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Research Article

PBIB-Designs and 2-Steiner Distance Eigenvalues of Split Graphs

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Abstract. In this paper, we have considered split graph $K_n \circ K_1$ with exactly four distinct adjacency eigenvalues. Since these graphs are not regular, it is laborious to construct block designs out of them. In this paper, we have shown the existence (hence construction) and non-existence of block designs arising out of split graphs $K_n \circ K_1$. In the second half of this paper, we have partially tried an NP-hard problem of finding 2-Steiner distance matrix of the split graph $K_n \circ K_1$. Furthermore, we have given interlacing for adjacency, distance and 2-Steiner distance spectra of split graph $K_n \circ K_1$. Time complexity and algorithm for finding rank of a matrix is given. A conjecture concludes the paper.

Keywords. PBIB-design, Split graph, Steiner distance matrix, Spectrum

Mathematics Subject Classification (2020). 05C12, 05C50, 92E10

1. Introduction

A certain class of graphs which were introduced by mathematicians Földes and Hammer [15,16], and independently studied by Tyshkevich and Chernyak [29] in late 1970s called as polar graphs by them, later came to be known as split graphs. In graph theoretical terms, a split graph can be defined as a class of graphs whose vertex set can be partitioned into a clique and an independent set. In other words, a graph G is split if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs – cycle graphs C_4, C_5 or disjoint union of complete graphs $2K_2$. From the definition, it is evident that the complement and every induced subgraph of a split graph is also split. It can be easily observed that diameter of split graph is atmost three. A complete split graph is the one in which every vertex of independent set is adjacent with all the vertices of clique of G. Partitioning of vertex set of a split graph into a clique and an independent set is not unique as can be seen in the following example. For instance, a path $a-b-c$ may have the following partitions of its vertex set:

(i) clique $\{a,b\}$ and independent set $\{c\}$;

(ii) clique $\{b,c\}$ and independent set $\{a\}$;

(iii) clique $\{b\}$ and independent set $\{a,c\}$.

Many characterizations and properties of split graphs have been discovered over past few decades. Some of them are characterisation of split graphs based on the number of eigenvalues of the adjacency matrix, Laplacian matrix, signless Laplacian matrix and so on (see, Banerjee [4], Ghorbani and Azimi [18], Goldberg *et al.* [19], Li and Sun [23], and Song *et al.* [27]). Split graphs are solely recognised from their degree sequences. One of the interesting problems posed was to check whether a graph is split from their degree sequence. Motivated by this, Merris [24] investigated the arrangement of split degree sequences in the lattice of partitions. Split graphs satisfy all the properties exhibited by chordal graphs as almost all chordal graphs are split graphs. Therefore, split graph also belongs to the class of perfect graphs as chordal graphs are perfect. These graphs find various practical applications especially in biochemistry, in modeling and analysis of molecular networks and genetic interactions. In particular, they are used to model biological networks and analyze their structural and functional properties. They find applications in optimization problems as well. Owing to its wide spectrum of applications and properties, split graphs have become an important class of graphs for various research studies, e.g., Angaline and Mary [1], Collins *et al.* [11], Couto *et al.* [13], Grippo and Moyano [20], Guo and Lin [21], and Sethuraman and Nithya [26].

Study of block designs dates back to middle of nineteenth century with the work by Colbourn [10] and Stinson [28], although Euler studied Latin squares in late eighteenth century. Main focus in combinatorial design theory has been in proving the existence and construction of different types of designs. Balanced incomplete block (BIB)-designs and partially balanced incomplete block (PBIB)-designs are two major subfields finding a wide range of applications. PBIB-designs have a long history and have been extensively used in agriculture and industrial experiments (Bose and Nair [6]). In literature, many new block designs have been constructed from graphs. In most of these studies, graphs considered are regular (see, Huilgol and Vidya, [22], and Walikar *et al.* [30]). Since split graphs are not always regular, construction of block designs from vertex sets of split graphs is tricky. Hence, in this paper we have constructed a new class of PBIB-designs arising from certain class of split graphs whose blocks comprise of vertices of diametral paths.

