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Research Article

Characteristics of Soft Semitrees: Enhancing the Soft Semigraph Framework

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Abstract. Molodtsov pioneered the concept of soft sets, offering a method to classify elements of a universe based on a specified set of parameters. This approach serves to model vagueness and uncertainty. Semigraphs are a generalised form of graphs introduced by Sampathkumar *et al.* (*Semigraphs and Their Applications*, Academy of Discrete Mathematics and Applications, Mysore, India, 337 pages (2019)). Integrating soft set theory into semigraphs led to the creation of soft semigraphs. Due to its adeptness in handling parameterisation, the field of soft semigraph theory is rapidly evolving. In this study, we introduce the concept of soft semitrees and explore some of their characteristics.

Keywords. Semigraph, Soft semigraph, Soft semitree

Mathematics Subject Classification (2020). 05C99

1. Introduction

Traditional methods in formal modelling, reasoning, and computation are generally deterministic, clear, and precise. However, fields such as engineering, medicine, economics, and social sciences often deal with data that is not clearly defined. This introduces various uncertainties that challenge conventional methods. The fuzzy set theory addresses one type of uncertainty, known as "*Fuzziness*", which occurs when elements partially belong to a set. Although it effectively manages uncertainties related to vagueness or partial membership, it does not cover all uncertainties in real-world problems. In 1999, Molodtsov [9] introduced soft set theory, which offers a more practical approach than established theories like probability or fuzzy set theory due to its versatility. For instance, fuzzy set theory lacks adequate parameterization tools. Researchers like Maji *et al.* [7,8] and Saleh *et al.* [11,12] have expanded on soft set theory to solve decision-making problems.

Thumbakara and George [17] introduced the concept of soft graphs. Akram and Nawas [1,3] modified the definition of soft graphs, and they made further advancements by introducing fuzzy soft graphs, strong fuzzy soft graphs, complete fuzzy soft graphs, and regular fuzzy soft graphs, exploring their properties and potential applications (see [2] and references cited therein). Akram and Zafar [5] pioneered the concepts of soft trees and fuzzy soft trees. The fuzzy soft theory combines the characteristics of fuzzy sets and soft sets to handle problems with uncertain data. Nawaz and Akram [10] explored the applications of fuzzy soft graphs in analyzing oligopolistic competition among wireless internet service providers in Malaysia. Additionally, Akram and Shahzadi [4] proposed a decision-making approach using Pythagorean Dombi fuzzy soft graphs.

Thenge *et al.* [14–16] have contributed to the study of soft graphs, a growing field in graph theory due to their utility in handling parameterization. George *et al.* [6] have studied various concepts in soft graphs and introduced soft hypergraphs, soft directed graphs, soft directed hypergraphs, and soft disemigraphs, examining their properties. They extended the idea of graph products to soft graphs and explored various product operations in soft graphs and soft directed graphs. Sampathkumar *et al.* [13] introduced semigraphs, a broader version of graphs, where the order of vertices within edges is maintained. When represented on a plane, semigraphs resemble conventional graphs. In 2023, George *et al.* [6] introduced soft semigraphs by applying soft set principles to semigraphs and defined several soft semigraph operations. They also introduced product operations, connectedness, and various degrees, graphs, and matrices associated with soft semigraphs. Furthermore, they presented Eulerian and Hamiltonian soft semigraphs, the closure of a soft semigraph and various kinds of isomorphisms among soft semigraphs, including s-isomorphism, sev-isomorphism, se-isomorphism, and sa-isomorphism. In this paper, we introduce the concept of soft semitrees and explore some of their characteristics.

2. Preliminaries

In this section, we lay the foundation for comprehending soft sets, semigraphs, and soft semigraphs. We define fundamental concepts such as partial edges and p-part, which are crucial to the structure of soft semigraphs. Finally, we provide a brief overview of topics, including connectedness in soft semigraphs and bipartite soft semigraphs.

2.1 Semigraph

The notion of a semigraph was introduced by Sampathkumar *et al.* [13] as follows. A *semigraph* G is a pair (V,X) where V is a nonempty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions:

(1) Any two edges have at most one vertex in common.

(2) Two edges $(v_1, v_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_m)$ are considered to be equal if and only if

(a) $m = n$, and

(b) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Let $G = (V,X)$ be a semigraph and $E = (v_1, v_2, \dots, v_n)$ be an edge of G. Then, v_1 and v_n are the *end vertices* of E and v_i, $2 \leq i \leq n-1$ are the *middle vertices*(*or m-vertices*) of E. If a vertex v of a semigraph G appears only as an end vertex, then it is called an *end vertex*. If a vertex v is only a middle vertex then it is a *middle vertex* or *m-vertex* while a vertex v is called *middle-cum-end vertex or* (m,e)*-vertex* if it is a middle vertex of some edge and an end vertex of some other edge. A *subedge* of an edge $E = (v_1, v_2, \dots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$. We say that the subedge E' is *induced* by the set of vertices $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. A *partial edge* of $E = (v_1, v_2, \dots, v_n)$ is a $(j-i+1)$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, \dots, v_j)$, where $1 \leq i < j \leq n$. $G' = (V', X')$ is a *partial semigraph* of a semigraph G if the edges of G' are partial edges of G. Two vertices u and v in a semigraph G are said to be *adjacent* if they belong to the same edge. If u and v are adjacent and consecutive in order, then they are said to be *consecutively adjacent*. u and v are said to be *e-adjacent* if they are the end vertices of an edge and 1*e-adjacent* if both the vertices u and v belong to the same edge and at least one of them is an end vertex of that edge.

Theorem 2.1 ([13])**.** *Let G be a semigraph with p vertices and q edges E_i, $1 \leq i \leq q$ and k p-parts. Then, G contains no cycles if and only if $p + q = \sum_{i=1}^{q} |E_i| + k$.*

2.2 Soft Set

In 1999, Molodtsov [9] initiated the concept of soft sets. Let U be an initial universe set and let A be a set of parameters. A pair (F,A) is called a soft set (over U) if and only if F is a mapping of A into the set of all subsets of the set U. That is, $F : A \to \mathcal{P}(U)$.

