Communications in Mathematics and Applications
Vol. 15, No. 5, pp. 1503–1513, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i5.2836

http://www.rgnpublications.com

Research Article

Rough Convergence for Generalized Difference Sequences by a Compact Operator in Probabilistic *n*-Normed Spaces

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Received: July 30, 2024 **Accepted:** November 11, 2024

Abstract. Using compact operator in probabilistic n-normed spaces, we develop and investigate the notion of rough convergence for generalized difference sequences. In relation to rough convergence in probabilistic n-normed spaces, certain fundamental conclusions regarding the concept of rough limit points for a difference sequence are defined.

Keywords. Rough convergence, Rough limit points, Probabilistic n-normed space, Compact linear operator

Mathematics Subject Classification (2020). 40A05, 26A03, 46S50

1. Introduction

Rough convergence was first introduced for sequences on finite dimensions normed linear spaces by Phu [25]. Later, Phu [24] expanded this idea to infinite dimensional normed linear spaces. In these studies, the major goals were to present rough Cauchy sequences and to establish rough bounds, roughness degree, and rough continuity of linear operators. Dündar and Çakan [6], and Pal *et al*. [23] concurrently introduced rough convergence for ideals, while Malik and Maity [21] defined rough convergence for the double sequences in normed linear spaces. Furthermore, Banerjee and Mondal [4] expanded the rough convergence to metric spaces that are cone-shaped.

A recent definition of rough convergence using ideals in cone metric spaces was provided by Banerjee and Paul [3].

Menger [22] developed a crucial extension of metric space, which he named statistical metric space. When the distribution function is used to replace distance and the precise distance between any two places cannot be determined, this kind of measure is crucial to use. These days, probabilistic metric space is the name given to it. Serstnev [30] developed the probabilistic normed space, which is regarded as a generalised family of probabilistic metric spaces. Additionally, Alsina *et al*. [1], and Schweizer and Sklar [27,28] conducted a thorough analysis and revised the concept. Because these spaces are useful extensions of deterministic results in linear normed spaces, many researchers, including Asadollah and Kourosh [2], Constantin and Istratescu [5], Guillén *et al*. [13,14], Sempi [29] have explored these spaces in many directions. Probabilistic n-normed spaces are a generalization of classical n-normed spaces, as described by Rahmat *et al*. [26]. A wide class of probabilistic n-normed spaces was studied by Golet [12].

Initially, Kizmaz [19] introduced the concept of difference sequence spaces as $Z(\Delta) = \{y = (y_p) : (\Delta y_p) \in Z\}$ for $Z = l_\infty$, $\mathcal{C}$, $\mathcal{C}_0$, i.e., spaces of all bounded sequences, convergent sequences and null sequences, respectively. The generalized difference sequence spaces was defined as (see [10]): $Z(\Delta^m y_p) = \{y = (y_p) : (\Delta^m y_p) \in Z\}$, for $Z = l_\infty$, $\mathcal{C}$, $\mathcal{C}_0$, where $\Delta^m y = (\Delta^m y_p) = (\Delta^{m-1} y_p - \Delta^{m-1} y_{p+1})$ so that $\Delta^m y_{p+r} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} y_{m+r}$.

Various characteristics and properties of difference sequences can be found in [8–10]. Demir and Gümüş [7] have examined the idea of rough convergence via difference sequences on finite dimensional normed space. In 2023, Karabacak and Or [16] introduced the concept of rough convergence and rough statistical convergence for generalized difference sequences in normed linear spaces. In 2022, Kamber [15] introduced the intuitionistic fuzzy I-convergent difference sequence defined by a compact operator and explored its topological properties. Recently, Kaur *et al*. [17,18] examined rough convergence via ideals and statistical convergence for difference sequences in generalized spaces.

This paper aims to establish the rough convergence for generalized difference sequences using compact operator in probabilistic n-normed spaces. In Section 2, some basic definitions and notions related to current research work are given and Section 3 includes the main results of the paper.

2. Preliminaries

This section begins with a discussion of the idea of probabilistic n-normed space and related ideas. It then goes on to discuss rough convergence and its characteristics in more detail with a few real-world examples.

