mu(v,u)>0$, for all $u,v\in V$.

The following definitions are taken from [11].

Definition 2.2. Let $I=[0,1]$, X be any non empty set and I^X be a set of all fuzzy sets in X. A fuzzy topology on X is a family τ of members of I^X satisfying following conditions:

(i) $0,1 \in \tau$.

(ii) For any finite members $\{A_i\}_{i=1}^n$ of τ, $\bigcap_{i=1}^{n} A_i \in \tau$.

(iii) For any arbitrary family I of members of τ, $\bigcup_{A \in I} \{A\} \in \tau$.

Definition 2.3. A fuzzy topological space (X,τ) is said to be a fuzzy T_0 if for any pair of fuzzy points $p,q \in I^X$, $p \neq q$, $\exists\, u,v \in \tau$ such that $p \in u \subseteq co(q)$ or $q \in v \subseteq co(p)$.

Definition 2.4. A fuzzy topological space (X,τ) is said to be a fuzzy T_1 if for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open set $u,v \in \tau$ such that $p \in u$, $p \notin v$, $q \in v$, $q \notin u$.

Definition 2.5. A fuzzy topological space (X,τ) is said to be a fuzzy T_2 if for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open set $u,v \in \tau$ such that $p \in u$, $q \in v$ and $u \cap v = 0$.

Definition 2.6 ([3])**.** A fuzzy bitopological space (X,s,t) is said to be a fuzzy pairwise T_0 iff for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open set $u \in s \cup t$ such that $p \in u$ and $q \cap u = 0$ or $q \in u$ and $p \cap u = 0$.

Definition 2.7 ([10])**.** A fuzzy bitopological space (X,s,t) is said to be a fuzzy pairwise T_1 iff for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open sets $u,v \in s \cup t$ such that $p \in u$, $q \cap u = 0$ or $q \in v$, $p \cap v = 0$.

Definition 2.8 ([2])**.** A fuzzy bitopological space (X,s,t) is said to be a fuzzy pairwise T_2 iff for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open set $u \in s$ and open set $v \in t$ such that $p \in u$, $q \in v$ and $u \cap v = 0$.

Definition 2.9 ([13])**.** A fuzzy bitopological space (X,s,t) is said to be a fuzzy weakly pairwise T_2 iff for any two distinct fuzzy points $p,q \in I^X$ there exist fuzzy open set $u \in s$ and open set $v \in t$ such that $p \in u$, $q \in v$ or $q \in u$, $p \in v$ and $u \cap v = 0$.

3. Path Fuzzy Topological Spaces

In this section, we define a two type of fuzzy topological spaces on vertex set of fuzzy directed graph by using path relation.

Definition 3.1. Let $\mathcal{D} = (V,\alpha,\beta)$ be a fuzzy directed graph. Then for each $(x,\alpha) \in V$ we define $\mathcal{R}[(x,\alpha)]\uparrow$ as

$$\mathcal{R}[(x,\alpha)]\uparrow = \{(y,\alpha) \in V : \text{if there is path from } y \text{ to } x\}.$$

Let $\mathcal{S}\uparrow = \{\mathcal{R}[(x,\alpha)]\uparrow : \text{for each } (x,\alpha) \in V\}$ and Let $\mathcal{B}\uparrow$ be the fuzzy set of all finite intersection of members of $\mathcal{S}\uparrow$. Then the fuzzy set $\mathcal{B}$ generates a fuzzy topology on $\mathcal{D}$. We called it as a up path fuzzy topological space associated to a fuzzy directed graph $\mathcal{D}$. We denote it by $(V,\mathcal{T}\uparrow)$. A subset of $\mathcal{T}\uparrow$ is called as fuzzy up path open sets.

Similarly, for each $(x,\alpha) \in V$ we define $\mathcal{R}[(x,\alpha)]\downarrow$ as

$$\mathcal{R}[(x,\alpha)]\downarrow = \{(y,\alpha) \in V : \text{if there is path from } x \text{ to } y\}.$$

Let $\mathcal{S}\downarrow = \{\mathcal{R}[(x,\alpha)]\downarrow$: for each $(x,\alpha)\in V\}$ and Let $\mathcal{B}\downarrow$ be the fuzzy set of all finite intersection of members of $\mathcal{S}\downarrow$. Then, the fuzzy set $\mathcal{B}\downarrow$ generates a fuzzy topology on $\mathcal{D}$. We called it as a down path fuzzy topological space associated to a fuzzy directed graph $\mathcal{D}$. We denote it by $(V,\mathcal{T}\downarrow)$. A subset of $\mathcal{T}\downarrow$ is called as fuzzy down path open sets.

