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Research Article

Second Hankel Determinant for Certain Subclass of *p*-Valent Analytic Function

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Abstract. The aim of this paper is to find sharp upper bound for the Second Hankel determinant and Fekete-Szegö functional for certain subclass of p-valent analytic function.

Keywords. p-valent analytic function, Second Hankel determinant, Fekete-Szegö functional

Mathematics Subject Classification (2020). 30C45, 30C50

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions f of the form

$$f(z) = z^p + a_{1+p}z^{1+p} + a_{2+p}z^{2+p} + \dots \tag{1.1}$$

in the unit open disc $U = \{z : |z| < 1\}$. Let S be the subclass of $\mathcal{A}_1 = \mathcal{A}$, consisting of univalent functions.

The Hankel determinant for $k \ge 1$ and $n \ge 1$ was defined by Pommerenke [13] as follows:

$$H_k(n) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{n+k-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}.$$

This Hankel determinant has been studied by many researchers in the literature. For example, Janteng *et al.* [5] studied the Hankel determinant for the classes of starlike and convex functions. Also, Janteng *et al.* [6] discussed the Hankel determinant problems for the functions whose derivative has a positive real part. Yavuz [17] studied the analytic function defined by Ruscheweyh derivative [14] and got an upper bound for the second Hankel determinant for it in the unit disc. Krishna and Ramreddy [7] obtained an upper bound on the second Hankel determinant for 'p-valent' starlike and convex functions by using Toeplitz determinants. Kund and Mishra [8] studied a class of analytic functions related to the Carlson-Shaffer operator [1] in the unit disc and estimated the second Hankel determinant [11] for this class.

We note in particular that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

$H_2(1) = a_1a_3 - a_2^2$ is the well known Fekete-Szegö functional. Several authors have obtained $H_2(2) = a_2a_4 - a_3^2$ (second Hankel determinant) for different subclasses of univalent and multivalent functions.

Definition 1.1 ([4])**.** The q-differential operator was introduced by Jackson [8] is defined as

$$D_qf(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \in U. \tag{1.2}$$

In addition, the q-derivative at zero is $D_qf(0) = D_{q^{-1}}f(0)$, for $|q| > 1$. Equivalently, eq. (1.2) can be written as

$$D_qf(z) = [p]_q z^{p-1} + \sum_{n=1+p}^{\infty} [n]_q \cdot a_n \cdot z^{n-1}, \quad z \neq 0, \tag{1.3}$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1. \end{cases}$$

In the present paper, we have applied q-difference operator and we introduce some new subclass of p-valent functions as given in Definition 1.2 and obtained $H_2(1+p)$ given by,

$$H_2(1+p) = \begin{vmatrix} a_{1+p} & a_{2+p} \\ a_{2+p} & a_{3+p} \end{vmatrix} = a_{1+p}a_{3+p} - a_{2+p}^2,$$

and we seek sharp upper bound to the functional $|a_{1+p}a_{3+p} - a_{2+p}^2|$ for function f in (1.1), when it belongs to the new subclass, given in Definition 1.2.

Definition 1.2. We say that a function $f \in A_p$ exist in the class $\widetilde{RT}_{p,q}$ with $p \in N$ consisting of p-valent functions, if it satisfies the condition

$$\text{Re}\left[\frac{[p]_q \cdot z^{p-1}}{D_qf(z)}\right] > 0, \quad \forall\, z \in U. \tag{1.4}$$

The following lemmas are needed to prove our result.

2. Lemmas

Let $h \in Q$, where Q is a class of functions, such that

$$h(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{2.1}$$

where $h(z)$ is regular in the open unit disc U. Also, $\operatorname{Re} h(z) > 0$ for any $z \in U$. Here we observe that $h(z)$ is a 'Carathéodory' function.