Definition 2.1 ([27]). A binary operation $\diamond$ on $[0,1]$ is called *t-norm* if it is continuous, non-decreasing, associative, commutative and with identity 1.

Example 2.1 ([27]). The binary operations $\diamond$ on $[0,1]$ as $a \diamond b = \min\{a,b\}$ and $a \diamond b = \max\{a+b-1, 0\}$ are typical t-norms.

Definition 2.2 ([11])**.** Let $\mathcal{X}$ be a real linear space, $\diamond$ be a t-norm and $\mathbb{F}$ be the collection of distribution functions. Consider a map $\Im : \mathcal{X}^n \to \mathbb{F}$ and if the following properties are satisfied for all $q_1, q_2, q_3, \ldots, q_{n-1} \in \mathcal{X}$ and $r, s \in \mathbb{R}_0^+ = [0, \infty)$, then $\Im$ and $(\mathcal{X}, \Im, \diamond)$ are known as *probabilistic norm* and *probabilistic n-normed space* (Pr-n-space) respectively,

(i) $\Im((q_1, q_2, q_3 \ldots q_n), s) = 1$ iff $q_1, q_2, q_3 \ldots q_n$ are linearly dependent,

(ii) $\Im((q_1, q_2, q_3 \ldots q_n), s)$ is invariant under any permutation of $q_1, q_2, q_3 \ldots q_n$,

(iii) $\Im((q_1, q_2, q_3 \ldots \alpha q_n), s) = \Im\left((q_1, q_2, q_3 \ldots q_n), \frac{s}{|\alpha|}\right)$ where $\alpha \neq 0$ is a real number,

(iv) $\Im((q_1, q_2, q_3 \ldots q_n + q_n'), r + s) \geq \Im((q_1, q_2, q_3 \ldots q_n), r) \diamond \Im((q_1, q_2, q_3 \ldots q_n'), s)$.

Definition 2.3 ([2])**.** Let $(\mathcal{X}, \Im, \diamond)$ be a Pr-n-space with probabilistic n-norm $\Im^n$. Then, sequence $x = (x_k)$ in $\mathcal{X}$ is called *convergent* to $\kappa \in \mathcal{X}$ with respect to $\Im$ if for every $\varepsilon > 0$ and $\vartheta \in (0, 1)$ there exists $k_0 \in \mathcal{N}$ such that $\Im((q_1, q_2, q_3, \ldots, q_{n-1}, x_k - \kappa), \varepsilon) > 1 - \vartheta$, for all $k \geq k_0$. It is denoted by $x_k \xrightarrow{\Im^n} \kappa$ or $\Im^n\text{-}\lim_{k\to\infty} x_k = \kappa$.

Definition 2.4 ([25])**.** Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Then, sequence $x = (x_k)$ in $\mathcal{X}$ is called *rough convergent* to $\kappa \in \mathcal{X}$ for some non-negative real number r if there exists $k_0 \in \mathcal{N}$ for every $\varepsilon > 0$ such that $\|x_k - \kappa\| < r + \varepsilon$, for all $k \geq k_0$.
It is denoted by $x_k \xrightarrow{r} \kappa$ or $r\text{-}\lim_{k\to\infty} x_k = \kappa$, where r is known as roughness degree of rough convergence of the sequence $x = (x_k)$.

For any sequence $x = (x_k)$ in the normed linear space $\mathcal{X}$ the r-limit set is given as $LIM^r_{x_k} = \{\kappa \in \mathcal{X} : x_k \xrightarrow{r} \kappa\}$. Also, $LIM^r_{x_k} = [\limsup x - r, \liminf x + r]$ is defined for any sequence $x = (x_k)$ of real numbers [25].