Example 3.1. Let $\mathcal{D}$ be a fuzzy directed graph with vertex set,

$$V=\{(x,0.6),(y,0.7),(z,0.3),(u,0.2),(v,0.8)\}.$$

From Figure 1, we have

$$\mathcal{R}[(x,\alpha)]\uparrow = \{(x,0.6),(z,0.3)\},$$
$$\mathcal{R}[(y,\alpha)]\uparrow = \{(y,0.7),(x,0.6),(u,0.2),(v,0.8),(z,0.3)\}=V,$$
$$\mathcal{R}[(z,\alpha)]\uparrow = \{(z,0.3)\},$$
$$\mathcal{R}[(u,\alpha)]\uparrow = \{(u,0.2),(v,0.8),(z,0.3)\},$$
$$\mathcal{R}[(v,\alpha)]\uparrow = \{(v,0.8)\},$$
$$\mathcal{S}\uparrow = \{\{(z,0.3)\},\{(v,0.8)\},\{(x,0.6),(z,0.3)\},\{(u,0.2),(v,0.8),(z,0.3)\},V\},$$
$$\mathcal{B}\uparrow = \{0,\{(z,0.3)\},\{(v,0.8)\},\{(x,0.6),(z,0.3)\},\{(u,0.2),(v,0.8),(z,0.3)\},V\}.$$

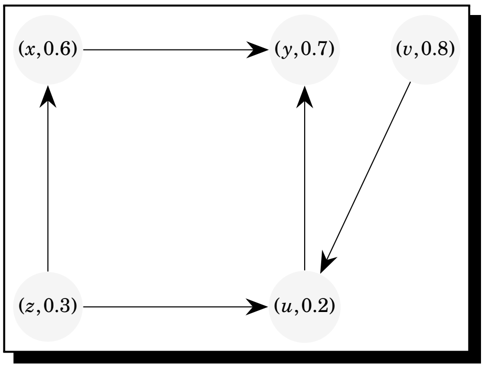

Figure 1. Fuzzy directed graph $\mathcal{D}$

Then, the up path fuzzy topological space is,

$$\mathcal{T}\uparrow = \{0,\{(z,0.3)\},\{(v,0.8)\},\{(z,0.3),(v,0.8)\},\{(x,0.6),(z,0.3)\},\{(x,0.6),(z,0.3),(v,0.8)\},\{(u,0.2),(v,0.8),(z,0.3)\},\{(x,0.6),(u,0.2),(v,0.8),(z,0.3)\},V\}.$$

Again from Figure 1, we have

$$\mathcal{R}[(x,\alpha)]\downarrow = \{(x,0.6),(y,0.7)\},$$
$$\mathcal{R}[(y,\alpha)]\downarrow = \{(y,0.7)\},$$
$$\mathcal{R}[(z,\alpha)]\downarrow = \{(z,0.3),(x,0.6),(y,0.7),(u,0.2)\},$$
$$\mathcal{R}[(u,\alpha)]\downarrow = \{(u,0.2),(y,0.7)\},$$
$$\mathcal{R}[(v,\alpha)]\downarrow = \{(v,0.8),(u,0.2),(y,0.7)\}$$
$$\mathcal{S}\downarrow = \{\{(y,0.7)\},\{(x,0.6),(y,0.7)\},\{(u,0.2),(y,0.7)\},\{(v,0.8),(u,0.2),(y,0.7)\},\{(z,0.3),(x,0.6),(y,0.7),(u,0.2)\}\},$$
$$\mathcal{B}\downarrow = \{0,\{(y,0.7)\},\{(x,0.6),(y,0.7)\},\{(u,0.2),(y,0.7)\},\{(v,0.8),(u,0.2),(y,0.7)\},\{(z,0.3),(x,0.6),(y,0.7),(u,0.2)\}\}.$$

