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Research Article

Exploring Matroid Varieties: A Tropical Perspective on Combinatorial Structures and Intersection Products

Tushar Atole* and Namrata Kaushal

Department of Mathematics, Mansarovar Global University, Sehore, Madhya Pradesh, India
***Corresponding author:** tushar.pradip.hct@gmail.com

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Abstract. Matroid varieties give a wide range of tropicalizations and a complex way to understand their combinatorial structure when applied to classical linear spaces. In this paper, we delve into the intricate connections between matroid theory and tropical geometry, highlighting how matroid varieties inherit a natural fan structure that elegantly organises themselves based on the flats of the matroid they stem from. This connection facilitates a seamless translation of matroid operations into the tropical realm, akin to speaking the same language in a different mathematical dialect. We focus on constructing an intersection product of cycles on matroid varieties, akin to understanding how loops interact within the landscape. By demonstrating the utility and consistency of this operation, we pave the way for further exploration and application in both matroid and tropical geometry.

Keywords. Matroid varieties, Tropical geometry, Intersection products, Combinatorial structures, Fan structure

Mathematics Subject Classification (2020). 05B35, 14M15, 14C17, 05Exx

1. Introduction

In the realm of mathematical exploration, the intricate interplay between combinatorial structures and geometric representations has long fascinated researchers (see Eur *et al*. [8], Hampe [12], Maclagan and Sturmfels [13], and Speyer and Sturmfels [19]). One such area of study that exemplifies this fascinating relationship is the domain of matroid varieties within tropical geometry [5]. Matroids, which are combinatorial objects capturing the essence of independence and dependency relationships, offer a rich framework for understanding the structural properties of various mathematical constructs, including linear spaces (François and Rau [9]). Tropical geometry, on the other hand, provides a geometric lens through which to view these combinatorial structures, offering insights into their underlying geometry through tropicalizations (Morelli [14]). The concept of matroid varieties extends this relationship further, offering a sophisticated extension of tropicalizations applied to classical linear spaces (Dickenstein *et al*. [6]). These varieties serve as a bridge between the abstract world of matroid theory and the geometric realm of tropical geometry, allowing researchers to explore the combinatorial structure of linear spaces in a novel and insightful manner (Speyer [19]). At the heart of matroid varieties, a natural fan structure lies, which elegantly organizes itself based on the flats of the matroid from which it stems. This fan structure offers a visual representation of the interconnections within matroids (Shaw [18]), akin to arranging puzzle pieces where each flat corresponds to a piece, and the fan structure emerges from their interconnections.

2. Method

Matroid varieties offer a fascinating extension of tropicalizations applied to classical linear spaces. Think of them as a sophisticated way of capturing the combinatorial structure of these spaces. What's intriguing is how they inherit a natural fan structure, which essentially means they organise themselves neatly based on the flats of the matroid they stem from. It is like arranging puzzle pieces where each flat corresponds to a piece, and the fan structure emerges from their interconnections.

This connection with matroid theory enables seamless translation of matroid operations into the tropical realm. It is akin to speaking the same language but in a different mathematical dialect. This synergy not only enriches our understanding of tropical geometry but also offers a pathway to explore smooth tropical varieties. These varieties, akin to classical smooth varieties but in a tropical setting, can be thought of as landscapes with softly curved hills and valleys, each contour representing a different matroid variety.

In this paper, the focus shifts towards constructing an intersection product of cycles on matroid varieties. This operation essentially allows us to calculate how these cycles, or loops in the landscape, interact with each other. It is a bit like figuring out where different paths cross in hilly terrain. By demonstrating that this product aligns with our expectations, we solidify its utility and pave the way for further exploration and application in both matroid and tropical geometry.

Notation 2.1. *Let us consider a set E and a subset A of E. If $x \in E$, then in this context, $A \cup x$ is typically understood as the union of A and the singleton set containing x, denoted as $A \cup \{x\}$. Similarly, $A \setminus x$ signifies the set obtained by removing x from A, expressed as*

$A \setminus \{x\}$,

$A \cup x = A \cup \{x\}$,

$A \setminus x = A \setminus \{x\}$.

Definition 2.1. Matroid $M = (E,B)$ consists of a finite set E together with a non-empty set B of subsets of E satisfying the basis exchange property. This property states that if B_1 and B_2 are in B and x is an element of B_1 but not in B_2, then there exists an element y in B_2 but not in B_1 such that replacing x with y in B_1 results in another set in B [3].

The elements of B are termed as bases of M, and the set E, sometimes denoted by $E(M)$, is called the ground set of M. Furthermore, each basis of M has the same number of elements, known as the rank $r(M)$ of M [17].