One of the most basic concepts associated with a graph is distance. In particular, if G is a connected graph and u and v are two vertices of G, then the distance $d(u,v)$ between u and v is the length of a shortest path connecting u and v. The Steiner distance of a graph, introduced by Chartrand *et al.* [9] is a natural generalization of the concept of classical graph distance. The Steiner distance of a set S of vertices in a connected graph G is the number of edges in a smallest connected subgraph of G containing S, called a Steiner tree for S. If $|S|=2$, then Steiner distance reduces to the usual distance between the two vertices of S. Steiner trees have many applications in multiprocessor computer networks. For example, in connecting a certain set of processors with a subnetwork that uses the fewest communication links, a Steiner tree that connects the processors of the subnetwork can be employed. A survey on Steiner distance by Mao summarizes known results on the Steiner distance parameters, including Steiner distance, Steiner diameter, Steiner center, Steiner median, Steiner interval, Steiner distance hereditary graphs, Steiner distance stable graphs, average Steiner distance and Steiner-Wiener index. Besides, it contains some conjectures and open problems. k-Steiner distance matrix of graph G is a $\binom{n}{k} \times \binom{n}{k}$ ordered matrix, where n is the order of the graph G, whose rows and columns are indexed by k-element subsets of the vertex set of G, denoted as $D_k(G)$. Suppose X_1 and X_2 are two k-subsets of G, then (X_1,X_2) entry of $D_k(G)$ is $d_G(X_1 \cup X_2)$, that is the minimum size among all the connected subgraphs of G whose vertex set is $X_1 \cup X_2$. Steiner tree problem falls in a class of combinatorial optimization problems which is a blend of two famous combinatorial optimization problems - shortest path problem and minimum spanning tree problem. There are many variations in Steiner tree problems. Finding k-Steiner distance matrix of a graph is NP-hard. Its an open problem to find the inverse and rank of k-Steiner distance matrix of a graph. In literature 2-Steiner distance matrix and rank of only caterpillar graphs and trees have been found (Azimi and Sivasubramanian [2], and Azimi *et al.* [3]). In this paper, we have made an attempt to find the rank of 2-Steiner distance matrix of split graph $K_n \circ K_1$.

An interesting branch of mathematics that deals with the study of graphs using algebraic properties of matrices associated with it is algebraic graph theory and in particular spectral graph theory. This branch studies the relation between graph properties and spectra of various matrices associated with it. Spectral graph theory gains its significance due to various applications of eigenvalues of a graph. The second largest adjacency eigenvalue of a graph gives information about expansion and randomness properties. Independence number and chromatic number of a graph can be determined from its smallest adjacency eigenvalue. Interlacing of eigenvalues gives information about its substructures. A lot of research has been undertaken and is still going on in finding eigenvalues, the bounds of eigenvalues, interlacing of eigenvalues, e.g., Bhunia *et al.* [5], Brouwer and Haemers [7], Constantine [12], Garren [17], Wolkowicz and Styan [32], and Zhan [33], etc.

The paper is organised as follows. First, we start off with a small introductory section, followed by preliminaries in Section 2. Results associated with designs are included in first half of the Section 3 and the latter half deals with 2-Steiner distance matrix, its rank and spectrum followed by conclusion in the last section.

2. Preliminaries

Before proceeding into the main results of this paper, let us first see some basic definitions that are required for better understanding of the results proved in this paper. Undefined graph theoretical and design theoretical terms are used in the sense of Buckley and Harary [8] and Colbourn [10], respectively.

Definition 2.1 ([25])**.** A balanced incomplete block (BIB)-design is a set of v elements arranged in b blocks of k elements each in such a way that each element occurs in exactly r blocks and every pair of unordered elements in λ blocks. The combinatorial configuration so obtained is called a (v,b,r,k,λ)-design. A BIB-design satisfies the following conditions:

(i) $vr = bk$,

(ii) $\lambda(v-1) = r(k-1)$,

(iii) $b \geq v$.

Definition 2.2 ([22]). Given a set $\{1,2,3,\dots,v\}$ of v elements, a relation satisfying the following conditions is said to be an association scheme with m classes:

- Any two elements α and β are ith associates for some i with $1 \leq i \leq m$ and this relation of being ith associates is symmetric.
- The number of ith associates of each element is n_i.
- If α and β are two elements which are ith associates, then the number of elements which are jth associates of α and kth associates of β is p^i_{jk} and is independent of the pair of ith associates α and β.

Definition 2.3 ([6]). Consider a set $V = \{1,2,\dots,v\}$ and an association scheme with m classes on V. A partially balanced incomplete block (PBIB)-design represented as $(v,b,r,k,\lambda_1,\dots,\lambda_m)$ is a collection of b subsets of V called blocks, each of them containing k elements ($k < v$) such that every element occurs in r blocks and any two elements α and β which are ith associates occur together in λ_i blocks, the number λ_i being independent of the choice of the pair α and β. The numbers v,b,r,k,λ_i ($i = 1,2,\dots,m$) are called the parameters of first kind and n_i's and p^i_{jk} are called the parameters of second kind.