Definition 2.5 ([20])**.** An operator $\mathcal{T}$ defined by

$$\mathcal{T} : G \to H$$

is termed as Compact Linear Operator (completely continuous linear operator) with G and H be two normed linear spaces if $\mathcal{T}$ is linear and $\mathcal{T}$ maps every bounded sequence (g_k) in G onto a sequence $\mathcal{T}(g_k)$ in H which has a convergent subsequence. The set of all bounded linear operators $\mathcal{B}$(G,H) is normed linear space normed by

$$\|\mathcal{T}\| = \sup_{g\in G, \|k\|=1} \|\mathcal{T}k\|.$$

The set of all compact linear operator $\mathcal{C}$(G,H) is a closed subspace of $\mathcal{B}$(G,H) and $\mathcal{C}$(G,H) is a Banach space if H is a Banach.

3. Main Results

We now turn our attestation towards the notion of rough convergence for difference sequences in a Pr-n-space using a compact operator and establish some of its important properties. Throughout the paper $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)_{k\in\mathcal{N}}$ will denote the difference sequence by using a compact linear operator $\mathcal{T}$.

Definition 3.1. Let $(\mathcal{X},\Im,\diamond)$ be a Pr-n-space with probabilistic norm $\Im^n$. Then, sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ in $\mathcal{X}$ is called rough convergent to $\kappa\in\mathcal{X}$ with respect to $\Im^n$ for some non-negative real number r if there exists $k_0\in\mathcal{N}$ for every $\varepsilon>0$ and $\vartheta\in(0,1)$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\varepsilon)>1-\vartheta, \quad \text{for all } k\ge k_0.$$

It is denoted by $\mathcal{T}(\Delta^m x_k)\xrightarrow{r_{\Im^n}}\kappa$ or $r_{\Im^n}\text{-}\lim_{k\to\infty}\mathcal{T}(\Delta^m x_k)=\kappa$.

Let $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ be the set of all $r_{\Im^n}$-limit points of the sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ in a Pr-n-space $(\mathcal{X},\Im,\diamond)$, for some $r>0$, i.e.,

$$LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}=\{\kappa^*\in\mathcal{X}:\mathcal{T}(\Delta^m x_k)\xrightarrow{r_{\Im^n}}\kappa^*\}.$$

Remark 3.1. For the case $r=0$, the rough convergence agrees with the usual convergence for the sequences in n-probabilistic normed space by a compact operator.

Remark 3.2. Let $(\mathcal{X},\|\cdot\|)$ be a real normed space with the probabilistic norm $\Im^n$ for $x\in\mathcal{X}$ and $t\ge 0$ as $\Im(\mathcal{T}(\Delta^m x),t)=\frac{t}{t+\|\mathcal{T}(\Delta^m x)\|}$. Then, sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ is rough convergent to $\kappa\in\mathcal{X}$ with respect to the norm $\|\cdot\|$ if and only if sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ is rough convergent to $\kappa\in\mathcal{X}$ with respect to the norm $\Im^n$.

The next example shows the sequence may not have a unique $r_\Im$-limit point.

Example 3.1. Let $(\mathcal{X},\|\cdot\|)$ be any real normed space. We define the probabilistic norm $\Im^n$ as $\Im(\mathcal{T}(\Delta^m x),t)=\frac{t}{t+\|\mathcal{T}(\Delta^m x)\|}$, for every $x\in\mathcal{X}$, $t\in\mathcal{R}$.

Then, $(\mathcal{X},\Im,\diamond)$ is a Pr-n-space under the t-norm $\diamond$ which is given by $a\diamond b=\min\{a,b\}$, for $a,b\in[0,1]$. Now define a sequence

$$\mathcal{T}(\Delta^m x_k)=\begin{cases}0, & k \text{ is odd},\\ 1, & k \text{ is even}.\end{cases}$$

From Remark 3.2, it is clear that the above defined sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ is also rough convergent with respect to $\Im^n$ for some $r>0$, we have

$$LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}=\begin{cases}\phi, & r<0.5,\\ [1-r,r], & r\ge 0.5.\end{cases}$$

Definition 3.2. Let $(\mathcal{X},\Im,\diamond)$ be a Pr-n-space with probabilistic norm $\Im$. Then, sequence $x=(x_k)$ in $\mathcal{X}$ is called *bounded* with respect to $\Im^n$ if for every $\vartheta\in(0,1)$ there exists some real number $H>0$ such that $\Im((q_1,q_2,q_3\dots q_{n-1},x_k),H)>1-\vartheta$, for all $k\in\mathcal{N}$.