Lemma 2.1 ([12, 15]). *If $h \in Q$ given by* (2.1)*, the following sharp estimate holds:*

$$|b_k| \le 2, \quad \text{for } k \ge 1,$$

and for any complex number μ, we have

$$|b_2 - \xi b_1^2| \le 2\max\{1, |2\xi - 1|\}. \tag{2.2}$$

Lemma 2.2 ([3]). *If $h \in Q$, then*

$$2b_2 = b_1^2 + x(4 - b_1^2) \tag{2.3}$$

and

$$4b_3 = b_1^3 + 2b_1(4 - b_1^2)x - b_1(4 - b_1^2)x^2 + 2(4 - b_1^2)(1 - |x|^2)z, \tag{2.4}$$

for some complex valued x, z such that $|x| \le 1$ and $|z| \le 1$.

3. Main Results

Theorem 3.1. *If $f(z) \in \widetilde{RT}_{p,q}$, with $p \in N$, then*

$$a_{1+p} = -\frac{b_1[p]_q}{[1+p]_q}, \tag{3.1}$$

$$a_{2+p} = \frac{(b_1^2 - b_2)[p]_q}{[2+p]_q}, \tag{3.2}$$

$$a_{3+p} = -[p]_q \frac{(b_3 - 2b_1 b_2 + b_1^3)}{[3+p]_q}. \tag{3.3}$$

Proof. Let $f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n \in \widetilde{RT}_{p,q}$. By (1.4), there exists $h \in Q$ with $\operatorname{Re}(h(z)) > 0$ such that

$$[p]_q \cdot z^{p-1} = h(z) \cdot D_q f(z). \tag{3.4}$$

Now by substituting $h(z)$ and $D_q f(z)$ in (3.4), we get

$$[p]_q \cdot z^{p-1} = \left\{1 + \sum_{n=1}^{\infty} b_n z^n\right\}\left\{[p]_q z^{p-1} + \sum_{n=1+p}^{\infty} [n]_q a_n z^{n-1}\right\}.$$

Simple computation, we have

$$\begin{aligned} 0 &= \{b_1[p]_q + [1+p]_q a_{1+p}\}z^p + \{[2+p]_q a_{2+p} + b_1[1+p]_q a_{1+p}\}z^{1+p} \\ &\quad + \{b_3[p]_q + b_2[1+p]_q a_{1+p} + b_1[2+p]_q a_{2+p} + [3+p]_q a_{3+p}\}z^{2+p} + \cdots. \end{aligned} \tag{3.5}$$

□

On comparing the coefficients of like powers of z^p, z^{1+p} and z^{2+p} respectively in (3.5), we get

$$a_{1+p} = -\frac{b_1[p]_q}{[1+p]_q}, \quad a_{2+p} = \frac{(b_1^2-b_2)[p]_q}{[2+p]_q}, \quad a_{3+p} = -[p]_q\frac{(b_3-2b_1b_2+b_1^3)}{[3+p]_q}. \tag{3.6}$$

By using Lemma 2.1 we get the desired inequalities.

Theorem 3.2. *If $f(z)\in\widetilde{RT}_{p,q}$, with $p\in N$, then*

$$|a_{2+p}-\mu a_{1+p}^2| \le \frac{2[p]_q}{[p+2]_q}\max\left\{1,\left|1+\frac{2\mu[p]_q[p+2]_q}{([p+1]_q)^2}\right|\right\}.$$

Proof. Using a_{1+p} and a_{2+p}, we get

$$|a_{2+p}-\mu a_{1+p}^2| = \left|\frac{(b_1^2-b_2)[p]_q}{[2+p]_q}-\mu\frac{b_1^2[p]_q^2}{[1+p]_q^2}\right|$$
$$= \frac{[p]_q}{[p+2]_q}\left|b_2-b_1^2\left(1-\frac{\mu[p]_q[p+2]_q}{([p+1]_q)^2}\right)\right|.$$

