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Research Article

On Bivariate Distributions with N Deleted Areas: Mathematical Definition

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Abstract. A new definition of deleting N-intervals from the domain of two random variables which have known bivariate distribution (N-truncated bivariate distribution) has been provided. This type of distribution is useful in real life applications, where it gives us the opportunity to delete N-intervals of data. This will increase the accuracy of statistical analysis, which facilitates decision-making. As a result of this multiple truncation, changes occur in the statistical properties of the bivariate distribution from its original version. We have thoroughly processed these properties and applied this definition to the bivariate normal distribution.

Keywords. Bivariate distribution, Statistical properties, N deleted areas, Moments

Mathematics Subject Classification (2020). 60E05

1. Introduction

One of the most important applications of truncated distributions is detecting missing targets on a line, surface, or space. It is necessary to determine the probability distribution of the target position at a specific period on the line or a specific area on the level surface. It is possible that information may be available indicating the probability of the target's presence in a group of periods or regions. The main goal of the various search models for lost targets which have been studied in Alamri and El-Hadidy [1], and the references cited therein, is to minimize the cost or maximize the probability of detecting them. This requires deleting some intervals or areas with the lowest probability of the target's presence and distributing that probability to the periods or regions with the highest probability. To do such, the truncation method is an adequate methodology. Put differently, truncation occurs when we need to remove values from a dataset that fall outside of a typical range. For example, you may limit the values of the random variable in some probability distributions to the two values, a and b. Three types of truncated distributions are recognized:

(i) truncated from above, where high random variable values are removed, extending your range from negative infinity to a maximum value;

(ii) truncated from below, where low random variable values are removed, extending your range from a minimum value to positive infinity; and

(iii) double truncation, in which the random variable's high and low values are removed.

Some works as in Ali and Nadarajah [2], and Zaninetti and Ferraro [8] presented truncated forms of many distribution types and discussed statistical properties of them.

Therefore, a new concept must be introduced about truncating periods of random variables that explain the location of the target. Where the previous definition, as in Ali and Nadarajah [2], and Zaninetti and Ferraro [8], is concerned with truncating an interval of distribution from the right side, the left side, or both together. Based on the change in the shape of the distribution, they studied the statistical properties of the distribution. In the case of searching for the target in a group of periods on a straight line, it requires cutting off some of the periods from the inside. El-Hadidy [3] presented a new definition to truncate several different periods of the probability distribution field of the target when the target is located on a straight line. This definition was able to distribute the probability of the truncated periods in equal proportion to the remaining periods. The effect of this definition became clear, as in Alamri and El-Hadidy [1], where the continuous linear search problem was transformed into a discrete one. Through this, the probability of detecting the target increased as it was present in one of a group of different periods. This definition also made new contributions to studying the behaviour of stock prices in specific time periods, as in El-Hadidy and Alfreedi [4]. They used this definition to delete some stochastic volatility intervals from the domain of the random variable. In addition, they provided a comprehensive treatment of the statistical properties of the new distribution.

In the plane, we face a difficult problem in maximizing the detection probability or minimizing the detection cost because the searching process done in all the area without neglecting the small areas which has a low target probability. Based on available information about the probability of the target being present in specific parts of the plane surface, we need a probability distribution that determines the spread of the target in those parts. The closest distribution to this is the bivariate distribution, which is truncated in two respects as in Nadarajah and Kotz [6]. This distribution is only suitable for searching for a target in a specific area. In order to obtain a suitable distribution for this type of search problem, here we will present a new definition to cut off several parts of the original target distribution. This definition is the generalization of the existing double truncation of the bivariate distribution to truncation with multiple areas.

This paper is organized as follows: in Section 2, we define the N-area truncated distribution and go over some of its statistical properties. We apply the results obtained in Section 2 to the bivariate normal distribution as in Section 3. The study concludes with some closing thoughts and future directions in Section 4.

2. *N* Deleted Areas From a Bivariate Distribution

Assuming that the two random variables H and Z have a bivariate distribution with a joint *probability density function* (PDF) $g_{H,Z}(h,z)$ and *cumulative distribution function* (CDF) $G_{H,Z}(h,z)$, where $(h,z) \in R^2$. Depending on some available data, we are in the process of excluding some intervals from the random variables definition domain. Thus, we consider the sets ℓ and $\wp$ of finite sequences of mesh-points on h and z coordinates respectively which define as follows,

$$\ell = \{-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_{N-1} < \beta_{N-1} < \alpha_N < \beta_N < +\infty\}, \text{ and} \tag{1}$$

$$\wp = \{-\infty < \vartheta_1 < \zeta_1 < \vartheta_2 < \zeta_2 < \ldots < \vartheta_{N-1} < \zeta_{N-1} < \vartheta_N < \zeta_N < +\infty\}. \tag{2}$$