2.3 Soft Semigraph

George *et al.* [6] introduced soft semigraph by applying the concept of soft set in semigraph as follows: Let $G^* = (V,X)$ be a semigraph having vertex set V and edge set X. Consider a subset V_1 of V. Then a partial edge formed by some or all vertices of V_1 is said to be a *maximum partial edge* or *mp edge* if it is not a partial edge of any other partial edge formed by some or all vertices of V_1. Let X_p be the collection of all partial edges of the semigraph G and A be a nonempty set. Let a subset R of $A \times V$ be an arbitrary relation from A to V. We define a mapping Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy\}$, $\forall\, x \in A$, where $\mathcal{P}(V)$ denotes the power set of V. Then the pair (Q,A) is a soft set over V. Also, define a mapping W from A to $\mathcal{P}(X_p)$ by

$W(x) = \{mp\ edges\langle Q(x)\rangle\}$, where $\{mp\ edges\langle Q(x)\rangle\}$ denotes the set of all mp edges that can be formed by some or all vertices of $Q(x)$ and $\mathcal{P}(X_p)$ denotes the power set of X_p. The pair (W,A) is a soft set over X_p. Then, we can define a soft semigraph as follows:

The 4-tuple $G = (G^*,Q,W,A)$ is called a *soft semigraph* of G^* if the following conditions are satisfied:

(1) $G^* = (V,X)$ is a semigraph having vertex set V and edge set X,

(2) A is the nonempty set of parameters,

(3) (Q,A) is a soft set over V,

(4) (W,A) is a soft set over X_p,

(5) $H(a) = (Q(a),W(a))$ is a partial semigraph of G^*, $\forall\ a \in A$.

Let $G^* = (V,X)$ be a semigraph and $G = (G^*,Q,W,A)$ be a soft semigraph of G^* which is also given by $\{H(x) : x \in A\}$. Then, the partial semigraph $H(x)$ corresponding to any parameter x in A is called a *p-part* of the soft semigraph G. An edge present in a soft semigraph G of G^* is called an *f-edge*. It may be a partial edge of some edge in G^* or an edge in G^*. A partial edge of any f-edge of a soft semigraph G is called a *p-edge* of G. An f-edge is a p-edge of itself. An f-edge or a p-edge of a soft semigraph G is called an *fp-edge* of G. Let $H(x)$ be any p-part of the soft semigraph G, and let v be any vertex in $H(x)$. Then, the *p-part degree* of v in $H(x)$ denoted by $\deg v[H(x)]$, is defined as the number of f-edges having v as an end vertex in $H(x)$. *Degree* of a vertex v in a soft semigraph G, denoted by $\deg v$ is defined as $\deg v \max\{\deg v[H(x)] : x \in A\}$, where $\deg v[H(x)]$ denotes the p-part degree of v in $H(x)$.

2.4 Connectedness in Soft Semigraph

George *et al.* introduced the concept of connectedness in soft semigraph as follows: A *soft walk* or an *s-walk* in a soft semigraph G is an alternating sequence $v_0E_1v_2E_2\ldots v_{n-1}E_nv_n$ of vertices and *fp*-edges, beginning with the vertex v_0 and ending with the vertex v_n such that v_{i-1} and v_i are the end vertices or partial end vertices of the *fp*-edge $E_i, 1 \le i \le n$. This s-walk is called a $v_0 - v_n$ s-walk. Here, v_0 is called the *origin* and v_n is called the *terminus* of the s-walk. A $v_0 - v_n$ s-walk is *closed* if $v_0 = v_n$. Otherwise, it is called *open*. Also, we can denote a $v_0 - v_n$ s-walk by writing the vertices of the *fp*-edge E_i instead of E_i. In other words, an s-walk can be represented by a sequence of vertices like $v_0v_1v_2v_3\ldots v_{n-1}v_n$ in which the vertices v_i and v_{i-1} are consecutively adjacent. An s-walk is called *trivial* if it has no *fp*-edges. An s-walk $v_0E_1v_2E_2\ldots v_{n-1}E_nv_n$ is called a *soft trail* or an *s-trail*, if the *fp*-edges $E_1,E_2,\ldots,E_n$ are such that $E_i \ne E_j$ or E_i is not a partial edge of E_j, $\forall\ i,j = 1,2,\ldots,n$. In an s-trail, vertices may be repeated. Also, note that the *fp*-edges in the form $(v_1,v_2,\ldots,v_{n-1},v_n)$ and $(v_n,v_{n-1},\ldots,v_2,v_1)$ are the same. Suppose $E = (v_1,v_2,\ldots,v_i,v_{i+1},\ldots v_r,v_{r+1},v_{n-1},v_n)$ is an f-edge of the soft semigraph G. Then, we treat $(v_i,v_{i+1},\ldots,v_r,v_{r+1})$ and $(v_{r+1},v_r,\ldots,v_{i+1},v_i)$ as the same partial edge of E. Keep this in mind while verifying the conditions for an s-trail. For example, if $E = (v_1,v_3,v_4,v_5)$ is an f-edge of G, then $E_1 = (v_3,v_4,v_5)$ and $E_2 = (v_4,v_3)$ are partial edges of E. Also, E_2 is a partial edge of E_1. A $v_0 - v_n$ *soft path* or a $v_0 - v_n$ *s-path* is a $v_0 - v_n$ s-trail, in which all the vertices are distinct. A s-path will also be an s-trail. A non-trivial closed s-trail $v_0E_1v_2E_2\ldots v_{n-1}E_nv_0$ is called a *soft cycle* or an *s-cycle* if its origin v_0 and internal vertices are distinct. That is, the closed s-trail $T = v_0E_1v_2E_2\ldots v_{n-1}E_nv_0$ is an s-cycle if, T contains at least three *fp*-edges (by the definition of semigraph) and the vertices $v_0, v_1, v_2, \cdots, v_{n-1}$ are distinct. Two vertices u and v are *soft connected* or *s-connected* in G, if there is an s-path between u and v in at least one of the p-parts $H(x)$ of G. A p-part $H(x)$ of G for some $x \in A$ is said to be *s-connected* if every two of its vertices are s-connected. That is, the p-part $H(x)$ is s-connected if for every two vertices in $Q(x)$, there is an s-path between them in that p-part. A soft semigraph G is said to be *soft connected* or *s-connected* if every two of its vertices are s-connected. That is, for every two vertices in $\cup_{x\in A} Q(x)$, there is an s-path between them in at least one p-part $H(x)$ of G. An *fp*-edge E in $H(x)$ is said to be a *p-part bridge* or a *p-part cut edge* of $H(x)$ if $\omega[H(x)-E] > \omega[H(x)]$ and E must be such minimum *fp*-edge. The term "minimum *fp*-edge" in the definition indicates that a partial edge E' of E cannot satisfy the condition $\omega[H(x)-E] > \omega[H(x)]$ in $H(x)$. An *fp*-edge E in G is said to be a *p-part bridge* or a *p-part cut edge* of G, if E is a *p-part bridge* or a *p-part cut edge* of at least one of the p-parts $H(x)$.