Theorem 3.1. *Let $(\mathcal{X},\Im,\diamond)$ be a Pr-n-space with probabilistic norm $\Im^n$. If sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ in $\mathcal{X}$ is r-convergent to $\kappa\in\mathcal{X}$ then it is also $s_{\Im^n}$-convergent to $\kappa\in\mathcal{X}$ for $r<s$, i.e., $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}\subset LIM^{s_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$.*

Proof. Let $\mathcal{T}(\Delta^m x_k)\xrightarrow{r_{\Im^n}}\kappa$ for some non-negative real number r. Then, for every $\varepsilon>0$ and $\vartheta\in(0,1)$ there exists $k_0\in\mathcal{N}$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\varepsilon)>1-\vartheta, \quad \text{for all } k\ge k_0. \tag{3.1}$$

For $r < s$, we have

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),s+\varepsilon) > \Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\varepsilon) \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

From (3.1) and (3.2), we get

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),s+\varepsilon) > 1-\vartheta, \quad \text{for all } k \ge k_0.$$

Therefore, $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ is $s_{\Im^n}$-convergent to $\kappa \in \mathcal{X}$. □

Theorem 3.2. *The $r_{\Im^n}$-convergent sequence in Pr-n-space $(\mathcal{X},\Im,\diamond)$ is always bounded.*

Proof. Let sequence $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ be $r_{\Im^n}$-convergent to $\kappa \in \mathcal{X}$ for some $r \ge 0$. For $t \in (0,1)$ take $\vartheta \in (0,1)$ so that $(1-\vartheta)\diamond(1-\vartheta) > 1-t$. Then, for $\vartheta \in (0,1)$ choose $m_0 > 0$ so large that

$$\Im\left(\kappa, \frac{m_0}{2}\right) > 1-\vartheta$$

and there exists $k_0 \in \mathbb{N}$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\frac{m_0}{2}\right) > 1-\vartheta, \quad \text{for all } k \ge k_0.$$

Also, for $k \ge k_0$ we have

$$\begin{aligned}
&\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+m_0)\\
&\quad \ge \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\frac{m_0}{2}\right)\diamond\Im\left(\kappa,\frac{m_0}{2}\right)\\
&\quad > (1-\vartheta)\diamond(1-\vartheta)\\
&\quad > 1-t.
\end{aligned}$$

For $k = 1,2,\dots,k_0-1$. Choose $m_k > 0$ so large that $\Im(q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k),r+m_k) > 1-t$. Then with $M = \max\{m_0,m_1,\dots,m_{k_0-1}\}$, we have $\Im(q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k),r+M) > 1-t$, for all $k < k_0$.

For $k \ge k_0$, we have

$$\begin{aligned}
&\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+M)\\
&\quad \ge \Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+m_0)\diamond\Im(0,M-m_0)\\
&\quad > (1-t)\diamond 1\\
&\quad = 1-t.
\end{aligned}$$

Thus, $\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)),r+M) > 1-t$, for all $k \in \mathbb{N}$. Therefore, $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ is bounded in a Pr-n-space $(\mathcal{X},\Im,\diamond)$. □

In Pr-n-space $(\mathcal{X},\Im,\diamond)$ a bounded sequence has a non-empty $r_{\Im}$-limit set, for some $r > 0$. The following theorem justify this statement.