We need to delete some certain range of values from the domain of H and Z, respectively. The new bivariate distribution of N intervals truncated from the domain of the random variables at the points $\alpha_j, \beta_j, \vartheta_j, \zeta_j$, $j = 1,2,\ldots,N$, is defined by:

Definition 2.1. *If H and Z are two random variables with known $g_{H,Z}(h,z)$, then we define X and Y as a corresponding N intervals truncated version of the random variables X and Y with a joint PDF $f_{X,Y}(x,y)$ (piecewise function) define as follows:*

$$f_{X,Y}(x,y) = \begin{cases} \Omega^{-1} g_{X,Y}(x,y), & \textit{if } x \in (-\infty,\alpha_1) \cup (\beta_1,\alpha_2) \cup (\beta_2,\alpha_3) \cup \ldots \cup (\beta_{N-1},\alpha_N) \cup (\beta_N,\infty) \\ & \textit{and } y \in (-\infty,\vartheta_1) \cup (\zeta_1,\vartheta_2) \cup (\zeta_2,\vartheta_3) \cup \ldots \cup (\zeta_{N-1},\vartheta_N) \cup (\zeta_N,\infty), \\ 0, & \textit{otherwise}, \end{cases} \tag{3}$$

where $\Omega = 1 - \sum_{j=1}^{N+1} [G_{X,Y}(\alpha_j,\vartheta_j) - G_{X,Y}(\beta_{j-1},\vartheta_j) - G_{X,Y}(\alpha_j,\zeta_{j-1}) + G_{X,Y}(\beta_{j-1},\zeta_{j-1})]$, $\alpha_{N+1} = \vartheta_{N+1} = \infty$ *and* $\beta_0 = \zeta_0 = -\infty$.

In this definition, we distribute the probability of the deleted areas on the remaining parts with equal proportions. If $N = 1$ (delete one bounded area) for any j such that $\alpha_j \le x \le \beta_j$ and $\vartheta_j \le y \le \zeta_j$ then we get the same definition of the bivariate truncation which obtained by Nadarajah and Kotz [6].

2.1 Cumulative Distribution, Survival, Hazard, Reversed Hazard Functions and Some Important Features

The following piecewise function represents *the cumulative distribution function* (CDF) of the N intervals truncated random variable:

$$F_{X,Y}(x,y) = \begin{cases} \Omega^{-1}(\Gamma_0 + G_{X,Y}(x,y)), \quad \text{if } x \in (-\infty,\alpha_1),\ y \in (-\infty,\vartheta_1), \\ \Omega^{-1}(\Gamma_1 + G_{X,Y}(x,y) - G_{X,Y}(\beta_1,y) - G_{X,Y}(x,\zeta_1) + G_{X,Y}(\beta_1,\zeta_1)), \\ \qquad \text{if } x \in (\beta_1,\alpha_2),\ y \in (\zeta_1,\vartheta_2), \\ \Omega^{-1}(\Gamma_2 + G_{X,Y}(x,y) - G_{X,Y}(\beta_2,y) - G_{X,Y}(x,\zeta_2) + G_{X,Y}(\beta_2,\zeta_2)), \\ \qquad \text{if } x \in (\beta_2,\alpha_3),\ y \in (\zeta_2,\vartheta_3), \\ \Omega^{-1}(\Gamma_3 + G_{X,Y}(x,y) - G_{X,Y}(\beta_3,y) - G_{X,Y}(x,\zeta_3) + G_{X,Y}(\beta_3,\zeta_3)), \\ \qquad \text{if } x \in (\beta_3,\alpha_4),\ y \in (\zeta_3,\vartheta_4), \\ \quad \vdots \\ \Omega^{-1}(\Gamma_{N-1} + G_{X,Y}(x,y) - G_{X,Y}(\beta_{N-1},y) - G_{X,Y}(x,\zeta_{N-1}) + G_{X,Y}(\beta_{N-1},\zeta_{N-1})), \\ \qquad \text{if } x \in (\beta_{N-1},\alpha_N),\ y \in (\zeta_{N-1},\vartheta_N), \\ \Omega^{-1}(\Gamma_N + G_{X,Y}(x,y) - G_{X,Y}(\beta_N,y) - G_{X,Y}(x,\zeta_N) + G_{X,Y}(\beta_N,\zeta_N)), \\ \qquad \text{if } x \in (\beta_N,\infty),\ y \in (\zeta_N,\infty), \end{cases} \tag{4}$$