We can assign each subset A of E a rank by setting it equal to the maximum number of linearly independent elements in A, denoted by $r(A)$. In other words, $r(A)$ is the size of the largest independent subset of A. This rank function allows us to characterize the structure of the matroid and understand its properties in terms of the independence of subsets.

In mathematical terms, the rank of a subset A of E is defined as:

$$r(A) = \max\{|X| : X \subseteq A\}.$$

The rank function $r : P(E) \to \mathbb{Z}_{20}$ associated with a matroid $M = (E,B)$ satisfies several properties, which are crucial for understanding the structure of the matroid. These properties are as follows:

(1) $r(\theta) = 0$: The rank of the empty set is zero.

(2) If $A \subseteq E$ and $x \in E$, then $r(A) \le r(A \cup x) \le r(A) + 1$: Adding an element x to a subset A can increase the rank by at most one [12].

(3) If $A \subseteq E$ and $x, y \in E$ such that $r(A \cup x) = r(A \cup y) = r(A)$, then $r(A \cup x \cup y) = r(A)$: This property essentially states that if adding either x or y individually to A does not increase the rank, then adding both simultaneously also does not increase the rank. The third property; also known as the exchange property, is a consequence of the basis exchange property in matroid theory. It ensures that the rank function behaves consistently and reflect the structure of the matroid [19].

In mathematical terms, the third property can be illustrated as follow: Assume $x, y \in A$ and $r(A) = r(A \cup x) = r(A \cup y) = r(A \cup x \cup y) - 1$.

Then, we can choose B and B' such that:

$$|B \cap A| = r(A) - 1, \ \ \{x, y\} \subseteq B, \ \ |B' \cap A| = r(A), \ \ B' \cap \{x, y\} = \theta.$$

If $z' \in B' \backslash (B \cup A)$, then our assumptions imply that only elements of $B \backslash (B' \cup A \cup \{x, y\})$ can potentially be added to $B' \backslash z'$ to form a basis of M. However, $|B| = |B'|$ implies that $|B' \rangle (B \cup A)| = |B \backslash (B' \cup A \cup \{x, y\})| + 1$, which leads to a contradiction.

On the other hand, it can be shown that any function $r : P(E) \to \mathbb{Z}_{20}$ fulfilling these conditions is the rank function of a matroid, where the bases are all subsets $A \subseteq E$ satisfying $|A| = r(A) = r(E)$.

Matroid M, the closure $\mathrm{cl}(A)$ of a subset $A \subseteq E$ is defined as:

$$\mathrm{cl}(A) = \{x \in E : r(A \cup x) = r(A)\}.$$

In simpler terms, the closure of A is the largest subset of E that contains A and has the same rank as A. If $A_1 \subseteq A_2$, then this definition implies that for all $x \in E$:

$$r(A_2 \cup x) - r(A_2) = 1 \Rightarrow r(A_1 \cup x) - r(A_1) = 1.$$

Hence, $\mathrm{cl}(A_1) \subseteq \mathrm{cl}(A_2)$. Sets A where $A = \mathrm{cl}(A)$ are called flats of M.

Understanding the properties of flats, we observe:

- E itself is a flat of M.
- The intersection of two flats is again a flat.

If $\{F_1, \ldots, F_p\}$ are the minimal flats strictly containing a flat F, then $E \setminus F$ is the disjoint union of $F_i \setminus F$.

Given these properties, the set of flats uniquely defines a rank function and hence a matroid. This rank function is defined as:

$$r(A) = \max\{i : \mathrm{cl}(\emptyset) = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i = \mathrm{cl}(A)\}.$$

Here, $\mathrm{cl}(A)$ represents the minimal flat of M containing A. This chain of flats characterizes the rank function, allowing us to uniquely define the matroid based on its flats.

Remark 2.1. A matroid M can be characterized by its independent sets, circuits, and hyperplanes. Independent sets, subsets of bases, represent collections of elements not dependent on each other, with inclusion-maximal independent sets corresponding to bases. Circuits are minimal dependent sets, where no proper subset is dependent, revealing essential configurations of dependence. Hyperplanes, flats of rank $r(M) - 1$, signify subsets whose removal decreases the matroid's rank by one, providing critical insights into the matroid's structural integrity and independence. These alternative descriptions offer varied perspectives on the underlying structure and behaviour of the matroid.

Definition 2.2. In the context of a matroid M, a loop is an element A in the ground set $E(M)$ such that the singleton set $\{A\}$ has a rank of 0, indicating that A is not part of any basis of M [12]. Essentially, a loop represents an element that cannot contribute to the generation of independent subsets within the matroid. On the other hand, a co-loop is an element A in $E(M)$ that is present in every basis of M, signifying its essential role in constructing independent sets [17]. In mathematical terms:

- A loop: $A \in E(M)$ with $r(\{A\}) = 0$.
- A co-loop: $A \in E(M)$ such that A is in every basis of M.