Definition 2.4 ([14]). Given a block design $(\mathcal{V},\mathcal{B})$, its associated block intersection graph is the graph on vertex set $\mathcal{B}$, of blocks, for which two vertices or blocks B_1 and B_2 are adjacent if and only if $|B_1 \cap B_2| \geq 1$.

Definition 2.5 ([7]). Spectra of a graph G is the set of eigenvalues of a matrix associated with the graph.

Definition 2.6 ([8]). A vertex and an edge are said to cover each other if they are incident. A vertex cover in graph G is the set of vertices that cover all the edges of G. An edge cover for a graph G is the set of edges that cover all the vertices of G.

Definition 2.7 ([8]). A set of vertices is said to be independent if no two vertices belonging to the same set are adjacent to each other. On similar lines, a set of edges is said to be independent if no two of them are incident with each other.

Definition 2.8 ([8]). In graph theory, geodesic between two vertices is a shortest path with these vertices as its end vertices.

Definition 2.9 ([8]). The geodesic interval $I(u,v)$ between vertices u and v is the set of vertices on all shortest $u-v$ paths. Given a set $S \subseteq V(G)$, its geodetic closure $I[S]$ is the set of all vertices lying on some shortest path joining two vertices of S. Thus,

$$I[S] = \{u \in V(G) \mid v \in I[x,y], x,y \in S\} = \bigcup_{x,y} I(x,y).$$

Definition 2.10 ([8]). A geodetic set or geodetic cover of a graph G is a set $S \subset V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. Equivalently, a set S is called a geodetic set in G if $I[S] = V(G)$, that is, every vertex in G lies on some geodesic between two vertices from S.

Definition 2.11 ([31]). A dominating set for a graph G is the subset D of vertices, such that any vertex of G has a neighbour in D.

Below we give a few already existing results that help in establishing our proof.

Theorem 2.1 ([12]). *Consider a $n \times n$ real symmetric matrix M whose entries are in the interval $[a,b]$. Let $\lambda_1(M) \geq \lambda_2(M) \geq \lambda_3(M) \geq \cdots \geq \lambda_n(M)$ be eigenvalues of matrix M. Then,*

$$\lambda_n(M) \geq \begin{cases} n(a-b)/2, & \text{if } n \text{ is even}, \\ (na - \sqrt{(a^2 + (n^2-1)b^2})/2, & \text{if } n \text{ is odd}, \end{cases}$$

for $n \geq 2$ and $a < b$.

This gives the lower bound for the eigenvalues of any real symmetric matrix which has entries in the interval $[a,b]$.

Theorem 2.2 ([17]). *For any arbitrary matrix M, the largest possible eigenvalue modulus is less than or equal to matrix norm M, that is, $\lambda_{\max}(M) \leq \|M\| = \max \sum_{i=1}^{n} m_{ki}$ where m_{ki} is the matrix element. In other words, $\|M\|$ is the largest row sum of matrix M.*

3. Results

This section can be formally divided into two subsections. Section 3.1 deals with the existence and non-existence of block designs associated with split graph $K_n \circ K_1$, and distance based concepts such as Steiner distance matrix, rank and spectra of 2-Steiner distance matrix of $K_n \circ K_1$ are discussed in Section 3.2.

3.1 Designs Associated With Split Graphs

First, we define association scheme for the design constructed in Theorem 3.1.

First associate of a pendant vertex is the vertex at distance 1 in clique and remaining all $2n-2$ vertices are its second associates. For a vertex in the clique, its first associate is the pendant vertex which is at distance 1 and the remaining vertices are its second associates. λ_1 gives the number of blocks containing a pair of vertices which are first associates and λ_2 gives the number of blocks containing a pair of vertices which are second associates.

Using the association scheme described above, we construct a PBIB-design with the parameters as given in Theorem 3.1.

Theorem 3.1. *There exists a PBIB-design with parameters $v = 2n$, $b = \binom{n}{2}$, $r = n-1$, $k = 4$, $\lambda_1 = n-1$ and $\lambda_2 = 1$ arising from split graph $K_n \circ K_1$ where blocks are vertices of diametral paths in the graph.*