where $\Gamma_N = \sum_{j=1}^{N}[G_{X,Y}(\alpha_j,\vartheta_j) - G_{X,Y}(\beta_{j-1},\vartheta_j) - G_{X,Y}(\alpha_j,\zeta_{j-1}) + G_{X,Y}(\beta_{j-1},\zeta_{j-1})]$, $\Gamma_0 = 0$ and $G_{X,Y}(-\infty,y) = G_{X,Y}(x,-\infty) = G_{X,Y}(-\infty,-\infty) = 0$. Also, its survival function is given by the piecewise function:

$$\widetilde{F}_{X,Y}(x,y) = \begin{cases} 1-\Omega^{-1}(\Gamma_0 + G_{X,Y}(x,y)), \quad \text{if } x \in (-\infty,\alpha_1),\ y \in (-\infty,\vartheta_1), \\ 1-\Omega^{-1}(\Gamma_1 + G_{X,Y}(x,y) - G_{X,Y}(\beta_1,y) - G_{X,Y}(x,\zeta_1) + G_{X,Y}(\beta_1,\zeta_1)), \\ \qquad \text{if } x \in (\beta_1,\alpha_2),\ y \in (\zeta_1,\vartheta_2), \\ 1-\Omega^{-1}(\Gamma_2 + G_{X,Y}(x,y) - G_{X,Y}(\beta_2,y) - G_{X,Y}(x,\zeta_2) + G_{X,Y}(\beta_2,\zeta_2)), \\ \qquad \text{if } x \in (\beta_2,\alpha_3),\ y \in (\zeta_2,\vartheta_3), \\ 1-\Omega^{-1}(\Gamma_3 + G_{X,Y}(x,y) - G_{X,Y}(\beta_3,y) - G_{X,Y}(x,\zeta_3) + G_{X,Y}(\beta_3,\zeta_3)), \\ \qquad \text{if } x \in (\beta_3,\alpha_4),\ y \in (\zeta_3,\vartheta_4) \\ \quad \vdots \\ 1-\Omega^{-1}(\Gamma_{N-1} + G_{X,Y}(x,y) - G_{X,Y}(\beta_{N-1},y) - G_{X,Y}(x,\zeta_{N-1}) + G_{X,Y}(\beta_{N-1},\zeta_{N-1})), \\ \qquad \text{if } x \in (\beta_{N-1},\alpha_N),\ y \in (\zeta_{N-1},\vartheta_N), \\ 1-\Omega^{-1}(\Gamma_N + G_{X,Y}(x,y) - G_{X,Y}(\beta_N,y) - G_{X,Y}(x,\zeta_N) + G_{X,Y}(\beta_N,\zeta_N)), \\ \qquad \text{if } x \in (\beta_N,\infty),\ y \in (\zeta_N,\infty). \end{cases} \tag{5}$$

In life time analysis, the hazard rate function is incredibly helpful. It is provided by a piecewise function:

$$\Theta_{X,Y}(x,y) = \begin{cases} \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_0+G_{X,Y}(x,y))}, & \text{if } x \in (-\infty,\alpha_1),\ y \in (-\infty,\vartheta_1), \\ \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_1+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1))}, & \text{if } x \in (\beta_1,\alpha_2),\ y \in (\zeta_1,\vartheta_2), \\ \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_2+G_{X,Y}(x,y)-G_{X,Y}(\beta_2,y)-G_{X,Y}(x,\zeta_2)+G_{X,Y}(\beta_2,\zeta_2))}, & \text{if } x \in (\beta_2,\alpha_3),\ y \in (\zeta_2,\vartheta_3), \\ \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_3+G_{X,Y}(x,y)-G_{X,Y}(\beta_2,y)-G_{X,Y}(x,\zeta_2)+G_{X,Y}(\beta_2,\zeta_2))}, & \text{if } x \in (\beta_3,\alpha_4),\ y \in (\zeta_3,\vartheta_4), \\ \quad \vdots & \\ \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_{N-1}+G_{X,Y}(x,y)-G_{X,Y}(\beta_2,y)-G_{X,Y}(x,\zeta_2)+G_{X,Y}(\beta_2,\zeta_2))}, & \text{if } x \in (\beta_{N-1},\alpha_N),\ y \in (\zeta_{N-1},\vartheta_N), \\ \frac{\Omega^{-1}g_{X,Y}(x,y)}{1-\Omega^{-1}(\Gamma_{N-1}+G_{X,Y}(x,y)-G_{X,Y}(\beta_2,y)-G_{X,Y}(x,\zeta_2)+G_{X,Y}(\beta_2,\zeta_2))}, & \text{if } x \in (\beta_N,\infty),\ y \in (\zeta_N,\infty), \end{cases} \tag{6}$$