Theorem 2.2. *Let u and v be any two consecutively adjacent vertices of a p-part $H(x)$ of a soft semigraph G. Then, the fp-edge $E=(u,v)$ of $H(x)$ is a p-part bridge if and only if E is not part of any s-cycle in that $H(x)$.*

2.5 Bipartite Soft Semigraph

Let $G^* = (V,X)$ be a semigraph and $G = (G^*, Q, W, A)$ be a soft semigraph of G^* represented by $\{H(x) : x \in A\}$. Then, G is called a bipartite soft semigraph if each of its p-parts $H(x)$ is a bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are independent for all x in A. That is, no f-edge in $W(x)$ is an *mp* edge$\langle Q_1(x)\rangle$ or an *mp* edge$\langle Q_2(x)\rangle$, for all x in A. The term *mp* edge$\langle Q_i(x)\rangle$ denotes a maximum partial edge that can be formed by some or all vertices of $Q_i(x)$. The soft semigraph G is called an e-bipartite soft semigraph if each of its p-parts $H(x)$ is an e-bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are e-independent for all x in A. That is, no two end vertices or partial end vertices of an f-edge in $W(x)$ belong to $Q_1(x)$ or $Q_2(x)$, for all x in A. G is called a strongly bipartite soft semigraph if each of its p-parts $H(x)$ is a strongly bipartite partial semigraph of G^*. That is, $Q(x)$ can be partitioned into sets $\{Q_1(x), Q_2(x)\}$ such that both $Q_1(x)$ and $Q_2(x)$ are strongly independent for all x in A. That is, no two adjacent vertices in $H(x)$ belong to $Q_1(x)$ or $Q_2(x)$, for all x in A.

3. Soft Semitree

Definition 3.1. Let $G = (G^*, Q, W, A)$ be a soft semigraph given by $\{H(x) : x \in A\}$. The p-part $H(x)$ of G is called a semitree p-part if it is connected and contains no s-cycles.

Definition 3.2. A soft semigraph $G = (G^*, Q, W, A)$ given by $\{H(x) : x \in A\}$ is called a soft semitree if all of its p-parts are semitree p-parts, i.e., every two vertices in $H(x)$ are connected by an s-path and contains no s-cycles, $\forall x \in A$.

Example 3.1. Let $G^* = (V,X)$ be a semigraph given in Figure 1, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ and $X = \{(v_1,v_2,v_3),(v_2,v_5,v_6,v_8),(v_1,v_4,v_7),(v_{12},v_{13},v_{14}),(v_4,v_9),(v_5,v_{13}),(v_9,v_{10},v_{11})\}$.

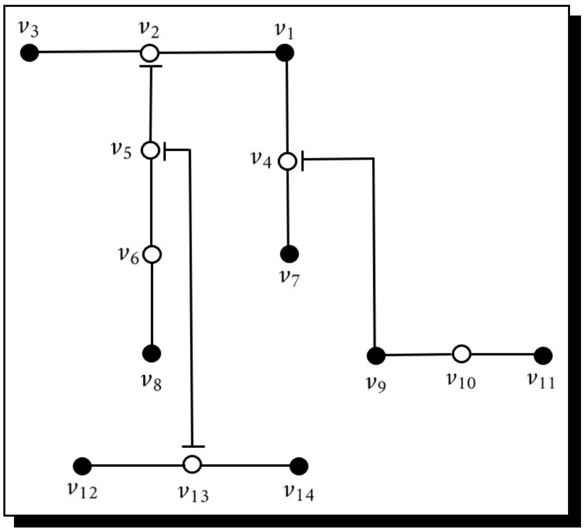

Figure 1. Semigraph $G^* = (V, X)$

Let $A = \{v_2, v_4\} \subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy \Leftrightarrow x = y$ or x and y are adjacent$\}$, $\forall\ x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp$-edges$\langle Q(x)\rangle\}$, $\forall\ x \in A$. That is, $Q(v_2) = \{v_1, v_2, v_3, v_5, v_6, v_8\}$ and $Q(v_4) = \{v_1, v_4, v_7, v_9\}$. Also, $W(v_2) = \{(v_1, v_2, v_3), (v_2, v_5, v_6, v_8)\}$ and $W(v_4) = \{(v_1, v_4, v_7), (v_4, v_9)\}$. Here $H(v_2) = (Q(v_2), W(v_2))$ and $H(v_4) = (Q(v_4), W(v_4))$ are partial semigraphs of G^* as shown below in Figure 2. Hence $G = \{H(v_2), H(v_4)\}$ is a soft semigraph of G^*.