Proof. Let G be the graph $K_n \circ K_1$ which has a complete graph K_n with a pendant vertex attached to each vertex in K_n. Clearly, it has $2n$ vertices. It is obvious that the diameter of G is 3 and thus each diametral path will have four vertices giving block size $k = 4$. For a pendant vertex, its eccentric vertices are all the remaining pendant vertices which are at distance 3 and for a vertex in the clique, its eccentric vertices are non-adjacent pendant vertices which are at distance 2 from them. Hence, diametral paths are all those shortest paths between every pair of pendant vertices. So now on we consider any pair of pendant vertices as a pair of eccentric vertices. From the structure of G, it is clear that there exists a unique diametral path between every pair of eccentric vertices which are at distance three. Thus there are $n-1$ diametral paths arising from any pendant vertex. Since there are n pendant vertices, we get $n(n-1)$ diametral paths in $K_n \circ K_1$. But each of them is counted twice from each of its end vertices. Hence, there are $\frac{n(n-1)}{2}$ distinct diametral paths in $K_n \circ K_1$. It is evident that each vertex appears in $n-1$ diametral paths giving repetition number of the design as $n-1$. A pair of vertices forming the pendant edge are first associates. Clearly, any pair of first associates appear together in $n-1$ diametral paths giving the value of λ_1 as $n-1$. Since there is a unique diametral path between every pair of eccentric vertices, λ_2 is 1. Taking the vertices of diametral paths as blocks, PBIB-design with parameters $(v,b,r,k,\lambda_1,\lambda_2) = \left(2n, \binom{n}{2}, n-1, 4, n-1, 1\right)$ is obtained along with parameters of second kind as $n_1 = 1$, $n_2 = 2n-2$ with $P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2n-2 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2n-4 \end{bmatrix}$. □

Theorem 3.2. *Block intersection graph arising from $K_n \circ K_1$ with blocks as vertices of diametral paths is triangular graph which is strongly regular graph with parameters $(v,k,\lambda,\mu) = \left(\binom{n}{2}, 2(n-2), n-2, 4\right)$, where $n \geq 4$.*

Proof. From Theorem 3.1, it is obvious that there are $\binom{n}{2}$ number of diametral paths in $K_n \circ K_1$ which forms the vertices of block intersection graph. Each block contains two pendant edges. Any two blocks are adjacent if they share a pendant edge.

Consider a block, say, B_1 containing two pendant edges $x-y$ and $z-w$. There are $n-2$ other blocks containing pendant edge $x-y$ and other distinct $n-2$ blocks containing pendant edge $z-w$. Thus, there are $2n-4$ number of blocks adjacent to block B_1 giving the regularity of block intersection graph as $2(n-2)$. There are $n-1$ blocks containing a particular edge say, $x-y$. Consider any two adjacent blocks say, B_1 and B_2, such that they share a common edge. Blocks which are adjacent to both B_1 and B_2 should contain the same common edge in it. There are $n-3$ blocks containing the common edge and another block containing the pair of edges which are not common in the considered blocks B_1 and B_2. Thus, there are $n-2$ blocks adjacent to both B_1 and B_2. Consider a pair of non-adjacent blocks say, B_1 and B_3. Composition of blocks adjacent to both B_1 and B_3 are such that they contain one pendant edge from B_1 and the other from B_3. Hence, four blocks are adjacent to B_1 and B_3. Thus, block intersection graph is a strongly regular graph with parameters $(v,d,p,q) = \left(\binom{n}{2}, 2(n-2), n-2, 4\right)$ where $n \geq 4$ which are also parameters of a class of strongly regular graph called triangular graph. □

Lemma 3.1. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are maximum independent set.*

Proof. Clearly, the size of maximum independent set would be n. We can get n such maximum independent sets containing $n-1$ pendant vertices and the other vertex from the clique not adjacent to any of the pendant vertices chosen and a maximum independent set with all pendant vertices. Clearly, pendant vertices are repeated more than the vertices of the clique in these $n+1$ maximum independent sets. Thus repetition number of vertices are not unique. Hence, taking vertices of maximum independent sets as blocks does not yield a block design. □

Lemma 3.2. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are vertices of minimum dominating set.*

Proof. From the structure of the graph $K_n \circ K_1$, it is evident that maximum independent sets and minimum dominating sets are same. Hence proof follows from Lemma 3.2. □

Lemma 3.3. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are geodetic sets of the graph.*

Proof. There exist only one geodetic set of minimum size n containing all pendant vertices. Hence, we cannot construct a design with just one block. □

Lemma 3.4. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are vertices of vertex covers of the graph.*

Proof. There are $n+1$ vertex covers in $K_n \circ K_1$ each of size n. Of these $n+1$ vertex covers, n vertex covers have $n-1$ vertices from K_n and a pendant vertex not adjacent to those vertices of the clique that has been chosen and the other vertex cover contains all vertices of the clique. Hence, clearly repetition number of vertices from the clique and pendant vertices are not same and thus no design exists. □