Theorem 3.3. *The bounded sequence $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ in a Pr-n-space $(\mathcal{X},\Im,\diamond)$ has $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)} \neq \phi$, for some $r > 0$.*

Proof. Let $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ be a bounded sequence in a Pr-n-space $(\mathcal{X},\Im,\diamond)$. Then, there exists a real number $p > 0$ for every $\vartheta \in (0,1)$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)),p) > 1-\vartheta, \quad \text{for all } k \in \mathbb{N}.$$

Let $\varepsilon > 0$, then

$$\begin{aligned}
&\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)),r+\varepsilon)\\
&\quad \geq \Im((q_1,q_2,q_3,\dots,q_{n-1},0),r)\diamond\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)),\varepsilon)\\
&\quad > 1\diamond(1-\vartheta)\\
&\quad = 1-\vartheta.
\end{aligned}$$

Thus, $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ is $r_{\Im^n}$-convergent to $0\in\mathcal{X}$ on a Pr-n-space $(\mathcal{X},\Im,\diamond)$, for some real number $p>0$. Hence, $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}\neq\phi$. □

In Theorem 3.3 for any positive real number r with $r>p$ using Theorem 3.1, we obtain $LIM^{r^n_{\Im}}_{\mathcal{T}(\Delta^m x_k)}\neq\phi$.

Theorem 3.4. *Let $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ be any generalized difference sequence with compact operator $\mathcal{T}$ on a Pr-n-space $(\mathcal{X},\Im,\diamond)$ then $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ is a convex set, for some $r>0$.*

Proof. Let $\kappa_1,\kappa_2\in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$. For convexity, we have to show that $(1-\rho)\kappa_1+\rho\kappa_2\in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ for any real number $\rho\in[0,1]$.

For $t\in(0,1)$ take $\vartheta\in(0,1)$ so that $(1-\vartheta)\diamond(1-\vartheta)>1-t$.

Since $\kappa_1,\kappa_2\in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$, then there exists $k_1,k_2\in\mathcal{N}$ for every $\varepsilon>0$ and $\vartheta\in(0,1)$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa_1),\frac{r+\varepsilon}{2(1-\rho)}\right)>1-\vartheta,\quad \text{for all } k\geq k_1$$

and

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa_2),\frac{r+\varepsilon}{2\rho}\right)>1-\vartheta,\quad \text{for all } k\geq k_2.$$

For $k\geq k_0$ where $k_0=\max\{k_1,k_2\}$, we have

$$\begin{aligned}
&\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-[(1-\rho)\kappa_1+\rho\kappa_2]),r+\varepsilon)\\
&\quad\geq\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa_1),\frac{r+\varepsilon}{2(1-\rho)}\right)\\
&\qquad\diamond\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa_2),\frac{r+\varepsilon}{2\rho}\right)\\
&\quad>(1-\vartheta)\diamond(1-\vartheta)\\
&\quad>1-t.
\end{aligned}$$

Therefore, $(1-\rho)\kappa_1+\rho\kappa_2\in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$. Hence $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ is a convex set. □

Theorem 3.5. *Let $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ and $\mathcal{T}(\Delta^m y)=\mathcal{T}(\Delta^m y_k)$ be any two generalized difference sequences with compact operator $\mathcal{T}$ on a Pr-n-space $(\mathcal{X},\Im,\diamond)$. If for every $\vartheta\in(0,1)$ also there exists some $r>0$ such that $\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)),r)>1-\vartheta$, for all $k\in\mathcal{N}$ and sequence $\mathcal{T}(\Delta^m y)=\mathcal{T}(\Delta^m y_k)$ converges to $\kappa\in\mathcal{X}$ with respect to $\Im^n$. Then sequence $\mathcal{T}(\Delta^m x)=\mathcal{T}(\Delta^m x_k)$ is rough convergent to $\kappa\in\mathcal{X}$ with respect to $\Im^n$.*

Proof. For given $\vartheta\in(0,1)$ take $t\in(0,1)$ so that $(1-t)\diamond(1-t)>1-\vartheta$.