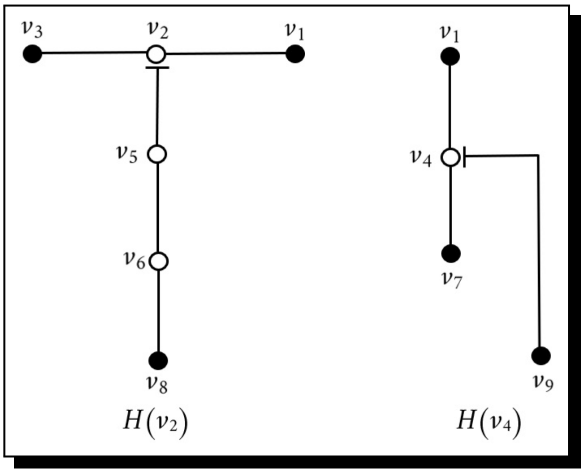

Figure 2. Soft semigraph $G = \{H(v_2), H(v_4)\}$

Here, G has two p-parts $H(v_2)$ and $H(v_4)$ and both are semitree p-parts. Hence, G is a soft semitree.

Theorem 3.1. *Every semitree $G^* = (V, X)$ is a soft semitree of itself.*

Proof. Let $G^* = (V, X)$ be a semitree. Then G^* will be a connected and acyclic semigraph. Let them v be any vertex in G^*. Take a parameter set $A = \{v\} \subseteq V$. Define $Q : A \to \mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy \Leftrightarrow x$ and y are connected$\}$, for all x in A. Then, (Q, A) is a soft set over V. Here, $Q(v)$ is the set of all vertices in G^* since G^* is a connected semigraph. Also, define

$W : A \to \mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}$, for all x in A. That is, $W(v)$ contains all edges in the semigraph G^*. Clearly, (W,A) is a soft set over X_p. If we consider $H(v) = (Q(v),W(v))$, it will be a partial semigraph of G^* since $H(v)$ is G^* itself, and a semigraph is a partial semigraph of itself. Also, G is a soft semitree since its only p-part is G^*, and it is a semitree. That is, the soft semitree $G = (G^*,Q,W,A)$, which is also represented by $\{H(v)\}$ will be the semitree G^* itself. □

Example 3.2. Let $G^* = (V,X)$ be a semitree given in Figure 3 having vertex set $V = \{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$ and edge set $X = \{(v_2,v_4,v_5),(v_5,v_6,v_7),(v_1,v_2),(v_3,v_4)\}$.

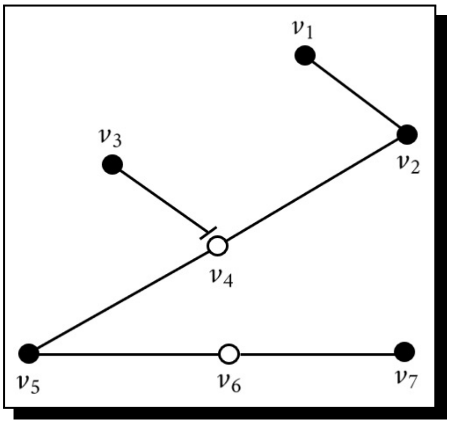

Figure 3. Semitree $G^* = (V,X)$

Let the parameter set be $A = \{v_1\} \subseteq V$. Define $Q : A \to \mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy \Leftrightarrow x$ and y are connected in $G^*\}$, for all x in A and $W : A \to \mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}$, for all x in A. That is, $Q(v_1) = \{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$. Also, $W(v_1) = \{(v_2,v_4,v_5),(v_5,v_6,v_7),(v_1,v_2),(v_3,v_4)\}$. Then, (Q,A) and (W,A) are soft sets over V and X_p respectively. Also, $H(v_1) = (Q(v_1),W(v_1))$ is a partial semigraphs of G^* as shown in Figure 4. Hence, $G = \{H(v_1)\}$ is a soft semitree of G^*.

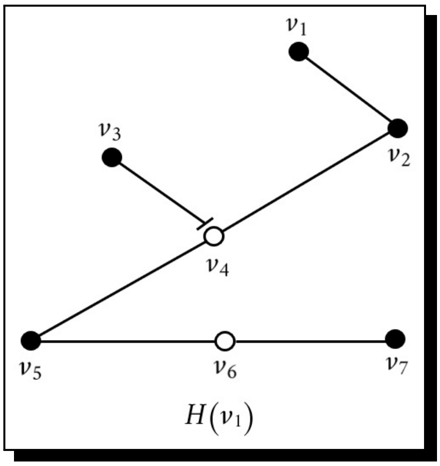

Figure 4. Soft semitree $G = \{H(v_1)\}$

Here, we can see that the soft semitree $G = (G^*,Q,W,A)$, which is also represented by $G = \{H(v_1)\}$ is exactly same as the semitree $G^* = (V,X)$.

Remark 3.1. If G is a soft semitree of G^*, then G^* need not be a semitree. This is clear from the following example.

Example 3.3. Let $G^* = (V, X)$ be a semigraph given in Figure 5, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $X = \{(v_1, v_2, v_3, v_4), (v_1, v_5, v_6, v_7), (v_3, v_6)\}$.

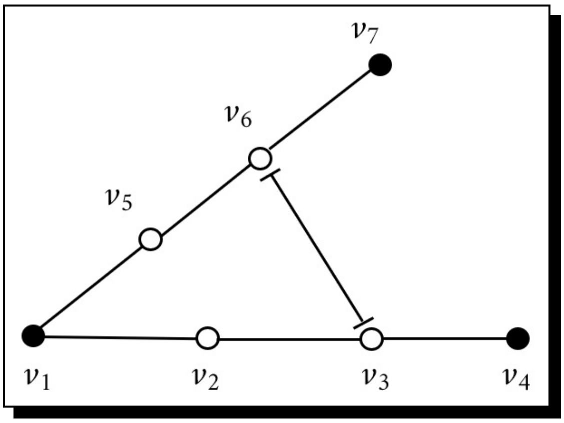

Figure 5. Semigraph $G^* = (V, X)$

Let $A = \{v_3, v_5\} \subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy \Leftrightarrow x = y$ or x and y are adjacent$\}$, $\forall\ x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\text{-edges}\langle Q(x)\rangle\}$, $\forall\ x \in A$. That is, $Q(v_3) = \{v_1, v_2, v_3, v_4, v_6\}$ and $Q(v_5) = \{v_1, v_5, v_6, v_7\}$. Also, $W(v_3) = \{(v_1, v_2, v_3, v_4), (v_3, v_6)\}$ and $W(v_5) = \{(v_1, v_5, v_6, v_7)\}$. Here $H(v_3) = (Q(v_3), W(v_3))$ and $H(v_5) = (Q(v_5), W(v_5))$ are partial semigraphs of G^* as shown below in Figure 6. Hence $G = \{H(v_3), H(v_5)\}$ is a soft semigraph of G^*.