Lemma 3.5. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are edge covers in the graph.*

Proof. From the graph construction it is obvious that there is only one minimum sized edge cover which contains all the pendent edges. Hence, no design can be constructed from a single block. □

Lemma 3.6. *There does not exist a block design arising from split graph $K_n \circ K_1$ where blocks are edge independent sets in the graph.*

Proof. For split graph $K_n \circ K_1$, maximum edge independent set and minimum edge covers are same. Hence proof follows from Lemma 3.6. □

3.2 Steiner Distance Matrix

Let G be a connected graph of order atleast 2 and S be a non-empty set of vertices of G. Then, the Steiner distance $d(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex set contains S. If H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d(S)$, then H is a tree. Such a tree is called a Steiner tree. 2-Steiner distance is the Steiner distance between two 2-element sets. 2-Steiner distance matrix of a graph G, denoted as $D_2(G)$ is $\binom{n}{2} \times \binom{n}{2}$ ordered symmetric square matrix whose rows and columns are indexed by 2-element subsets of the vertex set with entries as Steiner distance between the vertices contained in the row and column heading the entry. For clarity of the concept, let us illustrate this using an example.

Illustration: Consider the split graph $G : K_3 \circ K_1$ as shown in Figure 1:

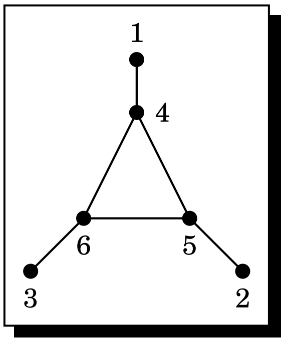

Figure 1. Split graph $K_3 \circ K_1$

Vertex set of G is $\{1,2,3,4,5,6\}$. The 2-element subsets of the vertex set is $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$. In the rest of the paper, we denote $\{1,2\}$ as simply 12. The rows and columns of 2-Steiner distance matrix $D_2(G)$ be indexed by these 2-element subsets. For example, suppose the sets are $\{1,4\}$ and $\{2,5\}$, then 2-Steiner distance of the set containing elements $\{1,2,4,5\}$ is 3, that is, the Steiner tree containing these four elements contains three edges, namely, 14, 25 and 45. On similar lines, all the entries of $D_2(G)$ can be obtained. In general we can give the ijth entry of 2-Steiner distance matrix of split graph $K_n \circ K_1$.

Let pendant vertices of the graph $K_n \circ K_1$ be labeled with $1,2,\dots,n$ and vertices of inner complete graph K_n be labeled with $n+1, n+2, \dots, 2n$ such that i is adjacent to $n+i$ where $1 \le i \le n$.

Let $1 \le i,j,k,l \le n$ and $n+1 \le p,q,r,s \le 2n$ be any four pendant vertices and vertices of complete graph, respectively, of $K_n \circ K_1$. Then entries of $D_2(G)$ are given as follows.

Diagonal entries:

$$\textit{Case (i)}: D_2[x,y](G) = \begin{cases} 1, & \text{if } x = (i, i+n) \text{ and } y = (i, i+n), \\ 1, & \text{if } x = (p,q) \text{ and } y = (p,q), \\ 2, & \text{if } x = (i,p),\ y = (i,p) \text{ and } p \neq i+n, \\ 3, & \text{if } x = (i,j),\ y = (i,j) \text{ and } j \neq i+n. \end{cases}$$

Non-diagonal entries:

$$\textit{Case (ii)}: D_2[(i,i+n),(y)](G)=\begin{cases}2, & \text{if } y=(i+n,p),\\ 2, & \text{if } y=(i,p) \text{and } p\neq i+n,\\ 3, & \text{if } y=(p,q),\ p\neq i+n \text{ and } q\neq i+n,\\ 3, & \text{if } y=(j,j+n),\\ 3, & \text{if } y=(j,i+n),\\ 3, & \text{if } y=(i,j),\\ 4, & \text{if } y=(j,p),\ p\neq i+n \text{ and } p\neq j+n,\\ 5, & \text{if } y=(j,k),\ i\neq j \text{ and } i\neq k.\end{cases}$$

$$\textit{Case (iii)}: D_2[(p,q),y](G)=\begin{cases}2, & \text{if } y=(p,r) \text{ or } (r,q),\\ 2, & \text{if } y=(i,q),\ p=i+n,\\ 3, & \text{if } y=(r,s),\\ 3, & \text{if } y=(i,r),\ p=i+n,\\ 3, & \text{if } y=(j,p) \text{ or } (j,q), \text{ and } p\neq j+n,\ q\neq j+n,\\ 3, & \text{if } y=(i,j),\ p=i+n \text{ and } q=j+n,\\ 4, & \text{if } y=(r,j),\ p\neq j+n,\ q\neq j+n,\ r\neq j+n,\\ 4, & \text{if } y=(i,k),\ p \text{ or } q=i+n \text{ and } p \text{ or } q\neq k+n,\\ 5, & \text{if } y=(i,j),\ p\neq i+n \text{ or } q\neq j+n \text{ and vice versa.}\end{cases}$$