Then the down path fuzzy topological space is,

$$\mathcal{T}\downarrow=\{0,\{(y,0.7)\},\{(x,0.6),(y,0.7)\},\{(u,0.2),(y,0.7)\},\{(x,0.6),(y,0.7),(u,0.2)\},\\ \{(v,0.8),(u,0.2),(y,0.7)\},\{(x,0.6),(y,0.7),(u,0.2),(v,0.8)\},\\ \{(z,0.3),(x,0.6),(y,0.7),(u,0.2)\},V\}.$$

Remark 3.1. If $(x,\alpha_1)\in V$ is an isolated vertex or if there is no path from (y,α_2) to (x,α_1), for all $(y,\alpha_2)\in V$ then we have $\mathcal{R}[(x,\alpha_1)]\uparrow=\{(x,\alpha_1)\}$. Similarly, if $(x,\alpha_1)\in V$ is an isolated vertex or if there is no path from (x,α_1) to (y,α_2), for all $(y,\alpha_2)\in V$ then we have $\mathcal{R}[(x,\alpha_1)]\downarrow=\{(x,\alpha_1)\}$.

Theorem 3.1. *If a fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ is a path then $\mathcal{T}\uparrow$ and $\mathcal{T}\downarrow$ are homeomorphic.*

Proof. Let $V=\{(x_1,\alpha_1),(x_2,\alpha_2),(x_3,\alpha_3),\dots,(x_n,\alpha_n)\}$ be set of vertex and let $\mathcal{D}=(V,\alpha,\beta)$ be a path from vertex (x_1,α_1) to (x_n,α_n). Then,

$\mathcal{R}[(x_1,\alpha_1)]\uparrow=\{(x_1,\alpha_1)\}$,

$\mathcal{R}[(x_2,\alpha_2)]\uparrow=\{(x_1,\alpha_1),(x_2,\alpha_2)\}$,

$\mathcal{R}[(x_3,\alpha_3)]\uparrow=\{(x_1,\alpha_1),(x_2,\alpha_2),(x_3,\alpha_3)\}$, and so on we get

$\mathcal{R}[(x_n,\alpha_n)]\uparrow=\{(x_1,\alpha_1),(x_2,\alpha_2),(x_3,\alpha_3),\dots,(x_n,\alpha_n)\}$.

Similarly,

$\mathcal{R}[(x_1,\alpha_1)]\downarrow=\{(x_1,\alpha_1),(x_2,\alpha_2),(x_3,\alpha_3),\dots,(x_n,\alpha_n)\}$,

$\mathcal{R}[(x_2,\alpha_2)]\downarrow=\{(x_2,\alpha_2),(x_3,\alpha_3),\dots,(x_n,\alpha_n)\}$,

$\mathcal{R}[(x_3,\alpha_3)]\downarrow=\{(x_3,\alpha_3),\dots,(x_n,\alpha_n)\}$, and so on we get

$\mathcal{R}[(x_n,\alpha_n)]\downarrow=\{(x_n,\alpha_n)\}$

from this it is observe that $\mathcal{S}\uparrow$ and $\mathcal{S}\downarrow$ are isomorphic. Hence the corresponding path fuzzy topological spaces $\mathcal{T}\uparrow$ and $\mathcal{T}\downarrow$ are homeomorphic. □

Remark 3.2. A fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ is null graph iff $\mathcal{R}[(x,\alpha)]\uparrow=\mathcal{R}[(x,\alpha)]\downarrow=(x,\alpha)$ for all $(x,\alpha)\in V$.

Theorem 3.2. *Let $(V,\mathcal{T}\uparrow)$ be the up path fuzzy topological space generated by a fuzzy connected directed graph $\mathcal{D}=(V,\alpha,\beta)$ and $A\subseteq\mathcal{T}\uparrow$ be any subset. If a vertex point $(x,\alpha)\in A$ then either $A=\mathcal{R}[(x,\alpha)]\uparrow$ or $A\subset\mathcal{R}[(z,\alpha_1)]\uparrow$ for some $(z,\alpha_1)\in V$ containing (x,α).*