for the bivariate truncation distribution with N deleted areas. Obtaining the reverse hazard rate function is the next step. This function is used to discuss the lifetime distribution with a reversed time scale and is helpful in the study of data when left-censored observations are present. If the time scale is turned around, the reverse hazard rate is only natural. Additionally, because it favors series systems and is better suited for researching parallel systems, this rate is crucial in the study of systems. The reverse hazard rate function will therefore have the following piecewise function form using our definition:

$$\widetilde{\Theta}_{X,Y}(x,y) = \begin{cases} \frac{g_{X,Y}(x,y)}{\Gamma_0+G_{X,Y}(x,y)}, & \text{if } x \in (-\infty,\alpha_1),\ y \in (-\infty,\vartheta_1), \\ \frac{g_{X,Y}(x,y)}{\Gamma_1+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1)}, & \text{if } x \in (\beta_1,\alpha_2),\ y \in (\zeta_1,\vartheta_2), \\ \frac{g_{X,Y}(x,y)}{\Gamma_2+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1)}, & \text{if } x \in (\beta_2,\alpha_3),\ y \in (\zeta_2,\vartheta_3), \\ \frac{g_{X,Y}(x,y)}{\Gamma_{N-1}+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1)}, & \text{if } x \in (\beta_{N-1},\alpha_N),\ y \in (\zeta_{N-1},\vartheta_N), \\ \frac{g_{X,Y}(x,y)}{\Gamma_N+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1)}, & \text{if } x \in (\beta_N,\infty),\ y \in (\zeta_N,\infty). \end{cases} \tag{7}$$

Some of the important features of the distribution can be used to describe some characteristics of it. Thus, if X and Y has joint PDF $f_{X,Y}(x,y)$ (3), then the product moment is given by:

$$E(X^kY^q) = \begin{cases} \sum_{j=1}^{N+1} \int_{\xi_{j-1}}^{\vartheta_j} \int_{\beta_{j-1}}^{\alpha_j} x^k y^q f_{X,Y}(x,y)dxdy & \text{if } X \text{ and } Y \text{ are continuous random variables,} \\ \sum_{j=1}^{N+1} \sum_{x=\beta_{j-1}}^{\alpha_j} \sum_{y=\xi_{j-1}}^{\vartheta_j} x^k y^q f_{X,Y}(x,y) & \text{if } X \text{ and } Y \text{ are discrete random variables,} \end{cases} \tag{8}$$

and the joint moment generating function of the N truncated areas is given by:

$$M_{X,Y}(t_1,t_2) = \begin{cases} \sum_{j=1}^{N+1} \int_{\zeta_{j-1}}^{\vartheta_j} \int_{\beta_{j-1}}^{\alpha_j} e^{t_1x+t_2y} f_{X,Y}(x,y)dxdy, & \text{if } X \text{ and } Y \text{ are continuous random variables,} \\ \sum_{j=1}^{N+1} \sum_{x=\beta_{j-1}}^{\alpha_j} \sum_{y=\zeta_{j-1}}^{\vartheta_j} e^{t_1x+t_2y} f_{X,Y}(x,y), & \text{if } X \text{ and } Y \text{ are discrete random variables.} \end{cases} \tag{9}$$

Similarly, the characteristic function of the N truncated areas is given by:

$$\Phi_{X,Y}(t_1,t_2) = \begin{cases} \sum_{j=1}^{N+1} \int_{\zeta_{j-1}}^{\vartheta_j} \int_{\beta_{j-1}}^{\alpha_j} e^{i(t_1x+t_2y)} f_{X,Y}(x,y)dxdy, & \text{if } X \text{ and } Y \text{ are continuous random variables,} \\ \sum_{j=1}^{N+1} \sum_{x=\beta_{j-1}}^{\alpha_j} \sum_{y=\zeta_{j-1}}^{\vartheta_j} e^{i(t_1x+t_2y)} f_{X,Y}(x,y), & \text{if } X \text{ and } Y \text{ are discrete random variables.} \end{cases} \tag{10}$$

3. Truncated Bivariate Normal Distribution With *N* Deleted Areas

We apply Definition 2.1 to the bivariate normal distribution. This distribution is frequently used to determine when simple mechanical and electronic equipment, components, or systems will fail. Here, we apply the aforementioned findings to this distribution in the truncated situations when two or three areas are deleted. We use MATHEMATICA 7 program to show the new curve of the distribution after applying the above definition.