Throughout the next case, we consider $p\neq i+n$,

$$\textit{Case (iv)}: D_2[(i,p),(y)](G)=\begin{cases}2, & \text{if } y=(i+n,p), \text{ and } p\neq i+n,\\ 2, & \text{if } y=(i,i+n),\\ 3, & \text{if } y=(i+n,q),\\ 3, & \text{if } y=(j,p),\ p=j+n,\\ 3, & \text{if } y=(p,q),\ q\neq i+n,\\ 3, & \text{if } y=(i,q),\ q\neq i+n,\\ 3, & \text{if } y=(j,q),\ q=i+n \text{ and } p=j+n,\\ 3, & \text{if } y=(i,j),\ p=j+n,\\ 4, & \text{if } y=(j,q),\ q=j+n \text{ and } p\neq j+n,\\ 4, & \text{if } y=(r,s),\ r\neq i+n \text{ and } s\neq i+n,\\ 4, & \text{if } y=(j,r),\ p=j+n \text{ and } r\neq i+n,\\ 4, & \text{if } y=(j,i+n),\ p\neq j+n,\\ 4, & \text{if } y=(j,p),\ p\neq j+n,\\ 4, & \text{if } y=(i,k),\ p\neq k+n,\\ 5, & \text{if } y=(j,q),\ p\neq j+n,\ q\neq i+n,\\ 5, & \text{if } y=(j,k),\ p=j+n \text{ or } p=k+n,\\ 6, & \text{if } y=(j,k),\ p\neq j+n \text{ and } p\neq k+n.\end{cases}$$

Case (v): $$D_2[(i,j),(y)](G) = \begin{cases} 3, & \text{if } y=(i,i+n),\\ 3, & \text{if } y=(i,p) \text{ and } p=j+n,\\ 3, & \text{if } y=(i+n,j+n),\\ 4, & \text{if } y=(i+n,q),\ q\neq j+n,\\ 4, & \text{if } y=(i,p) \text{ and } p\neq j+n,\\ 4, & \text{if } y=(p,j),\ p\neq i+n \text{ and } p\neq j+n,\\ 5, & \text{if } y=(i,k),\\ 5, & \text{if } y=(k,k+n),\\ 5, & \text{if } y=(k,p),\ p=i+n \text{ or } p=j+n,\\ 5, & \text{if } y=(p,q),\ p\neq i+n \text{ and } q\neq j+n,\\ 6, & \text{if } y=(p,k),\ p\neq i+n \text{ and } p\neq j+n \text{ and } p\neq k+n,\\ 7, & \text{if } y=(k,l). \end{cases}$$

Next, let us see interlacing of spectra of adjacency matrix, distance matrix and 2-Steiner distance matrix of split graph $K_n \circ K_1$.

Theorem 3.3. *Suppose α, β and γ are eigenvalues of adjacency matrix A, distance matrix D and 2-Steiner distance matrix D_2, respectively, of the split graph $K_n \circ K_1$, then $\gamma_{\min} < \beta_{\min} < \alpha_{\min} < \alpha_{\max} < \beta_{\max} < \gamma_{\max}$.*

Proof. In proving this theorem, we make use of Theorems 2.1 ([12]) and 2.2 ([17]) in obtaining the lower and upper bounds of 2-Steiner distance matrix, respectively.

Using above two results, we now determine the interlacing for eigenvalues of adjacency matrix, distance matrix and 2-Steiner distance matrix of split graphs $K_n \circ K_1$. Before finding the interlacing sequence, we start with computing bounds for eigenvalues of A, D and D_2.

Let G be the split graph $K_n \circ K_1$.

Case (i): Bounds for adjacency eigenvalues of G.
Let A be the adjacency matrix of graph G with eigenvalues $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \dots \ge \alpha_{2n}$. Since split graph $K_n \circ K_1$ contains $2n$ number of vertices, we consider only the case for even number of vertices in finding the lower bound of eigenvalues. Also the entries in A lie in the interval $[0,1]$. Hence the lower bound is $-n$, that is, $-n \le \alpha_{\min}$.