Proof. As $\mathcal{D}=(V,\alpha,\beta)$ is connected, for any $(x,\alpha)\in V$ is connected to for some $(z,\alpha_1)\in V$, that is, there is path from either (x,α) to (z,α_1) or (z,α_1) to (x,α), thus either $(x,\alpha)\in\mathcal{R}[(x,\alpha)]$ or $(x,\alpha)\in\mathcal{R}[(z,\alpha_1)]$ for some $(z,\alpha_1)\in V$. So for any $A\subseteq\mathcal{T}\uparrow$ containing (x,α), either $A=\mathcal{R}[(x,\alpha)]\uparrow$ or $A\subset\mathcal{R}[(z,\alpha_1)]\uparrow$ for some $(z,\alpha_1)\in V$ containing (x,α). □

Corollary 3.1. *Let $(V,\mathcal{T}\downarrow)$ be the down path fuzzy topological space generated by a fuzzy connected directed graph $\mathcal{D}=(V,\alpha,\beta)$ and $A\subseteq\mathcal{T}\downarrow$ be any subset. If a vertex point $(x,\alpha)\in A$ then either $A=\mathcal{R}[(x,\alpha)]\downarrow$ or $A\subset\mathcal{R}[(z,\alpha_1)]\downarrow$ for some $(z,\alpha_1)\in V$ containing (x,α).*

Proof. Proof is similar like Theorem 3.2. □

Definition 3.2. A fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is said to be strongly connected if for any $u, v \in V$, there is path from u to v and there is path from v to u.

Theorem 3.3. *If $\mathcal{D} = (V, \alpha, \beta)$ is strongly connected fuzzy directed graph then $\mathcal{T}\uparrow = \mathcal{T}\downarrow$ and both the fuzzy topologies are indiscrete.*

Proof. Let $(x, \alpha) \in V$ be any vertex. Since $\mathcal{D} = (V, \alpha, \beta)$ is a strongly connected fuzzy directed graph, there is path from every vertex of D to (x, α) and there is path from (x, α) to every vertex of D. Hence we have $\mathcal{R}[(x, \alpha)]\uparrow = \mathcal{R}[(x, \alpha)]\downarrow = V$, this implies $\mathcal{T}\uparrow = \mathcal{T}\downarrow = \{0, V\}$. □

Corollary 3.2. *If a fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is a cycle then $\mathcal{T}\uparrow = \mathcal{T}\downarrow$ and both the fuzzy topologies are indiscrete.*

Proof. Since every cycle is a strongly connected fuzzy directed graph hence by Theorem 3.3 result is true. □

Corollary 3.3. *If a fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is complete then $\mathcal{T}\uparrow = \mathcal{T}\downarrow$ and both the fuzzy topologies are indiscrete.*

Proof. Since every complete fuzzy directed graph is a strongly connected hence by Theorem 3.3 result is true. □

Definition 3.3. A fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is symmetric if $\mu(u, v) = \mu(v, u)$, for all $u, v \in V$.

Corollary 3.4. *If a fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is connected and symmetric then $\mathcal{T}\uparrow = \mathcal{T}\downarrow$ and both the fuzzy topologies are indiscrete.*

Proof. Since every connected and symmetric fuzzy directed graph is a strongly connected hence by Theorem 3.3 result is true. □

Definition 3.4. A fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is asymmetric if $\mathcal{D}$ has at most one fuzzy directed edge between every pair of vertices, may containing self loop.

Theorem 3.4. *If a fuzzy directed graph $\mathcal{D} = (V, \alpha, \beta)$ is connected and asymmetric then $\mathcal{T}\uparrow \cap \mathcal{T}\downarrow$ is an indiscrete fuzzy topological space, that is, $\mathcal{T}\uparrow$ and $\mathcal{T}\downarrow$ have no common proper path fuzzy open sets.*

Proof. Let $\mathcal{D} = (V, \alpha, \beta)$ is connected and asymmetric. If $\mathcal{D} = (V, \alpha, \beta)$ is strongly connected then by Theorem 3.3 result is true. Suppose $\mathcal{D} = (V, \alpha, \beta)$ is not strongly connected. Then, there exist a pair of vertices $(x, \alpha_1), (y, \alpha_2) \in V$ such that there is no path from either (x, α_1) to (y, α_2) or there is no path from (y, α_2) to (x, α_1). Suppose there is path (x, α_1) to (y, α_2) but there is no path from (y, α_2) to (x, α_1). Therefore, $(y, \alpha_2) \in \mathcal{R}[(x, \alpha_1)]\downarrow$ but $(y, \alpha_2) \notin \mathcal{R}[(x, \alpha_1)]\uparrow$. Also, $(y, \alpha_2) \notin \mathcal{R}[(z, \alpha_3)]\uparrow$ containing (x, α_1) except if $\mathcal{R}[(z, \alpha_3)]\uparrow \neq V$. Since D is connected and asymmetric, this implies, there exist at least one $(u, \alpha_4) \in V$ such that $\mathcal{R}[(u, \alpha_4)]\uparrow = V$ or there exist at least one $(v, \alpha_5) \in V$