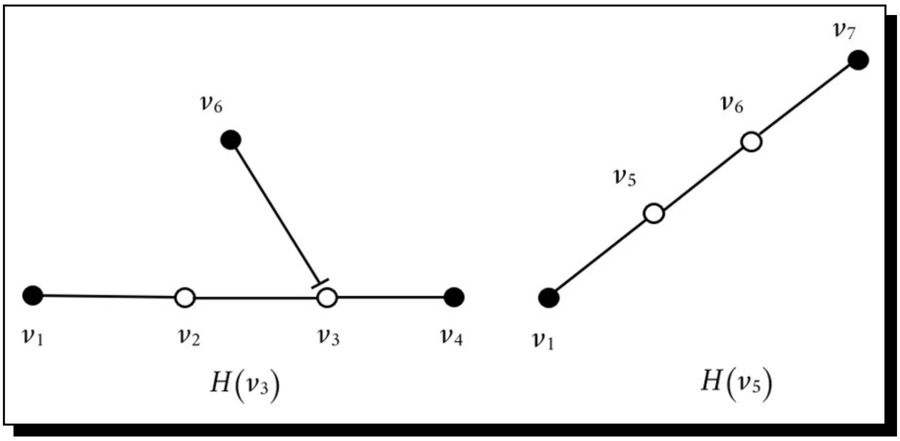

Figure 6. Soft semitree $G = \{H(v_3), H(v_5)\}$

Here, G has two p-parts $H(v_3)$ and $H(v_5)$ and both are semitree p-parts. Hence $G = \{H(v_3), H(v_5)\}$ is a soft semitree of G^*. But the semigraph G^* is not a semitree since it is not acyclic.

Theorem 3.2. *If u and v are any two vertices of a semitree p-part of a soft semigraph G, then there is precisely one s-path from u to v in that p-part.*

Proof. Let $H(x) = (Q(x), W(x))$ a semitree p-part of a soft semigraph $G = (G^*, Q, W, A)$. Then we assume the contrary. That is, we assume that there are two different s- paths from u to v, say $P_1 = uE_1v_1E_2v_2\ldots u_{n-1}E_nv$ and $P_2 = uF_1v_1F_2\ldots v_{m-1}F_mv$, where $u, v_1, v_2\ldots u_{n-1}, v, v_1, v_2, \ldots, v_{m-1}$ are vertices of $Q(x)$ given in some order and $E_1, E_2, \ldots, E_n, F_1, F_2, \ldots, F_m$ are *fp*-edges present in $H(x)$. Express the two paths by writing the vertices of the *fp*-edges E_i and F_j instead of them, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Also, suppose that E_i's and F_j's are different *fp*-edges. Let t be the first common vertex of two s-paths after the common vertex u. There exists at least one such vertex since v is common to both P_1 and P_2. We have $t = u_i = v_j$ for some i and j, then combining the $u - u_i$ portion of the s-path P_1 and $v_j - u$ portion of the s-path P_2 (both are s-paths) we get an s-cycle. Also, if some *fp*-edges at the beginning of P_1 and P_2 are the same, then take the last common vertex w of P_1 and P_2 after such common *fp*-edges. Suppose that $w = u_i = v_k$. Then find the next common vertex of P_1 and P_2 immediately after w. Such a vertex exists since v is common to both s-paths. Suppose $r = u_y = v_z$, then combining one $u_i - u_y$ portion of the s-path P_1, $v_z - v_k$ portion of the s-path P_2 (both are s-paths) we get an s-cycle. This is not possible since $H(x)$ is a p-part semitree. Therefore, our assumption was wrong. That is, there is precisely one s-path in $H(x)$ from u to v. □

Theorem 3.3. *Let G be a soft semitree with at least two vertices in each p-part. Then G has at least $2.w(G)$ number of vertices having degree one, if we count a vertex as many times, it appears in different p-parts with degree one.*

Proof. Let $G = (G^*, Q, W, A)$ be a soft semigraph of G^*, which is a soft semitree, and is given by $\{H(x) : x \in A\}$. Take a p-part $H(x)$ of G, which is a semitree p-part. Consider the longest s-path $P = v_0E_1v_1E_2\ldots v_{n-1}E_nv_n$ in $H(x)$. Suppose that $\deg v_0[H(x)] > 1$. The *fp*-edge $E_1 = v_0E_1v_1$ contributes one to the degree of v_0 in $H(x)$. Therefore, there must be another *fp*-edge from v_0 to a vertex u, which is different from the *fp*-edge E_1. If this vertex or any vertex present in the *fp*-edge E is the same as any one of the vertex u_i in the s-path P when we represent P in terms of vertices of E_i, $i = 1, 2, \ldots, n$, then we get an s-cycle $v_0\ldots v_i\ldots v_0$. This is a contradiction since $H(x)$ a p-part semitree. The remaining possibility is that all vertices present in the *fp*-edge E are different from all vertices present in the s-path P. Then the s-path $P_1 = uEv_0E_1v_1E_2\ldots v_{n-1}E_nv_n$ is an s-path having length one more than that of P which is a contradiction to our assumption that P is the longest s-path in $H(x)$. Therefore, there is no *fp*-edge E in $H(x)$ as we defined. So, $\deg v_0[H(x)] = 1$. Similarly, we can prove that $\deg v_n[H(x)] = 1$. That is, there exist at least two vertices in $H(x)$ having degree 1. Totally, G has $w(G)$ semitree p-parts. Therefore, G has at least $2w(G)$ number of vertices having degree 1 if we count a vertex as many times it appears in different p-parts with degree 1. □

Theorem 3.4. *Let G be a soft semigraph given by $\{H(x) : x \in A\}$ and let u and v be any two consecutively adjacent vertices of a connected p-part $H(x)$ of G. Then $H(x)$ is a semitree p-part if and only if every fp-edge $E = (u, v)$ is a p-part bridge.*

Proof. Assume that the p-part $H(x)$ of the soft semigraph G is a semitree p-part. Then $H(x)$ is connected, and it contains no s-cycles. Therefore, no *fp*-edge of $H(x)$ is part of an s-cycle.