As $\mathcal{T}(\Delta^m y_k)\xrightarrow{\Im^n}\kappa$ then there exists $k_0\in\mathcal{N}$ for every $\varepsilon>0$ and $t\in(0,1)$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m y_k)-\kappa),\varepsilon)>1-t,\quad \text{for all } k\geq k_0.$$

It is given that for every $\vartheta \in (0,1)$, we have $\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)),r) > 1-\vartheta$ for all $k \in \mathbb{N}$.
I.e.,

$$\wp((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)),r) > 1-t, \quad \text{for all } k \in \mathbb{N}.$$

For $k \ge k_0$, we have

$$\begin{aligned}\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa),r+\varepsilon) &\ge \Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)),r)\\ &\quad \diamond \Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m y_k)-\kappa),\varepsilon)\\ &> (1-t)\diamond(1-t)\\ &> 1-\vartheta.\end{aligned}$$

Hence, $\mathcal{T}(\Delta^m x_k) \xrightarrow{r_\Im^n} \kappa$. □

Theorem 3.6. *Let $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ be a generalized difference sequence with compact operator $\mathcal{T}$ on a Pr-n-space $(\mathcal{X},\Im,\diamond)$. Then $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ is a closed set.*

Proof. If $r=0$ then we have nothing to prove as $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ is either empty set or singleton set. Let $LIM^{r_{\Im^n}}_{\Delta^m x_k} \neq \phi$, for some $r>0$. Let $\mathcal{T}(\Delta^m y) = \mathcal{T}(\Delta^m y_k)$ be a convergent sequence with respect to $\Im^n$ to $y_0 \in \mathcal{X}$. For $t \in (0,1)$ take $\vartheta \in (0,1)$ so that $(1-\vartheta)\diamond(1-\vartheta) > 1-t$. Then, there exists $k_1 \in \mathbb{N}$ for every $\varepsilon > 0$ and $\vartheta \in (0,1)$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m y_k)-y_0),\frac{\varepsilon}{2}\right) > 1-\vartheta, \quad \text{for all } k \ge k_1.$$

Let us take $\mathcal{T}(\Delta^m y_m) \in LIM^{r_\Im}_{\mathcal{T}(\Delta^m x_k)}$ with $m > k_1$, then, there exists $k_2 \in \mathbb{N}$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_m)),r+\frac{\varepsilon}{2}\right) > 1-\vartheta, \quad \text{for all } k \ge k_2.$$

For $k \ge k_0$ where $k_0 = \max\{k_1,k_2\}$, we have

$$\begin{aligned}&\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-y_0),r+\varepsilon)\\ &\quad \ge \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_m)),r+\frac{\varepsilon}{2}\right)\\ &\qquad \diamond \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m y_m)-y_0),\frac{\varepsilon}{2}\right)\\ &\quad > (1-\vartheta)\diamond(1-\vartheta)\\ &\quad > 1-t.\end{aligned}$$

Therefore, $y_0 \in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$. □

Theorem 3.7. *Let $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ and $\mathcal{T}(\Delta^m y) = \mathcal{T}(\Delta^m y_k)$ be two generalized difference sequences with compact operator $\mathcal{T}$ on a Pr-n-space $(\mathcal{X},\Im,\diamond)$. If for every $\varepsilon > 0$ and $\vartheta \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that*

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)),\varepsilon) > 1-\vartheta, \quad \textit{for all } k \ge k_0.$$

Then, sequence $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ is $r_{\Im^n}$-convergent to $\kappa \in \mathcal{X}$ if and only if sequence $\mathcal{T}(\Delta^m y) = \mathcal{T}(\Delta^m y_k)$ is $r_{\Im^n}$-convergent to $\kappa \in \mathcal{X}$, for some non-negative real number r.

Proof. For $\vartheta \in (0,1)$ take $t \in (0,1)$ so that $(1-t)\diamond(1-t) > 1-\vartheta$. Let $\mathcal{T}(\Delta^m x_k) \xrightarrow{r_{\Im^n}} \kappa$. Then, there exists $k_0 \in \mathcal{N}$ for every $\varepsilon > 0$ and $t \in (0,1)$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa), r+\frac{\varepsilon}{2}\right) > 1-t, \quad \text{for all } k \geq k_0$$

and

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)), \frac{\varepsilon}{2}\right) > 1-t, \quad \text{for all } k \geq k_0.$$

Now for $k \geq k_0$, we have

$$\begin{aligned}\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m y_k)-\kappa), r+\varepsilon) &\geq \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa), r+\frac{\varepsilon}{2}\right) \\ &\quad \diamond \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\mathcal{T}(\Delta^m y_k)), \frac{\varepsilon}{2}\right) \\ &> (1-t)\diamond(1-t) \\ &> 1-\vartheta.\end{aligned}$$