such that $\mathcal{R}[(v,\alpha_5)]\downarrow = V$. Hence by our construction, $\mathcal{R}[(x,\alpha_1)]\uparrow \neq \mathcal{R}[(x,\alpha_1)]\downarrow$ and for any $(y,\alpha_2)\in V$, $\mathcal{R}[(y,\alpha_2)]\uparrow$ and $\mathcal{R}[(y,\alpha_2)]\downarrow$ are different. So, there intersection and union is also different. Therefore, $\mathcal{T}\uparrow$ and $\mathcal{T}\downarrow$ have no common proper path fuzzy open sets. □

4. Separation Axioms on Path Fuzzy Bitopological Space

In this section we define path fuzzy bitopological space by using path fuzzy topological spaces in Section 3, on fuzzy directed graph $\mathcal{D}$.

Definition 4.1. The path fuzzy topological spaces $(V,\mathcal{T}\uparrow)$ and $(V,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}$ forms a fuzzy bitopological space $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ on $\mathcal{D}$ we called it as path fuzzy bitopological space.

Example 4.1. From Example 3.1, $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ be a path fuzzy bitopological space on $\mathcal{D}$.

Remark 4.1. If $d^-(x,\alpha)=0$ then $\mathcal{R}[(x,\alpha)]\uparrow = \{(x,\alpha)\}$ and if $d^+(x,\alpha)=0$ then $\mathcal{R}[(x,\alpha)]\downarrow = \{(x,\alpha)\}$.

Theorem 4.1. *A path fuzzy bitopological space $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ is fuzzy pairwise T_2 iff $\mathcal{D}=(V,\alpha,\beta)$ is a null graph.*

Proof. Let $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_2 then for any distinct $(x,\alpha_1),(y,\alpha_2)\in V$ there exist $A\in\mathcal{T}\uparrow$ and $B\in\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in A$, $(y,\alpha_2)\in B$ and $A\cap B=0$, this implies, there is no path from (y,α_2) to (x,α_1) or there is no path from (x,α_1) to (y,α_2). Hence $d^-(x,\alpha)=0$ and $d^+(x,\alpha)=0$, for each $(x,\alpha)\in V$. This shows that every pair of distinct vertices in V is not connected by any path. Hence $\mathcal{D}$ must be null graph.

Conversely, let $\mathcal{D}=(V,\alpha,\beta)$ is a null graph, then for each $(x,\alpha)\in V$, $\mathcal{R}[(x,\alpha)]\uparrow = \{(x,\alpha)\}$ and $\mathcal{R}[(x,\alpha)]\downarrow = \{(x,\alpha)\}$. Therefore, for any distinct $(x,\alpha_1),(y,\alpha_2)\in V$ there exist $A=\mathcal{R}[(x,\alpha_1)]\uparrow\in\mathcal{T}\uparrow$, $B=\mathcal{R}[(y,\alpha_2)]\downarrow\in\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in A$, $(y,\alpha_2)\in B$ and $A\cap B=0$. Hence $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_2. □

Theorem 4.2. *If $\mathcal{D}=(V,\alpha,\beta)$ is fuzzy directed graph such that $d^-(x,\alpha)=0$ or $d^+(x,\alpha)=0$ for all $(x,\alpha)\in V$ then a path fuzzy bitopological space $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ is fuzzy pairwise T_0.*