Therefore, by Theorem 2.2 every *fp*-edge $E=(u,v)$ is a p-part bridge.

Conversely, suppose that the p-part $H(x)$ of G is connected and every *fp*-edge $E=(u,v)$ is a p-part bridge. Then the p-part $H(x)$ has no s-cycles since an *fp*-edge which is the part of the s-cycle is not a p-part bridge by Theorem 2.2, that is, $H(x)$ is connected and contents no s-cycles. Therefore $H(x)$ is a semitree p-part of G. □

Theorem 3.5. *A soft semitree having at least* 2 *vertices in each of its* p*-part is* e*-bipartite soft semigraph and hence bipartite soft semigraph*

Proof. Let $G=(G^*,Q,W,A)$ be a soft semitree having at least two vertices in each of its p-parts. Consider any p-part $H(x)$ of G, $x\in A$. Since G is a soft semitree, $H(x)$ is a semitree p-part. We prove that $H(x)$ is an e-bipartite partial semigraph by mathematical induction on the number of vertices of the semitree p-part $H(x)$, that is, on the number of elements in $Q(x)$. For $n=2$, the result is true. That is, when $n=2$, $H(x)$ is an e-bipartite partial semigraph, since we can find a partition $\{Q_1(x),Q_2(x)\}$ for $Q(x)$ where $Q_1(x)$ and $Q_2(x)$ contain one vertex each such that $Q_1(x)$ and $Q_2(x)$ are e-independent. Assume that the semitree p-part $H(x)$ having less than n vertices is e-bipartite. Then take a semitree p-part $H(x)$ having n vertices. Let v be the vertex in $H(x)$ such that $\deg v[H(x)]=1$. Such a vertex exists in $H(x)$ by Theorem 3.3. Also, let u be the end vertex or partial end vertex of the f edge in $H(x)$, which contributes 1 to $\deg v[H(x)]$. Then $H(x)-v$ is a p-part semitree having $n-1$ vertices and is an e-bipartite partial semigraph by our induction assumption. Suppose $\{Q_1(x),Q_2(x)\}$ is the e-independent bipartition of $Q(x)-v$ in $H(x)-v$. Then if $u\in Q_1(x)$, $\{Q_1(x),Q_2(x)\cup\{v\}\}$ is an e-independent bipartition of $Q(x)$ Also, if $u\in Q_2(x)$, then $\{Q_1(x)\cup\{v\},Q_2(x)\}$ is an e-independent bipartition of $Q(x)$. Therefore, $H(x)$ is an e-bipartite partial semigraph. So by mathematical induction, we can say that semitree p-parts $H(x)$ of G are e-bipartite partial semigraphs, $\forall\, x\in A$. Therefore, G is an e-bipartite soft semigraph and hence bipartite since every e-bipartite soft semigraph is also bipartite. □

Example 3.4. Let $G^*=(V,X)$ be a semigraph given in Figure 7, where $V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9,v_{10}\}$ and $X=\{(v_1,v_2,v_3),(v_3,v_4,v_5),(v_2,v_7,v_6),(v_4,v_{10},v_9),(v_8,v_9)\}$.

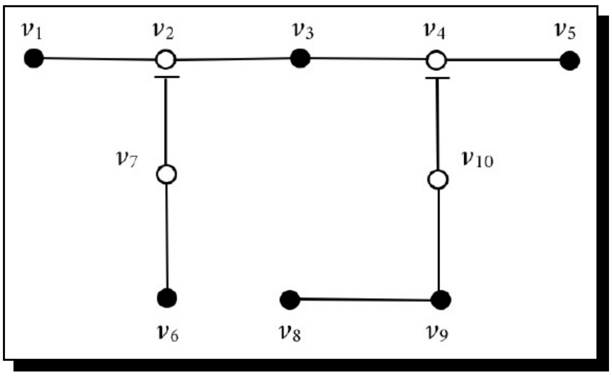

Figure 7. Semigraph $G^*=(V,X)$

Let $A=\{v_2,v_9\}\subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x)=\{y\in V\mid xRy \Leftrightarrow x=y$ or x and y are consecutively adjacent$\}$, $\forall\, x\in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x)=\{mp\ edges\langle Q(x)\rangle\}$, $\forall\, x\in A$. That is, $Q(v_2)=\{v_1,v_2,v_3,v_7\}$ and $Q(v_9)=\{v_8,v_9,v_{10}\}$.

Also, $W(v_2) = \{(v_2,v_7),(v_1,v_2,v_3)\}$ and $W(v_9) = \{(v_8,v_9),(v_9,v_{10})\}$. Then, $H(v_2) = (Q(v_2),W(v_2))$ and $H(v_9) = (Q(v_9),W(v_9))$ are partial semigraphs of G^* as shown below in Figure 8. Hence $G = \{H(v_2),H(v_9)\}$ is a soft semigraph of G^*.