Assuming that S and T are two random variables have a bivariate normal distribution with the following joint PDF:

$$q_{S,T}(s,t) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{s-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{s-\mu_1}{\sigma_1}\right)\left(\frac{t-\mu_2}{\sigma_2}\right) + \left(\frac{t-\mu_2}{\sigma_2}\right)^2\right)\right], \tag{11}$$

where $-\infty < s < \infty$, $\infty < t < \infty$, $\mu_1, \mu_2 < \infty$, $\sigma_1, \sigma_2 > 0$ and ρ is the correlation coefficient between S and T (Kenney and Keeping [5]). Without loss of generality, we consider S and T are independent random variables (i.e., $\rho = 0$) and $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, then the joint PDF (11) takes the form:

$$q_{S,T}(s,t) = q_S(s)\cdot q_T(t) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{s^2}{2}\right]\cdot\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{t^2}{2}\right] = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(s^2+t^2)\right], \tag{12}$$

where its joint CDF is given by:

$$Q_{S,T}(s,t) = Q_S(s)\cdot Q_T(t) = \frac{1}{4}\left(1+erf\left[\frac{s}{\sqrt{2}}\right]\right)\cdot\left(\left(1+erf\left[\frac{t}{\sqrt{2}}\right]\right)\right). \tag{13}$$

Firstly, in real life application we need to determine the boundaries of the area by using the definition of the double truncated distribution which presented by Ali and Nadarajah [2], Zaninetti [7], and Zaninetti and Ferraro [8]. Thus, the double truncated (from left and right) of the joint PDF (12) with two new independent random variables H and Z when $-a \le S \le a$ and $-b \le T \le b$ is,

$$\begin{aligned} g_{H,Z}(h,z) &= g_H(h)\cdot g_Z(z) \\ &= \frac{1}{2\pi}\frac{\exp\left[-\frac{h^2}{2}\right]}{(2Q_S(a)-1)}\cdot\frac{\exp\left[-\frac{z^2}{2}\right]}{(2Q_S(b)-1)}, \end{aligned} \tag{14}$$

and its CDF is,

$$\begin{aligned} G_{H,Z}(h,z) &= G_H(h)\cdot G_Z(z) \\ &= \frac{1}{4}\frac{\left(1+erf\left[\frac{h}{\sqrt{2}}\right]\right)}{(2Q_S(a)-1)}\cdot\frac{\left(\left(1+erf\left[\frac{z}{\sqrt{2}}\right]\right)\right)}{(2Q_S(b)-1)}. \end{aligned} \tag{15}$$

Consequently, from definition (1), the joint PDF of a double truncated bivariate normal distribution (which has PDF (14) and CDF (15)) with one deleted area (central region) (i.e., $N=1$) is given by:

$$\tilde{\tilde{f}}_{X,Y}(x,y) = \begin{cases} \Omega^{-1} g_{X,Y}(x,y), & \text{if } x \in (-a,\alpha_1)\cup(\beta_1,a) \text{ and } y\in(-b,\vartheta_1)\cup(\zeta_1,b), \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where $\Omega = 1-\sum_{j=1}^{2}[G_{X,Y}(\alpha_j,\vartheta_j)-G_{X,Y}(\beta_{j-1},\vartheta_j)-G_{X,Y}(\alpha_j,\zeta_{j-1})+G_{X,Y}(\beta_{j-1},\zeta_{j-1})]$, and its CDF is,

$$\tilde{\tilde{F}}_{X,Y}(x,y) = \begin{cases} \Omega^{-1} G_{X,Y}(x,y), & \text{if } x\in(-a,\alpha_1),\ y\in(-b,\vartheta_1), \\ \Omega^{-1}(\Gamma_1 + G_{X,Y}(x,y) - G_{X,Y}(\beta_1,y) - G_{X,Y}(x,\zeta_1) + G_{X,Y}(\beta_1,\zeta_1)), & \\ & \text{if } x\in(\beta_1,a),\ y\in(\zeta_1,b). \end{cases} \tag{17}$$

The corresponding marginal CDFs are,

$$\tilde{\tilde{F}}_X(x) = \begin{cases} \Omega^{-1} G_X(x), & \text{if } x\in(-a,\alpha_1), \\ \Omega^{-1}(G_{X,Y}(\alpha_1,\vartheta_1) + G_X(x) - G_{X,Y}(x,\zeta_1) + G_{X,Y}(\beta_1,\zeta_1)), & \text{if } x\in(\beta_1,a) \end{cases} \tag{18}$$

and

$$\tilde{\tilde{F}}_Y(y) = \begin{cases} \Omega^{-1} G_Y(y), & \text{if } y\in(-b,\vartheta_1), \\ \Omega^{-1}(G_{X,Y}(\alpha_1,\vartheta_1) + G_Y(y) - G_{X,Y}(\beta_1,y) + G_{X,Y}(\beta_1,\zeta_1)), & \text{if } y\in(\zeta_1,b). \end{cases} \tag{19}$$