Proof. If $\mathcal{D}=(V,\alpha,\beta)$ is a null graph, then result is obvious. Suppose $\mathcal{D}=(V,\alpha,\beta)$ is not a null graph and as for each $(x,\alpha)\in V$, $d^-(x,\alpha)=0$ or $d^+(x,\alpha)=0$, then for each $(x,\alpha)\in V$, $\mathcal{R}[(x,\alpha)]\uparrow = \{(x,\alpha)\}$ or $\mathcal{R}[(x,\alpha)]\downarrow = \{(x,\alpha)\}$. Therefore, for any distinct $(x,\alpha_1),(y,\alpha_2)\in V$ there exist $A=\{(x,\alpha_1)\}\subset\mathcal{T}\uparrow\cup\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in A$ and $(y,\alpha_2)\cap A=0$. Hence $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_0. □

Theorem 4.3. *If $\mathcal{D}=(V,\alpha,\beta)$ is fuzzy directed graph such that $d^-(x,\alpha)=0$ or $d^+(x,\alpha)=0$ for all $(x,\alpha)\in V$ then a path fuzzy bitopological space $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ is fuzzy pairwise T_1.*

Proof. If $\mathcal{D} = (V,\alpha,\beta)$ is a null graph, then result is obvious. Suppose $\mathcal{D} = (V,\alpha,\beta)$ is not a null graph and as for each $(x,\alpha) \in V$, $d^-(x,\alpha) = 0$ or $d^+(x,\alpha) = 0$, then for each $(x,\alpha) \in V$, $\mathcal{R}[(x,\alpha)]\uparrow = \{(x,\alpha)\}$ or $\mathcal{R}[(x,\alpha)]\downarrow = \{(x,\alpha)\}$. Therefore, for any distinct $(x,\alpha_1),(y,\alpha_2) \in V$ there exist $A = \{(x,\alpha_1)\}, B = \{(y,\alpha_2)\} \in \mathcal{T}\uparrow \cup \mathcal{T}\downarrow$ such that $(x,\alpha_1) \in A$ and $(y,\alpha_2) \cap A = 0$ or $(y,\alpha_2) \in B$ and $(x,\alpha_1) \cap B = 0$. Hence $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_1. □

Theorem 4.4. *If a path fuzzy topological spaces $(V,\mathcal{T}\uparrow)$ and $(V,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D} = (V,\alpha,\beta)$ are fuzzy T_0 then the corresponding path fuzzy bitopological spaces $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_0.*

Proof. As $(V,\mathcal{T}\uparrow)$ is fuzzy T_0. Therefore, for any distinct $(x,\alpha_1),(y,\alpha_2) \in V$ there exist $A,B \in \mathcal{T}\uparrow$ such that $(x,\alpha_1) \in A \subseteq co(y,\alpha_2)$ or $(y,\alpha_2) \in B \subseteq co(x,\alpha_1)$. Similarly, as $(V,\mathcal{T}\downarrow)$ is fuzzy T_0 then for any distinct $(x,\alpha_1),(y,\alpha_2) \in V$ there exist $C,D \in \mathcal{T}\downarrow$ such that $(x,\alpha_1) \in C \subseteq co(y,\alpha_2)$ or $(y,\alpha_2) \in D \subseteq co(x,\alpha_1)$, thus there exist $E = A \cup B \in \mathcal{T}\uparrow \cup \mathcal{T}\downarrow$ such that $(x,\alpha_1) \in E$ and $(y,\alpha_2) \cap E = 0$ or there exist $F = C \cup D \in \mathcal{T}\uparrow \cup \mathcal{T}\downarrow$ such that $(y,\alpha_2) \in F$ and $(x,\alpha_1) \cap F = 0$. Hence $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_0. □

Remark 4.2. Converse of Theorem 4.4, need not be true, that is, if $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_0 then $(V,\mathcal{T}\uparrow)$ and $(V,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D} = (V,\alpha,\beta)$ may not be fuzzy T_0. From Example 4.2, $(V,\mathcal{T}_1\uparrow)$ in equation (4.1) and $(V,\mathcal{T}_2\downarrow)$ in equation (4.2) are not fuzzy T_0 but the corresponding fuzzy bitopological space $(V,\mathcal{T}_1\uparrow,\mathcal{T}_2\downarrow)$ in equation (4.3) is fuzzy pairwise T_0.