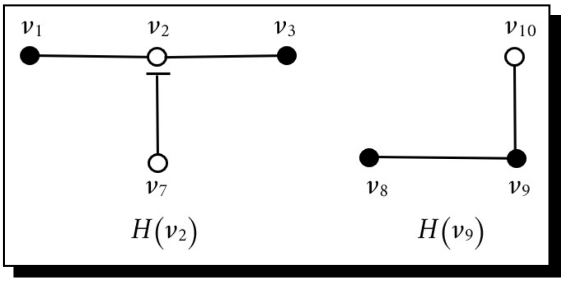

Figure 8. Soft semigraph $G = \{H(v_2),H(v_9)\}$

Here G has two p-parts $H(v_2)$ and $H(v_9)$ and both are semitree p-parts. Hence, G is a soft semitree. Also, each p-part has at least two vertices and $Q(v_2) = \{v_1,v_2,v_3,v_7\}$ can be partitioned into sets $\{Q_1(v_2),Q_2(v_2)\}$, where $Q_1(v_2) = \{v_1,v_2\}$ and $Q_2(v_2) = \{v_3,v_7\}$. Then, $Q_1(v_2)$ and $Q_2(v_2)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_2)$ belong to $Q_1(v_2)$ or $Q_2(v_2)$. Also, $Q(v_9) = \{v_8,v_9,v_{10}\}$ can be partitioned into sets $\{Q_3(v_9),Q_4(v_9)\}$, where $Q_3(v_9) = \{v_8,v_{10}\}$ and $Q_4(v_9) = \{v_9\}$. Then, $Q_3(v_9)$ and $Q_4(v_9)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_9)$ belong to $Q_3(v_9)$ or $Q_4(v_9)$. Therefore, $H(v_2)$ and $H(v_9)$ are e-bipartite partial semigraphs of G^* and hence, $G = \{H(v_2),H(v_9)\}$ is an e-bipartite soft semigraph. Again, $Q(v_2) = \{v_1,v_2,v_3,v_7\}$ can be partitioned into sets $\{Q_1(v_2),Q_2(v_2)\}$, where $Q_1(v_2) = \{v_1,v_3,v_7\}$ and $Q_2(v_2) = \{v_2\}$. Then, $Q_1(v_2)$ and $Q_2(v_2)$ are independent since, no edge in $W(v_2)$ is an *mp* edge$\langle Q_1(v_2)\rangle$ or an *mp* edge$\langle Q_2(v_2)\rangle$. Also, $Q(v_9) = \{v_8,v_9,v_{10}\}$ can be partitioned into sets $\{Q_3(v_9),Q_4(v_9)\}$, where $Q_3(v_9) = \{v_8,v_{10}\}$ and $Q_4(v_9) = \{v_9\}$. Then, $Q_3(v_9)$ and $Q_4(v_9)$ are independent since, no edge in $W(v_9)$ is an *mp* edge$\langle Q_3(v_9)\rangle$ or an *mp* edge$\langle Q_4(v_9)\rangle$. Therefore, $H(v_2)$ and $H(v_9)$ are bipartite partial semigraphs of G^* and hence, $G = \{H(v_2),H(v_9)\}$ is a bipartite soft semigraph.

Remark 3.2. If a soft semigraph G is e-bipartite (and hence bipartite), then G need not be a soft semitree. This is clear from the following example.

Example 3.5. Let $G^* = (V,X)$ be a semigraph given in Figure 9, where $V = \{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$ and $X = \{(v_1,v_2),(v_2,v_3,v_4,v_5),(v_5,v_6,v_7,v_8),(v_3,v_6)\}$.

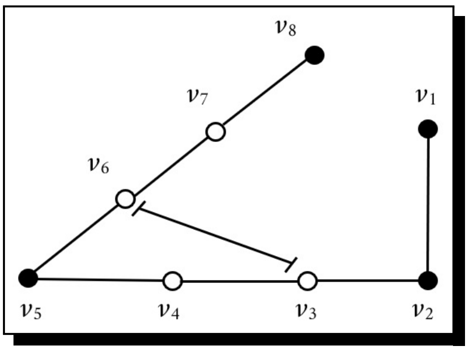

Figure 9. Semigraph $G^* = (V,X)$

Let $A = \{v_3, v_7\} \subseteq V$ be a parameter set. Define Q from A to $\mathcal{P}(V)$ by $Q(x) = \{y \in V \mid xRy \Leftrightarrow x = y$ or x and y are adjacent$\}$, $\forall\ x \in A$ and W from A to $\mathcal{P}(X_p)$ by $W(x) = \{mp\ edges\langle Q(x)\rangle\}$, $\forall\ x \in A$. That is, $Q(v_3) = \{v_2, v_3, v_4, v_5, v_6\}$ and $Q(v_7) = \{v_5, v_6, v_7, v_8\}$. Also, $W(v_3) = \{(v_2, v_3, v_4, v_5), (v_5, v_6), (v_3, v_6)\}$ and $W(v_7) = \{(v_5, v_6, v_7, v_8)\}$. Then, $H(v_3) = (Q(v_3), W(v_3))$ and $H(v_7) = (Q(v_7), W(v_7))$ are partial semigraphs of G^* as shown below in Figure 10. Hence $G = \{H(v_3), H(v_7)\}$ is a soft semigraph of G^*.