Application of (2.2), leads us to

$$|a_{2+p}-\mu a_{1+p}^2| \le \frac{2[p]_q}{[p+2]_q}\max\left\{1,\left|1+\frac{2\mu[p]_q[p+2]_q}{([p+1]_q)^2}\right|\right\}.$$ □

Corollary 3.3. *If $f(z)\in\widetilde{RT}_{p,q}$ and $\mu=1$, then*

$$|a_{2+p}-a_{1+p}^2| \le \frac{2[p]_q}{[p+2]_q}\left\{1+\frac{2[p]_q[p+2]_q}{([p+1]_q)^2}\right\}.$$

Theorem 3.4. *If $f(z)\in\widetilde{RT}_{p,q}$, with $p\in N$, then*

$$|a_{1+p}a_{3+p}-a_{2+p}^2| \le \left[\frac{2[p]_q}{[2+p]_q}\right]^2$$

and the inequality is sharp.

Proof. Substituting the values of a_{1+p}, a_{2+p} and a_{3+p} from (3.6) in $|a_{1+p}a_{3+p}-a_{2+p}^2|$ for $f\in\widetilde{RT}_{p,q}$ and on simplification, we get

$$|a_{1+p}a_{3+p}-a_{2+p}^2| = ([p]_q)^2\left|\frac{([2+p]_q)^2b_1b_3-2b_1^2b_2\{([2+p]_q)^2-[1+p]_q[3+p]_q\}}{[1+p]_q([2+p]_q)^2[3+p]_q}\right.$$
$$\left.-\frac{b_2^2[1+p]_q[3+p]_q}{[1+p]_q([2+p]_q)^2[3+p]_q}-\frac{b_1^4\{[1+p]_q[3+p]_q-([2+p]_q)^2\}}{[1+p]_q([2+p]_q)^2[3+p]_q}\right|$$

which is equivalent to

$$|a_{1+p}a_{3+p}-a_{2+p}^2| = \frac{([p]_q)^2|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4|}{[1+p]_q([2+p]_q)^2[3+p]_q}, \tag{3.7}$$

where

$$\left.\begin{aligned} d_1 &= ([2+p]_q)^2, & d_2 &= -2\{([2+p]_q)^2-[1+p]_q[3+p]_q\},\\ d_3 &= -[1+p]_q[3+p]_q, & d_4 &= -\{[1+p]_q[3+p]_q-([2+p]_q)^2\}. \end{aligned}\right\} \tag{3.8}$$

Substituting for b_2 and b_3 in (3.7), we get

$$\begin{aligned}|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4\\ =\Big|d_1b_1\times\frac{1}{4}\{b_1^3+2b_1(4-b_1^2)x-b_1(4-b_1^2)x^2+2(4-b_1^2)(1-|x^2|z)\}\\ +d_2b_1^2\times\frac{1}{2}\{b_1^2+x(4-b_1^2)\}+d_3\times\frac{1}{4}\{b_1^2+x(4-b_1^2)\}^2+d_4b_1^4\Big|.\end{aligned}$$

Since $|z|<1$ and applying triangle inequality the above expression reduces to

$$\begin{aligned}4|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4|\le|(d_1+2d_2+d_3+4d_4)b_1^4+2d_1b_1(4-b_1^2)\\ +2(d_1+d_2+d_3)b_1^2(4-b_1^2)|x|\\ -\{(d_1+d_3)b_1^2+2d_1b_1-4d_3\}(4-b_1^2)|x|^2|.\end{aligned}\tag{3.9}$$

From (3.8), we can write

$$\left.\begin{aligned}&d_1+2d_2+d_3+4d_4=([2+p]_q)^2-[1+p]_q[3+p]_q,\quad d_1=([2+p]_q)^2,\\ &d_1+d_2+d_3=-[([2+p]_q)^2-[1+p]_q[3+p]_q].\end{aligned}\right\}\tag{3.10}$$

Thus, we have

$$(d_1+d_3)b_1^2+2d_1b_1-4d_3=\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2+2([2+p]_q)^2b_1+4[1+p]_q[3+p]_q\,.\tag{3.11}$$

By writing,

$$\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2+2([2+p]_q)^2b_1+4[1+p]_q[3+p]_q$$

in the form $(b_1+a)(b_1+c)$.