If we assume that, $a=b=5, \alpha_1=\vartheta_1=-1$ and $\beta_1=\zeta_1=1$ then the joint PDF (16) becomes,

$$\tilde{\tilde{f}}_{X,Y}(x,y) = \begin{cases} \frac{\Omega^{-1}}{2\pi(2Q_S(5)-1)^2}\cdot\exp\left[-\frac{x^2+y^2}{2}\right], & \text{if } x,y\in(-5,-1)\cup(1,5), \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where $\Omega = 1-\sum_{j=1}^{2}[G_{X,Y}(\alpha_j,\vartheta_j)-G_{X,Y}(\beta_{j-1},\vartheta_j)-G_{X,Y}(\alpha_j,\zeta_{j-1})+G_{X,Y}(\beta_{j-1},\zeta_{j-1})]$, $\alpha_2=\vartheta_2=5$ and $\beta_0=\zeta_0=-5$, see Figure 1.

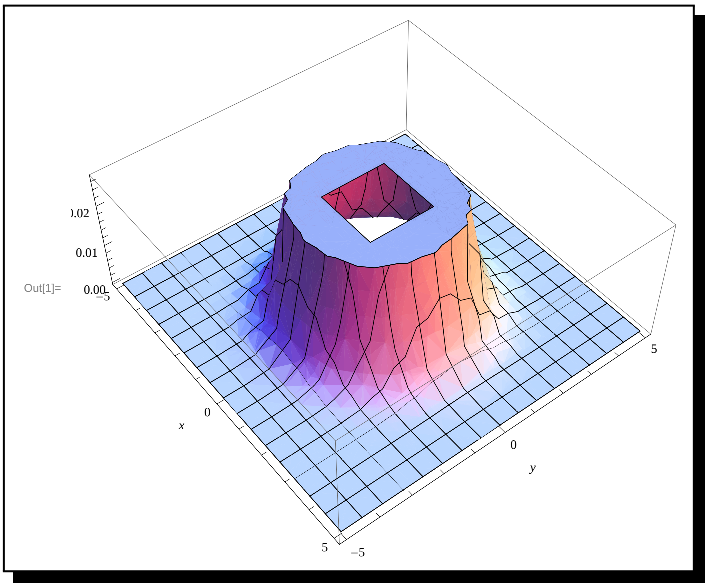

Figure 1. Joint double truncated PDF (20) with deleted one area

Also, the joint double truncated CDF with deleted one area becomes,

$$\tilde{\tilde{F}}_{X,Y}(x,y)=\begin{cases}\frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right), & \text{if } x,y\in(-5,-1),\\ \frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(\Gamma_1+\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)-\left(1+erf\left[\frac{\beta_1}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right),\right. & \\ \left.\quad -\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{\zeta_1}{\sqrt{2}}\right]\right)\right)+\left(1+erf\left[\frac{\beta_1}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{\zeta_1}{\sqrt{2}}\right]\right)\right)\right), & \text{if } x,y\in(1,5),\end{cases} \tag{21}$$

where $\Gamma_1=G_{X,Y}(-1,-1)-G_{X,Y}(-5,-1)-G_{X,Y}(-1,-5)+G_{X,Y}(-5,-5)$, and it is appeared as in Figure 2.

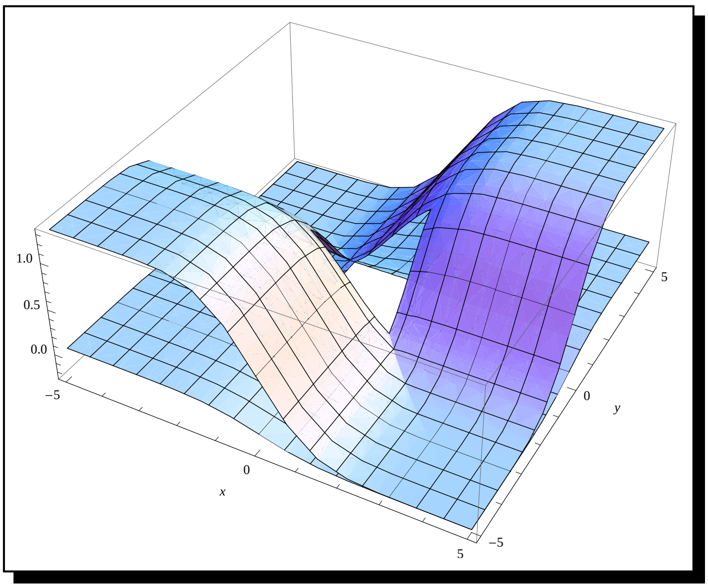

Figure 2. Joint double truncated CDF (21) with deleted one area

Also, from (5) and (6), the survival and the hazard rate functions are given by:

$$\Lambda_{X,Y}(x,y)=\begin{cases}1-\frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right), & \text{if } x,y\in(-5,-1),\\ 1-\frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(\Gamma_1+\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)-\left(1+erf\left[\frac{\beta_1}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)\right.\\ \quad\left.-\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{\zeta_1}{\sqrt{2}}\right]\right)\right)+\left(1+erf\left[\frac{\beta_1}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{\zeta_1}{\sqrt{2}}\right]\right)\right)\right), & \text{if } x,y\in(1,5),\end{cases} \tag{22}$$

and

$$\widetilde{\Lambda}_{X,Y}(x,y)=\begin{cases}\dfrac{\Omega^{-1}\exp\left[-\frac{x^2+y^2}{2}\right]}{2\pi(2Q_S(5)-1)^2\left(1-\frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)\right)}, & \text{if } x,y\in(-5,-1),\\ \dfrac{\Omega^{-1}\exp\left[-\frac{x^2+y^2}{2}\right]}{\left(\begin{array}{l}2\pi(2Q_S(5)-1)^2\left(1-\frac{\Omega^{-1}}{4(2Q_S(5)-1)^2}\left(\Gamma_1+\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)\right.\right.\\ \quad\left.\left.-\left(1+erf\left[\frac{1}{\sqrt{2}}\right]\right)\left[\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)-\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)+\left(\left(1+erf\left[\frac{1}{\sqrt{2}}\right]\right)\right)\right]\right)\right)\end{array}\right)}, & \text{if } x,y\in(-5,-1),\end{cases} \tag{23}$$

respectively, where $\Gamma_1=G_{X,Y}(-1,-1)-G_{X,Y}(-5,-1)-G_{X,Y}(-1,-5)+G_{X,Y}(-5,-5)$, see Figures 3 and 4.

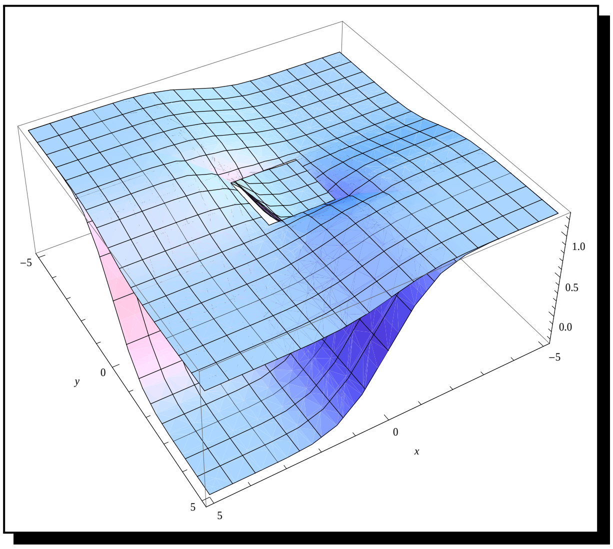

Figure 3. Joint survival truncated function (22) with deleted one area

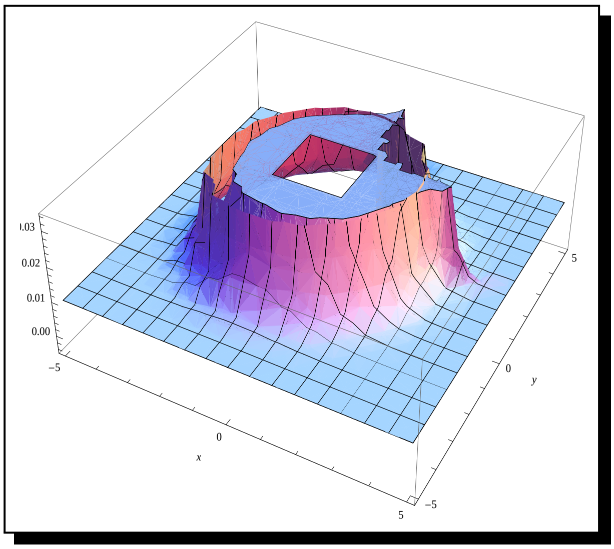

Figure 4. Joint hazard rate truncated function (23) with deleted one area

Also, from (7) the reversed hazard rate function is given by,

$$\widetilde{\Lambda}_{X,Y}(x,y)=\begin{cases}\dfrac{\exp\left[-\frac{x^2+y^2}{2}\right]}{\frac{\pi}{2}\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)}, & \text{if } x,y\in(-5,-1),\\ \dfrac{\exp\left[-\frac{x^2+y^2}{2}\right]}{\frac{\pi}{2}\left(\Gamma_1+\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)\left(\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)\right)-\left(1+erf\left[\frac{1}{\sqrt{2}}\right]\right)\left[\left(1+erf\left[\frac{y}{\sqrt{2}}\right]\right)-\left(1+erf\left[\frac{x}{\sqrt{2}}\right]\right)+\left(\left(1+erf\left[\frac{1}{\sqrt{2}}\right]\right)\right)\right]\right)}, & \text{if } x,y\in(-5,-1),\end{cases} \tag{24}$$

see Figure 5.