Example 4.2. Let $\mathcal{H}$ be a fuzzy directed graph with vertex set, $V = \{(x,0.6),(z,0.3),(u,0.2)\}$. Now from Figure 2, we have

$$\mathcal{R}[(x,\alpha)]\uparrow = \{(x,0.6),(z,0.3)\},$$
$$\mathcal{R}[(z,\alpha)]\uparrow = \{(z,0.3)\},$$
$$\mathcal{R}[(u,\alpha)]\uparrow = \{(u,0.2),((z,0.3)\}.$$

Then, the up path fuzzy topological space is

$$\mathcal{T}_1\uparrow = \{0,\{(z,0.3)\},\{(x,0.6),(z,0.3)\},\{(u,0.2),(z,0.3)\},V\}. \tag{4.1}$$

Again from Figure 2, we have

$$\mathcal{R}[(x,\alpha)]\downarrow = \{(x,0.6)\},$$
$$\mathcal{R}[(z,\alpha)]\downarrow = \{(z,0.3),(x,0.6),(u,0.2)\},$$
$$\mathcal{R}[(u,\alpha)]\downarrow = \{(u,0.2)\}.$$

Then, the down path fuzzy topological space is

$$\mathcal{T}_2\downarrow = \{0,\{(x,0.6)\},\{(u,0.2)\},\{(x,0.6),(u,0.2)\},V\}. \tag{4.2}$$

Hence, the path fuzzy bitopological space is

$$\begin{aligned}(V,\mathcal{T}_1\uparrow,\mathcal{T}_2\downarrow) = \{&0,\{(x,0.6)\},\{(z,0.3)\},\{(u,0.2)\},\{(x,0.6),(z,0.3)\},\\ &\{(x,0.6),(u,0.2)\},\{(u,0.2),(z,0.3)\},V\}.\end{aligned} \tag{4.3}$$

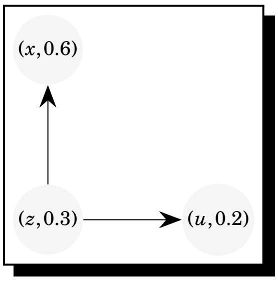

Figure 2. Fuzzy directed graph $\mathcal{H}$

Theorem 4.5. *If a path fuzzy topological spaces $(V,\mathcal{T}\uparrow)$ and $(V,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ are fuzzy T_1 then the corresponding path fuzzy bitopological spaces $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_1.*

Proof. As $(V,\mathcal{T}\uparrow)$ is fuzzy T_1. Therefore, for any distinct $(x,\alpha_1),(y,\alpha_2)\in V$ there exist $A,B\in\mathcal{T}\uparrow$ such that $(x,\alpha_1)\in A$, $(x,\alpha_1)\notin B$, $(y,\alpha_2)\in B$, $(y,\alpha_2)\notin A$. Similarly, as $(V,\mathcal{T}\downarrow)$ is fuzzy T_1 then for any distinct $(x,\alpha_1),(y,\alpha_2)\in V$ there exist $C,D\in\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in C$, $(x,\alpha_1)\notin D$, $(y,\alpha_2)\in D$, $(y,\alpha_2)\notin C$, thus there exist $E=A\cup B$, $F=C\cup D\in\mathcal{T}\uparrow\cup\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in E$ and $(y,\alpha_2)\cap E=0$ or $(y,\alpha_2)\in F$ and $(x,\alpha_1)\cap F=0$. Hence $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_1. □

Remark 4.3. Converse of Theorem 4.5, need not be true, that is, if $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy pairwise T_1 then $(V,\mathcal{T}\uparrow)$ and $(V,\mathcal{T}\downarrow)$ generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$ may not be fuzzy T_1. From Example 4.2, $(V,\mathcal{T}_1\uparrow)$ in equation (4.1) and $(V,\mathcal{T}_2\downarrow)$ in equation (4.2) are not fuzzy T_1 but the corresponding fuzzy bitopological space $(V,\mathcal{T}_1\uparrow,\mathcal{T}_2\downarrow)$ in equation (4.3) is fuzzy pairwise T_1.