In order to find upper bound of eigenvalues of A, we need to find norm of matrix A which is the largest row sum of A. As entries in adjacency matrix depend on degrees of vertices, largest row sum corresponds to the row indexed by vertex having maximum degree. Since degree of vertex in complete graph of G, which is n, is larger compared to that of pendant vertex, maximum row sum in A is n. Hence $\alpha_{\max} \le n$,

$$-n \le \alpha \le n.$$

Another formula for obtaining upper bound especially for adjacency matrix of a graph with edge number e and maximum clique size cl is $\sqrt{2e(cl-1)/cl}$. On substituting values for e as $\binom{n}{2}+n$ and cl as n we get yet another closer upper bound as $\alpha_{\max} \le \sqrt{2\left(\binom{n}{2}+n\right)(n-1)/n}$.

Case (ii): Bounds for distance eigenvalues of G.
Let D be the distance matrix of G with eigenvalues $\beta_1 \ge \beta_2 \ge \beta_3 \ge \dots \ge \beta_{2n}$. Since diameter of G is 3, entries of D vary in the interval $[0,3]$. On substituting values for a as 0, b as 3 and number of vertices which is even, we get the lower bound as $-3n$, that is, $-3n \le \beta_{\min}$.

It is obvious that row sum corresponding to pendant vertex in matrix D will be greater than the vertex of complete graph as pendant vertices have $n-1$ vertices at distance 3. Each pendant vertex has one vertex at distance 1, $n-1$ vertices at distance 2 which are in K_n and remaining $n-1$ pendant vertices at distance 3. Hence greatest row sum in D would be $5n-4$, that is, $\beta_{\max} \le 5n-4$,

$$-3n \le \beta \le 5n-4.$$

Case (iii): Bounds for 2-Steiner-distance eigenvalues of G.
Let D_2 denote 2-Steiner distance matrix of G having order $\binom{2n}{2} \times \binom{2n}{2}$ with eigenvalues $\gamma_1 \ge \gamma_2 \ge \gamma_3 \ge \dots \gamma_{\binom{2n}{2}}$. Here, entries of D_2 lie in the interval $[1,7]$ with largest element at the intersection of row and column indexed by all pendant vertex pairs. Therefore, on substituting values for a, b and the order of matrix, we get

$$\left\{\begin{array}{ll} -3\binom{2n}{2}, & \text{if } \binom{2n}{2} \text{ is even}, \\ \frac{\left(\binom{2n}{2}-\sqrt{1+49\left(\binom{2n}{2}^2-1\right)}\right)}{2}, & \text{if } \binom{2n}{2} \text{ is odd}. \end{array}\right\} \le \gamma_{\min}.$$

Maximum row sum correspond to the row containing the highest entry, that is, the row indexed by the pendant vertex pairs. Hence, largest row sum is $18+4(2(n-2)+2(n-2)) +5((n-2)+2(n-2)+(n-2)+(n-2)+\binom{n-2}{2})+6((n-2)(n-3))+7(\binom{n-2}{2})$, that is, $\gamma_{\max} \le 18+41(n-2) +6(n-2)(n-3)+12\binom{n-2}{2}$,

$$\left\{\begin{array}{l} -3\binom{2n}{2} \\ \frac{\left(\binom{2n}{2}-\sqrt{1+49\left(\binom{2n}{2}^2-1\right)}\right)}{2} \end{array}\right\} \le \gamma \le 18+41(n-2)+6(n-2)(n-3)+12\binom{n-2}{2}.$$

Clearly, on substituting various integral values for n, we infer that

$$\gamma_{\min} < \beta_{\min} < \alpha_{\min} < \alpha_{\max} < \beta_{\max} < \gamma_{\max}.$$ □

Further, we give an algorithm to obtain 2-Steiner distance matrix from a graph and then another algorithm to find rank of this matrix and hence determine its time complexity.

Algorithm 1. Algorithm to obtain 2-Steiner distance matrix D_2

Input:	Adjacency matrix A of graph G
Output:	2-Steiner distance matrix D_2
Step 1:	Start
Step 2:	Read A. $A \leftarrow$ Adjacency matrix of graph G
Step 3:	Define function $sd(a,b)$:= steiner_distance(a,b) in G
Step 4:	Declare variables i, j, p
Step 5:	Read the value for p
Step 6:	Initialize $D_2 \leftarrow [0]_{p\times p}$
Step 7:	Call function $sd()$
Step 8:	Repeat until $i \leftarrow 1$ to p, $j \leftarrow 1$ to p add $sd(i,j)$ to D_2
Step 9:	Display D_2
Step 10:	Stop

As already mentioned, Steiner tree problems and its variations are NP-hard problems. Algorithm 1 finds Steiner distance matrix of a graph G and it requires exponential time for running.