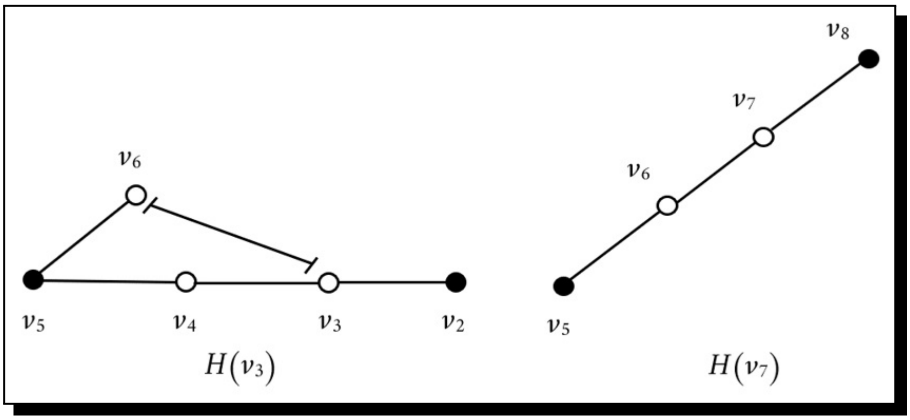

Figure 10. Soft Semigraph $G = \{H(v_3), H(v_7)\}$

Here G has two p-parts $H(v_3)$ and $H(v_7)$ and both are semitree p-parts. Hence, G is a soft semitree. Also, each p-part has at least two vertices and $Q(v_3) = \{v_2, v_3, v_4, v_5, v_6\}$ can be partitioned into sets $\{Q_1(v_3), Q_2(v_3)\}$, where $Q_1(v_3) = \{v_2, v_6\}$ and $Q_2(v_3) = \{v_3, v_4, v_5\}$. Then, $Q_1(v_3)$ and $Q_2(v_3)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_3)$ belong to $Q_1(v_3)$ or $Q_2(v_3)$. Also, $Q(v_7) = \{v_5, v_6, v_7, v_8\}$ can be partitioned into sets $\{Q_3(v_7), Q_4(v_7)\}$, where $Q_3(v_7) = \{v_5, v_7\}$ and $Q_4(v_7) = \{v_6, v_8\}$. Then, $Q_3(v_7)$ and $Q_4(v_7)$ are e-independent since no two end vertices or partial end vertices of an f-edge in $W(v_7)$ belong to $Q_3(v_7)$ or $Q_4(v_7)$. Therefore, $H(v_3)$ and $H(v_7)$ are e-bipartite partial semigraphs of G^* and hence, $G = \{H(v_3), H(v_7)\}$ is an e-bipartite soft semigraph. Again $Q_1(v_3)$ and $Q_2(v_3)$ are independent since, no edge in $W(v_3)$ is an *mp* edge$\langle Q_1(v_3)\rangle$ or an *mp* edge$\langle Q_2(v_3)\rangle$. Also $Q_3(v_7)$ and $Q_4(v_7)$ are independent since, no edge in $W(v_7)$ is an *mp* edge$\langle Q_3(v_7)\rangle$ or an *mp* edge$\langle Q_4(v_7)\rangle$. Therefore, $H(v_3)$ and $H(v_7)$ are bipartite partial semigraphs of G^* and hence, $G = \{H(v_3), H(v_7)\}$ is a bipartite soft semigraph. But G is not a soft semitree since $H(v_3)$ is not a semitree p-part.

Theorem 3.6. *Let $G = (G^*, Q, W, A)$ be a soft semigraph given by $\{H(x) : x \in A\}$. Then, $H(x)$ for some $x \in A$ is a semitree p-part if and only if $|Q(x)| + |W(x)| = \sum_{i=1}^{|W(x)|} |E_i| + 1$, where $|Q(x)|$ and $|W(x)|$ represents number of elements in $Q(x)$ and $W(x)$ respectively and $|E_i|$ represents number of vertices present in the f edge of E_i.*

Proof. Let G be a semigraph with p vertices and q edges E_i, $i \le i \le q$ and k p-parts. Then G contains no cycles if and only if $p + q = \sum_{i=1}^{q} |E_i| + k$, by Theorem 2.1. Here $H(x)$ is a partial semigraph of G^* having $p = |Q(x)|$, $q = |W(x)|$ and $k = 1$. Therefore,

$$|Q(x)| + |W(x)| = \sum_{i=1}^{|W(x)|} |E_i| + 1. \tag{3.1}$$

Conversely, assume that condition (3.1) is satisfied. Let $H(x)$ be the graph satisfying this condition. Then, by the above theorem, $H(x)$ contains no s-cycles and is connected since $k = 1$. Therefore, $H(x)$ is a semitree p-part. □

Theorem 3.7. *Let $G = (G^*, Q, W, A)$ be a soft semitree given by $\{H(x) : x \in A\}$. Then, $\sum_{x\in A} |Q(x)| + \sum_{x\in A} |W(x)| = \sum_{x\in A} \sum_{i=1}^{|W(x)|} |E_i| + |A|$, where $|E_i|$ denotes the number of vertices present in the edge E_i for other sets '| |' denotes 'the number of elements in'.*

Proof. Assume that G is a soft semitree given by $\{H(x) : x \in A\}$. Then each p-part $H(x)$ is a semitree p-part. Then, by Theorem 3.6, we have $|Q(x)| + |W(x)| = \sum_{i=1}^{|W(x)|} |E_i| + 1$, $\forall\ x \in A$. We have totally $|A|$ semitree p-parts, and this is true for all. Adding the terms for all $H(x)$, we get

$$\sum_{x\in A} |Q(x)| + \sum_{x\in A} |W(x)| = \sum_{x\in A} \sum_{i=1}^{|W(x)|} |E_i| + |A|.$$ □

Example 3.6. Consider the semigraph $G^* = (V, X)$ and soft semitree $G = \{H(v_3), H(v_5)\}$ given in Example 3.3. Here,

$$\sum_{x\in A} |Q(x)| + \sum_{x\in A} |W(x)| = (5+4) + (2+1) = 12.$$

Also,

$$\sum_{x\in A} \sum_{i=1}^{|W(x)|} |E_i| + |A| = (4+2) + (4) + 2 = 12.$$

That is,

$$\sum_{x\in A} |Q(x)| + \sum_{x\in A} |W(x)| = \sum_{x\in A} \sum_{i=1}^{|W(x)|} |E_i| + |A|.$$

4. Conclusion

This research has furthered the integration of soft set theory with semigraphs, resulting in a robust theoretical framework for soft semigraphs. By employing parameterization, soft semigraphs offer a versatile and nuanced approach to representing complex relationships within semigraphs. Our study has introduced soft semitrees within soft semigraphs, providing a foundation for further exploration of their algebraic properties and potential applications. Additionally, we have established the relation between various bipartite soft semigraphs and soft semitrees, which enhances our understanding of soft semigraphs. These advancements not only expand the theoretical base of soft semigraphs but also pave the way for practical applications in various fields requiring detailed graph analysis. This work underscores the importance of parameterization in graph theory and opens new directions for future research in the study of soft semigraphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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