Theorem 4.6. *Let $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ be a path fuzzy bitopological space generated by fuzzy directed graph $\mathcal{D}=(V,\alpha,\beta)$. If $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy weakly pairwise T_2 iff $\mathcal{D}$ contains no circuit.*

Proof. Let $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy weakly pairwise T_2. So for any distinct vertices (x,α_1) and (y,α_2) in V there exists fuzzy open sets say $A\in\mathcal{T}\uparrow$ and $B\in\mathcal{T}\downarrow$ such that $(x,\alpha_1)\in A$, $(y,\alpha_2)\in B$ or $(y,\alpha_2)\in A$, $(x,\alpha_1)\in B$ and $A\cap B=0$. On contradiction, suppose $\mathcal{D}$ contains a circuit. Let (x,α_1) and (y,α_2) be two distinct vertices on the circuit. Then $\mathcal{R}[(x,\alpha_1)]\uparrow$ and $\mathcal{R}[(x,\alpha_1)]\downarrow$ must contains (y,α_2) which is contradiction to $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy weakly pairwise T_2. Hence our assumption is wrong, so $\mathcal{D}$ contains no circuit.

Conversely, let $\mathcal{D}$ contains no circuit. Let (x,α_1) and (y,α_2) be any two distinct vertices in V. To show $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy weakly pairwise T_2, it is sufficient to show that $\mathcal{R}[(x,\alpha_1)]\uparrow\cap\mathcal{R}[(y,\alpha_2)]\downarrow=0$ or $\mathcal{R}[(y,\alpha_2)]\uparrow\cap\mathcal{R}[(x,\alpha_1)]\downarrow=0$. Suppose $\mathcal{R}[(x,\alpha_1)]\uparrow\cap\mathcal{R}[(y,\alpha_2)]\downarrow\neq 0$ and $\mathcal{R}[(y,\alpha)]\uparrow\cap\mathcal{R}[(x,\alpha)]\downarrow\neq 0$.

Let $(u,\alpha_3)\in\mathcal{R}[(x,\alpha_1)]\uparrow\cap\mathcal{R}[(y,\alpha_2)]\downarrow$ and $(v,\alpha_4)\in\mathcal{R}[(y,\alpha_2)]\uparrow\cap\mathcal{R}[(x,\alpha_1)]\downarrow$.

As $(u,\alpha_3)\in\mathcal{R}[(x,\alpha_1)]\uparrow\cap\mathcal{R}[(y,\alpha_2)]\downarrow$,

$\implies$ there is a path from (u,α_3) to (x,α_1) and a path from (y,α_2) to (u,α_3)

$\implies$ there is a path from (y,α_2) to (x,α_1).

Similarly, as $(v,\alpha_4)\in\mathcal{R}[(y,\alpha_2)]\uparrow\cap\mathcal{R}[(x,\alpha_1)]\downarrow$

$\implies$ there is a path from (v,α_4) to (y,α_2) and a path from (x,α_1) to (v,α_4)

$\implies$ there is a path from (x,α_1) to (y,α_2).

Hence $\mathcal{D}$ contains a circuit, which is contraction to our hypothesis.

Therefore, $(V,\mathcal{T}\uparrow,\mathcal{T}\downarrow)$ is fuzzy weakly pairwise T_2. □

5. Conclusion

We have introduced the concept of path fuzzy bitopological spaces associated with fuzzy directed graphs ($\mathcal{D}$). We began by defining the up path fuzzy topology $\mathcal{T}\uparrow$ and the down path fuzzy topology $\mathcal{T}\downarrow$ on fuzzy directed graphs using path relations between vertices. We then explored the relationships between $\mathcal{T}\uparrow$ and $\mathcal{T}\downarrow$ across various fuzzy digraphs. We introduced the concepts of strongly connected, symmetric and asymmetric fuzzy directed graphs, demonstrating that the corresponding path topological spaces are indiscrete. A significant result was established, showing that the path fuzzy bitopological space is fuzzy pairwise T_2 if and only if $\mathcal{D}$ is a null graph. Furthermore, we examined the relationships between path topological spaces and path bitopological spaces using the separation axioms of fuzzy pairwise T_0 and T_1.

Our work establishes a foundational connection between fuzzy directed graphs and path fuzzy bitopological spaces, offering a new perspective for analyzing complex fuzzy structures.

The study of fuzzy topological and bitopological spaces on fuzzy directed graphs has the potential to have applications in several fields like Network Analysis, Decision-Making Systems, Fuzzy Control Systems: Data Science and Machine Learning, Mathematical Modeling etc.

By extending these concepts, researchers can develop new methodologies for understanding and interpreting complex systems, thereby advancing both theoretical and applied mathematics.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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