Algorithm 2. Algorithm to find rank ρ of matrix D_2

Input: 2-Steiner distance matrix D_2
Output: Rank ρ of matrix D_2

Step 1: Start
Step 2: Initialise $M \leftarrow [0]_{p\times p}$
Step 3: Define function $ech(M)$:= row_echelon_form(M)
Step 4: Read D_2
Step 5: Call function $ech()$
add D_2 to M
Step 6: Calculate ρ. $\rho \leftarrow$ number of non-zero rows in $ech(D_2)$
Step 7: Display ρ
Step 8: Stop

Time complexity of finding echelon form of a $n\times n$ matrix is $O(n^2)$ and Algorithm 2, which computes rank of matrix takes $O(n^2)$ time for running a program. Time complexity of finding rank of matrix D_2, a square matrix of order $p\times p$ is $O(p^2)$. Since, p is $\binom{2n}{2}$ for D_2, time complexity is $O(n^4)$.

We now illustrate the above two algorithms.

Consider split graph $K_3\circ K_1$ (Figure 1). Suppose the rows and columns of $D_2(K_3\circ K_1)$ are indexed in the order $\{14,25,36,45,46,56,15,16,24,26,34,35,12,13,23\}$, then the 2-Steiner distance matrix of $K_3\circ K_1$, obtained as output of Algorithm 1 is

$$D_2(K_3\circ K1)=\begin{bmatrix}
1&3&3&2&2&3&2&2&3&4&3&4&3&3&5\\
3&1&3&2&3&2&3&4&2&2&4&3&3&5&3\\
3&3&1&3&2&2&4&3&4&3&2&2&5&3&3\\
2&2&3&1&2&2&2&3&2&3&3&3&3&4&4\\
2&3&2&2&1&2&3&2&3&3&2&3&4&3&4\\
3&2&2&2&2&1&3&3&3&2&3&2&4&4&3\\
2&3&4&2&3&3&2&3&3&4&4&4&3&4&5\\
2&4&3&3&2&3&3&2&4&4&3&4&4&3&5\\
3&2&4&2&3&3&3&4&2&3&4&4&3&5&4\\
4&2&3&3&3&2&4&4&3&2&4&3&4&5&3\\
3&4&2&3&2&3&4&3&4&4&2&3&5&3&4\\
4&3&2&3&3&2&4&4&4&3&3&2&5&4&3\\
3&3&5&3&4&4&3&4&3&4&5&5&3&5&5\\
3&5&3&4&3&4&4&3&5&5&3&4&5&3&5\\
5&3&3&4&4&3&5&5&4&3&4&3&5&5&3
\end{bmatrix}.$$

Considering the above matrix $D_2(K_3 \circ K_1)$ as the input for Algorithm 2, we obtain the output of Algorithm 2 as $\rho(D_2) = 7$ with the row echelon form given below.

$$\begin{bmatrix} 1&0&0&0&0&0&2&2&1&1&1&1&3&3&2\\ 0&1&0&0&0&0&1&1&2&2&1&1&3&2&3\\ 0&0&1&0&0&0&1&1&1&1&2&2&2&3&3\\ 0&0&0&1&0&0&3&2&3&2&3&3&5&5&5\\ 0&0&0&0&1&0&2&3&3&3&3&2&5&5&5\\ 0&0&0&0&0&1&3&3&2&3&2&3&5&5&5\\ 0&0&0&0&0&0&5&5&5&5&5&5&10&10&10\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Finally, we conjecture the proof of the result given below.

Conjecture 3.1. *Rank of* 2*-Steiner distance matrix* $D_2(K_n \circ K_1)$ *is* $2n+1$.

4. Conclusion

Construction and existence of block designs has always been an interesting area for researchers. In this paper, we have showed the existence of a new class of PBIB-designs arising from diametral paths of split graph $K_n \circ K_1$. We have also shown that considering other vertex subsets as blocks do not yield a block design. Further, we obtained 2-Steiner distance matrix of split graph $K_n \circ K_1$ and partially solved the problem of computing its rank. Finally, we have given an interlacing theorem for the spectra of adjacency matrix, distance matrix and 2-Steiner distance matrix of split graph $K_n \circ K_1$. The paper concludes with a conjecture on rank of $D_2(K_n \circ K_1)$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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