These concepts highlight the distinct roles that elements can play within the structure of a matroid, contributing to our understanding of independence and dependency relationships [19].

Example 2.1. A set of n non-zero vectors $v_1, \ldots, v_n$ in a vector space V over a field K. The function r, which maps a subset $A \subseteq \{1, \ldots, n\}$ to the dimension of the linear span of the set $\{v_i : i \in A\}$, serves as the rank function of a matroid. In this context, the independent sets of the matroid precisely correspond to the linearly independent sets among $\{v_1, \ldots, v_n\}$. Such matroids, which can be represented over the field K, are termed realizable or representable matroids.

Mathematically, this implies that:

The rank function r assigns to each subset A of indices the dimension of the subspace spanned by the corresponding vectors.

The independent sets in this matroid correspond exactly to the linearly independent sets among the given vectors $\{v_1,\dots,v_n\}$.

These realizable matroids provide a crucial link between vector spaces and matroid theory, offering insights into the geometric and algebraic properties of linearly independent sets.

Definition 2.3. The direct sum of two matroids M and N, denoted $M\oplus N$, creates a new matroid whose ground set is the disjoint union of the ground sets of M and N. The set of bases for this new matroid is formed by taking the union of bases from M and N, separately.
In mathematical terms, if BM and BN represent bases of M and N respectively, then a basis of $M\oplus N$ is formed by taking the union of these bases: $BM\cup BN$. It is important to note that a subset of the combined ground set $E(M)\cup E(N)$ is a flat of $M\oplus N$ if and only if it can be expressed as the disjoint union of a flat of M and a flat of N [17].

Example 2.2. The Fano matroid F_7 defined on the set $\{1,\dots,7\}\{1,\dots,7\}$ is a rank 3 matroid. Its rank-one flats are singletons, and its rank-two flats consist of seven specific triples: $\{1,2,3\}\{1,2,3\}$, $\{1,4,7\}\{1,4,7\}$, $\{1,5,6\}\{1,5,6\}$, $\{2,5,7\}\{2,5,7\}$, $\{3,4,5\}\{3,4,5\}$, $\{3,6,7\}\{3,6,7\}$, and $\{2,4,6\}\{2,4,6\}$. Conversely, the anti-Fano (or non-Fano) matroid F_7 on the same set is also rank 3. Its rank-one flats remain singletons, while its rank-two flats include the same seven triples as F_7, along with two additional sets: $\{2,4\}\{2,4\}$ and $\{2,6\}\{2,6\}$.
The realisability of F_7 and F_{-7} over a field K is dependent on the characteristic of $K: F_7$ is realisable if and only if K has characteristic 2, while F_{-7} is realisable if and only if the characteristic of K is not 2. Consequently, their direct sum $F_7\oplus F_{-7}$ is not realisable over any field.
Although $F_7\oplus F_{-7}$ is not the smallest non-realisable matroid, there exist rank-4 matroids on $\{1,\dots,8\}\{1,\dots,8\}$ with similar properties.

With these concepts in place, we can define a matroid fan and demonstrate its fulfillment of the balancing condition.

Definition 2.4. In the context of matroids, let M be a matroid defined on the ground set $E=\{1,\dots,n\}$. Consider the standard basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$.
The set $B(M)$ represents the fan of pure dimension $r(M)$, consisting of cones defined as follows:

$$\langle F\rangle=\left\{\sum_{i=1}^{p}\lambda_i\cdot VF_i:\lambda_1,\dots,\lambda_{p-1}\geq 0,\,\lambda_p\in\mathbb{R}\right\},$$

where $F=(\emptyset,F_1,\dots,F_{p-1},F_p=E)$ forms a chain of flats in M, and $VF=-\sum_{i\in F}e_i$ represents the vector corresponding to the flat F [3].
Weigh $B(M)$ by assigning each maximal cone the trivial weight 1. It is important to note that $B(M)$ has a lineality space of $\mathbb{R}\cdot(1,\dots,1)$ [19].
This equation describes the construction of the fan $B(M)$ based on the matroid M, incorporating the concept of flats and their corresponding vectors in $\mathbb{R}^n$, providing a mathematical framework for studying the structure of matroids [8].

Proposition 2.1. *A balanced matroid fan implies that each cone within the fan has an equal number of positively and negatively oriented facets. This balance ensures symmetry in the fan's*

structure, which is crucial for various applications and theoretical analyses.
In practical terms, balance signifies that for every flat F in the matroid M, there are an equal number of cones in $B(M)$ whose generating set includes F and those whose generating set excludes F. This equilibrium reflects the inherent symmetry and consistency in the geometric representation of the matroid.