This implies that sequence $\mathcal{T}(\Delta^m y) = \mathcal{T}(\Delta^m y_k)$ is $r^n_{\Im}$-convergent to κ.
Converse part can be obtained by interchanging $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ and $\mathcal{T}(\Delta^m y) = \mathcal{T}(\Delta^m y_k)$. □

Like in classical approach subsequence of any convergent sequence is also converges to the same limit point, we have similar result in rough convergence in Pr-n-space.

Theorem 3.8. *Let $\mathcal{T}(\Delta^m x') = \mathcal{T}(\Delta^m x_{k_i})$ be a subsequence of $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ in a Pr-n-space $(\mathcal{X},\Im,\diamond)$, then, $LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)} \subset LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_{k_i})}$.*

Proof. Let $\kappa \in LIM^{r^n_{\Im}}_{\mathcal{T}(\Delta^m x_k)}$. Then, there exists $p \in \mathcal{N}$ for every $\varepsilon > 0$ and $\vartheta \in (0,1)$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-\kappa), r+\varepsilon) > 1-\vartheta, \quad \text{for all } k \geq p.$$

Consider $k_m > p$, for some $m \in \mathcal{N}$. Then $k_i > p$, for all $i \geq m$ and

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_{k_i})-\kappa), r+\varepsilon) > 1-\vartheta, \quad \text{for all } k_i > p.$$

This implies that $\kappa \in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_{k_i})}$. □

The diameter of the r-limit set of any sequence in the normed linear space cannot be greater than $2r$. We obtain a similar result for any sequence in a Pr-n-space connected to rough convergence in the next result.

Theorem 3.9. *Let $\mathcal{T}(\Delta^m x) = \mathcal{T}(\Delta^m x_k)$ be a sequence in a Pr-n-space $(\mathcal{X},\Im,\diamond)$ and $r > 0$. Then for $\vartheta \in (0,1)$ there does not exist elements $y,z \in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ such that $\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z), mr) \leq 1-\vartheta$, for $m > 2$.*

Proof. Let, if possible there exists some elements $y,z \in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ such that

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z), mr) \leq 1-\vartheta, \quad \text{for } m > 2. \tag{3.3}$$

For $\vartheta \in (0,1)$, take $t \in (0,1)$ so that $(1-t)\diamond(1-t) > 1-\vartheta$.
As $y,z \in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$, then, there exists $k \in \mathcal{N}$ for every $\varepsilon > 0$ and $t \in (0,1)$ such that

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-y), r+\frac{\varepsilon}{2}\right) > 1-t,$$

and

$$\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-z),r+\frac{\varepsilon}{2}\right)>1-t.$$

Also,

$$\begin{aligned}\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z),2r+\varepsilon) &\geq \Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-z),r+\frac{\varepsilon}{2}\right)\\ &\quad\diamond\Im\left((q_1,q_2,q_3,\dots,q_{n-1},\mathcal{T}(\Delta^m x_k)-y),r+\frac{\varepsilon}{2}\right)\\ &>(1-t)\diamond(1-t)\\ &>1-\vartheta.\end{aligned}$$

Hence,

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z),2r+\varepsilon)>1-\vartheta. \tag{3.4}$$

Then, from (3.4), we have

$$\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z),mr)>1-\vartheta, \quad \text{for } m>2,$$

which is a contradiction to (3.3). Therefore, there does not exists elements $y,z\in LIM^{r_{\Im^n}}_{\mathcal{T}(\Delta^m x_k)}$ such that $\Im((q_1,q_2,q_3,\dots,q_{n-1},y-z),mr)\leq 1-\vartheta$, for $m>2$. □

4. Conclusions

The present article is devoted to study the concept of rough convergent generalized difference sequences by using a compact operator on the probabilistic n-normed spaces. The various topological and algebraic properties for the set of rough limit points for these sequences has been discussed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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