Since $b_1\in[0,2]$ and $(b_1+a)(b_1+c)\ge(b_1-a)(b_1-c)$, where $a,c\ge0$, then the above expression becomes

$$\begin{aligned}-[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2+2([2+p]_q)^2b_1+4[1+p]_q[3+p]_q]\\ \le-[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2-2([2+p]_q)^2b_1+4[1+p]_q[3+p]_q].\end{aligned}\tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned}-\{(d_1+d_3)b_1^2+2d_1b_1-4d_3\}\\ \le-[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2-2([2+p]_q)^2b_1+4[1+p]_q[3+p]_q].\end{aligned}\tag{3.13}$$

Using (3.10) and (3.13) in (3.9), we get

$$\begin{aligned}4|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4|\le|\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^4+2([2+p]_q)^2b_1(4-b_1^2)\\ -2\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2(4-b_1^2)|x|\\ -[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b_1^2-2([2+p]_q)^2b_1\\ +4[1+p]_q[3+p]_q](4-b_1^2)|x|^2|.\end{aligned}$$

Choose $b_1=b\in[0,2]$, using triangle inequality and replace $|x|$ by ρ on the right-hand side of the above inequality, we get

$$\begin{aligned}4|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4|\le[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^4+2([2+p]_q)^2b(4-b^2)\\ +2\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2(4-b^2)\rho\\ +[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2-2([2+p]_q)^2b\\ +4[1+p]_q[3+p]_q](4-b^2)\rho^2]\\ =G(b,\rho).\end{aligned}\tag{3.14}$$

Let

$$\begin{aligned}G(b,\rho) &= \{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^4+2([2+p]_q)^2b(4-b^2)\\ &\quad +2\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2(4-b^2)\rho\\ &\quad +[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2-2([2+p]_q)^2b\\ &\quad +4[1+p]_q[3+p]_q](4-b^2)\rho^2. \end{aligned} \tag{3.15}$$

Now we find $[G(b,\rho)]_{\max}$ in the region $[0,2]\times[0,1]$. Differentiating $G(b,\rho)$ partially with respect to ρ, we get

$$\begin{aligned}G_\rho &= 2[\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2+\{\{([2+p]_q)^2-[1+p]_q[3+p]_q\}b^2-2([2+p]_q)^2b\\ &\quad +4[1+p]_q[3+p]_q\}\rho](4-b^2). \end{aligned} \tag{3.16}$$

For $\rho\in(0,1)$, and for a fixed b with $b\in(0,2)$, $p\in N$, we observe in (3.16) that $G_\rho>(0)$. Hence, $G(b,\rho)$ will be an increasing function of ρ and thereby it will not attain a maximum value no where in the region $[0,2]\times[0,1]$. Moreover, for a fixed $b\in[0,2]$, we have

$$\max_{0\le\rho\le1} G(b,\rho)=G(b,1)=F(b). \tag{3.17}$$

Therefore,

$$\begin{aligned}G(b,1)=F(b) &= -2b^4\{([2+p]_q)^2-[1+p]_q[3+p]_q\}-4b^2\{-3([2+p]_q)^2+4[1+p]_q[3+p]_qt\}\\ &\quad +16[1+p]_q[3+p]_q, \end{aligned} \tag{3.18}$$

$$F'(b)=-8b[b^2\{([2+p]_q)^2-[1+p]_q[p+3]_q\}+\{-3([2+p]_q)^2+4[1+p]_q[3+p]_q\}]. \tag{3.19}$$