Proof. Let $r = r(M)$ represent the rank of the matroid M, and let $\tau = \langle \emptyset, F_1, F_2, \dots, F_{r-1} = E \rangle$ be an arbitrary cone of codimension 1 in $B(M)$, the matroid fan.
According to the given statement, there exists an index s such that $r(F_i) = i$ for $i \le s$ and $r(F_i) = i+1$ for $i \ge s+1$.
The facets around τ can be expressed as:

$$\langle \emptyset, F_1, \dots, F_s, F, F_{s+1}, \dots, F_{r-1} = E \rangle,$$

where F is a flat of M. Therefore, it is sufficient to prove the equality:

$$\sum F \text{ flat } V_{F_s}(F) = V_{F_{s+1}} + (|\{F : F \text{ flat with } F_s \subset F \subset F_{s+1}\}| - 1) \cdot V_{F_s} \in V\tau\,. \qquad \square$$

Example 2.3. The matroid fan B $(U_{3,4})$, depicted in the provided picture, we are essentially examining the geometric representation of the matroid $U_{3,4}$ modulo its linearity space $R \cdot (1,1,1,1)$. This fan offers us a visual insight into the combinatorial structure of the matroid. The fan B $(U_{3,4})$ captures various cones representing different combinations of flats within the matroid $U_{3,4}$. Each cone corresponds to a particular arrangement of these flats, giving us a glimpse into the possible independent sets and their relations in $U_{3,4}$.
Understanding this fan is crucial for comprehending the properties and behaviour of the underlying matroid. It allows us to analyse the interplay between different subsets of elements and sheds light on the combinatorial intricacies inherent in $U_{3,4}$.

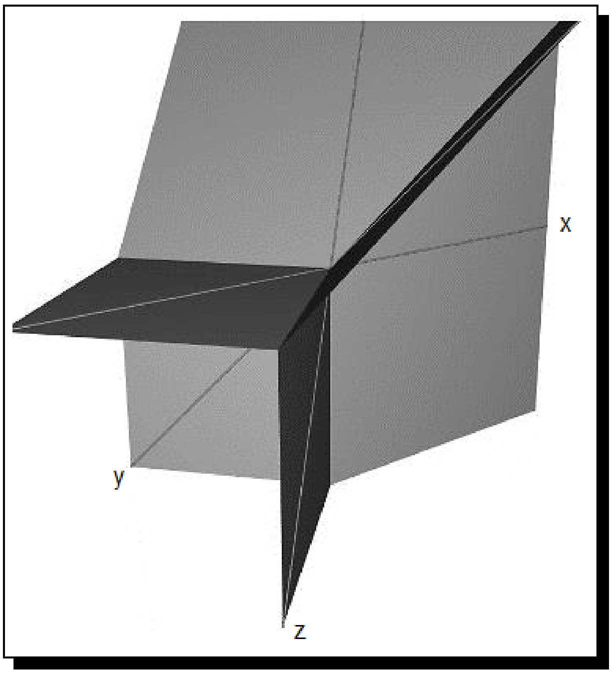

Figure 1. Matroid fan $B(U_{3,4})$

In the given scenario, let us denote $e_1 = (1,0,0,0)$. We observe that:

$$B(M - e_1) = \{\{1,2,3\},\{1,2,4\}\{1,3,4\}\},$$

$$B(M_{e_1}) = \{2,3,4\}.$$

We can infer that $-e_1$ is contained in $B(M)$, whereas e_1 is not. This is because when considering M_{e_1} the element 1 becomes a loop, thereby preventing e_1 from being included in $B(M)$.
This situation can be mathematically represented as follows:

$$B(M - e_1) = \{F \subset E : e_1 \in F\},$$

$$B(M_{e_1}) = \{F \subset E : e_1 \in F\}.$$

These equations illustrate the sets of flats in $M - e_1$ and M_{e_1}, respectively. The presence or absence of e_1 in these flats determines its inclusion in the respective matroid fans.

3. Conclusion

Our exploration into matroid varieties has unveiled a deep interplay between combinatorial structures and tropical geometry. By leveraging the inherent fan structure inherited from matroids, we have been able to construct intersection products that align with our expectations, solidifying their utility in understanding the interconnections within matroid landscapes. The synergy between matroid theory and tropical geometry not only enriches our understanding of both fields but also opens doors to exploring smooth tropical varieties. These varieties, akin to classical smooth varieties but in a tropical setting, offer a landscape rich with softly curved hills and valleys, each contour representing a different matroid variety. As we continue to delve into this fascinating realm, we anticipate further insights and applications that will deepen our understanding of combinatorial structures and their geometric manifestations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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