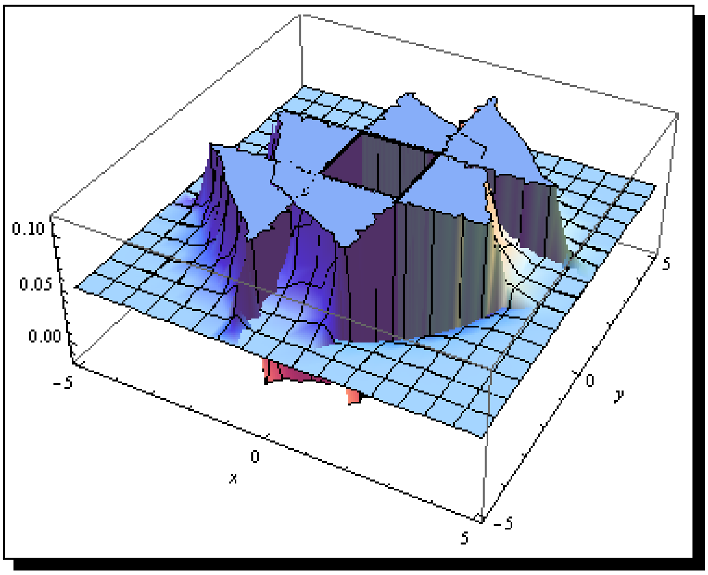

Figure 5. Joint reversed hazard rate truncated function (24) with deleted one area

And, the product moment is given by,

$$\begin{aligned}E(X^kY^q) &= \int_{-5}^{-1}\int_{-5}^{-1} x^k y^q \frac{\Omega^{-1}}{2\pi(2Q_S(5)-1)^2}\exp\left[-\frac{x^2+y^2}{2}\right]dx\,dy\\&\quad+\int_{1}^{5}\int_{1}^{5} x^k y^q \frac{\Omega^{-1}}{2\pi(2Q_S(5)-1)^2}\exp\left[-\frac{x^2+y^2}{2}\right]dx\,dy\\&=\frac{\Omega^{-1}}{2\pi(2Q_S(5)-1)^2}\left[\left(\int_{-5}^{-1} x^k\exp\left[-\frac{x^2}{2}\right]dx\right)\left(\int_{-5}^{-1} y^q\exp\left[-\frac{y^2}{2}\right]dy\right)\right.\\&\quad\left.+\left(\int_{1}^{5} x^k\exp\left[-\frac{x^2}{2}\right]dx\right)\left(\int_{1}^{5} y^q\exp\left[-\frac{y^2}{2}\right]dy\right)\right].\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}E(X^kY^q) &= \frac{\Omega^{-3}}{2\pi(2Q_S(5)-1)^2}\left[\left((-1)^kG_X(-1)-k\int_{-5}^{-1}x^{k-1}G_X(x)dx\right)\right.\\&\quad\cdot\left((-1)^qG_Y(-1)-q\int_{-5}^{-1}y^{q-1}G_Y(y)dy\right)\\&\quad+\left(5^k(G_X(5)-G_{X,Y}(5,1))-G_X(1)+G_{X,Y}(1,1)\right.\\&\quad\left.-k\int_{1}^{5}x^{k-1}(G_X(x)-G_{X,Y}(x,1)+G_{X,Y}(1,1))dx\right)\\&\quad\times\left(5^q(G_Y(5)-G_{X,Y}(5,1))-G_Y(1)+G_{X,Y}(1,1)\right.\\&\quad\left.\left.-q\int_{1}^{5}y^{q-1}(G_Y(y)-G_{X,Y}(1,y)+G_{X,Y}(1,1))dy\right)\right].\end{aligned}$$

Similarly, using (9) and (10) one can obtain the joint moment generating and the characteristic functions, respectively.

4. Concluding Remarks and Future Work

We provided a new truncation definition for the bivariate probability distributions, i.e., N areas deleted. Applications in engineering and industry can benefit from the improved distribution. Its purpose was to exclude certain data value intervals from a domain consisting of two independent or dependent random variables. We provide functions for distribution, likelihood, survival, hazard, reversed hazard, moment generating, and characteristic. This definition is crucial in determining how the probability of a deleted area is distributed across the remaining portions. We used the acquired results to the bivariate normal distribution, which we deemed as an application for the new concept. This new notion could find extensive use in lifespan modelling, particularly in the development of suitable search algorithms to detect the lost targets in bounded known areas.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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