In (3.19), we see that $F'(b)\le 0$, for every $b\in[0,2]$. Hence, $F(b)$ is a decreasing function in $[0,2]$, which attains maximum value at $b=0$ only. From (3.18), we have

$$F_{\max}=F(0)=16[p+1]_q[p+3]_q. \tag{3.20}$$

From (3.9) and (3.20), we get

$$|d_1b_1b_3+d_2b_1^2b_2+d_3b_2^2+d_4b_1^4|\le 4[p+1]_q[p+3]_q. \tag{3.21}$$

From (3.7) and (3.21), we obtain

$$|a_{1+p}a_{3+p}-a_{2+p}^2|\le\left[\frac{2[p]_q}{[2+p]_q}\right]^2. \tag{3.22}$$

By choosing $b_1=b=0$ and setting $x=1$ in (2.3) and (2.4), we see that $b_2=2$ and $b_3=0$. Substituting for b_1, b_2, and b_3 in (3.21) along with the values in (3.8), we see that equality is obtained. Therefore the result is sharp. Hence the proof. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM Journal on Mathematical Analysis* **15**(4) (1984), 737 – 745, DOI: 10.1137/0515057.

[2] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften series, Springer, New York, xiv + 384 pages (1983).

[3] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, University of California Press, Berkeley, 262 pages (1958), DOI: 10.1525/9780520355408.

[4] F. H. Jackson, XI – On q-functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh* **46**(2) (1909), 253 – 281, DOI: 10.1017/S0080456800002751.

[5] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *Journal of Inequalities in Pure & Applied Mathematics* **7**(2) (2006), Article 50, URL: https://eudml.org/doc/128719.

[6] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, *International Journal of Mathematical Analysis* **13**(1) (2007), 619 – 625.

[7] D. V. Krishna and T. Ramreddy, Hankel determinant for p-valent starlike and convex functions of order α, *Novi Sad Journal of Mathematics* **42**(2) (2012), 89 – 96, URL: https://sites.dmi.uns.ac.rs/nsjom/Papers/42_2/NSJOM_42_2_089_102.pdf.

[8] S. N. Kund and A. K. Mishra, The second Hankel determinant for a class of analytic functions associated with the Carlson-Shaffer operator, *Tamkang Journal of Mathematics* **44**(1) (2013), 73 – 82, DOI: 10.5556/j.tkjm.44.2013.963.

[9] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, *Proceedings of the American Mathematical Society* **87**(2) (1983), 251 – 257, DOI: 10.1090/S0002-9939-1983-0681830-8.

[10] T. H. MacGregor, Functions whose derivative have a positive real part, *Transactions of the American Mathematical Society* **104**(3) (1962), 532 – 537, DOI: 10.2307/1993803.

[11] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, *Transactions of the American Mathematical Society* **223** (1976), 337 – 346, DOI: 10.2307/1997533.

[12] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *Journal of the London Mathematical Society* **s1-41** (1996), 111 – 122, DOI: 10.1112/jlms/s1-41.1.111.

[13] C. Pommerenke, Univalent functions, in: *Studia Mathematica Mathematische Lehrbücher*, Vandenhoeck and Ruprecht (1975).

[14] B. Simon, *Orthogonal Polynomials on the Unit Circle: Part 1: Classical Theory; Part 2: Spectral Theory*, American Mathematical Society Colloquium Publications, Vol. 54, American Mathematical Society, Providence, RI (2005).

[15] S. Ruscheweyh, New criteria for univalent functions, *Proceedings of the American Mathematical Society* **49**(1) (1975), 109 – 115.

[16] D. V. Krishna, B. Venkateshwarulu and T. R. Reddy, Coefficient inequality for certain subclass of p-valent analytic functions whose reciprocal derivative has a positive real part, *Matematicki Vesnik* **68**(2) (2016), 87 – 92.

[17] T. Yavuz, Second Hankel determinant for analytic functions defined by Ruscheweyh derivative, *International Journal of Analysis and Applications* **8**(1) (2015), 63 – 68, URL: https://etamaths.com/index.php/ijaa/article/view/511.