Communications in Mathematics and Applications
Vol. 15, No. 3, pp. 1089–1098, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i3.2779

http://www.rgnpublications.com

Research Article

A Study of Coupled Fixed Point in Multiplicative *S*-Metric Space

Manjusha P. Gandhi*[1], Anushree A. Aserkar[1] and Shiney Chib[2]

[1] *Department of Applied Mathematics and Humanities, Yeshwantrao Chavan College of Engineering, Nagpur 441110, Maharashtra, India*
[2] *Datta Meghe Institute of Management Studies, Nagpur, Maharashtra, India*
***Corresponding author:** manjusha_g2@rediffmail.com

Received: June 18, 2024 **Accepted:** August 13, 2024

Abstract. This study presents a novel common coupled fixed point result for two mappings in multiplicative S-metric space. Uniqueness of this point is proven through the utilization of a compatibility condition, validated through a provided example. Additionally, an application to integral equations is discussed within this article.

Keywords. Coupled fixed point, Compatible mappings, Multiplicative S-metric space

Mathematics Subject Classification (2020). 47H10, 54H25

Copyright © 2024 Manjusha P. Gandhi, Anushree A. Aserkar and Shiney Chib. *This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.*

1. Introduction

In 1922, Polish mathematician, Banach [3], established a theorem guaranteeing the presence and uniqueness of a fixed point under suitable conditions. This finding is recognized as the Banach contraction principle. Over time, numerous researchers have expanded upon and develop this principle in various ways.

Bhaskar and Lakshmikantham [4] pioneered the idea of coupled fixed points, leveraging the mixed monotone property to show several theorems in this area. Subsequently, Lakshmikantham and Ćirić [14] improved and generalized this work by giving the notion of coupled coincidence points of functions applying the mixed g-monotone property.

For further exploration of coupled fixed point theory in diverse spaces like partial order metric spaces, modular function spaces, cone metric spaces, G-metric spaces and complex-valued metric spaces, see, Aydi *et al.* [2], Chifu and Petrusel [5], Cho *et al.* [6], Đorić *et al.* [7], Gandhi and Aserkar [8], Gu [9], He *et al.* [10], Jain *et al.* [11], Karapınar [13], Lakshmikantham and Ćirić [14], Mohammad *et al.* [15], Shatanawi *et al.* [17], and Singh *et al.* [18]. However, research in multiplicative S-metric spaces remains scarce, prompting the authors to explore into this area.

This study aims to create a unique common coupled fixed point theorem for two functions in multiplicative S-metric space by introducing a novel contraction condition. The uniqueness of coupled fixed point is proven by employing $\omega^{\#}$-compatibility. An illustration is provided for the validation of the theorem, and potential application is discussed in this paper.

Definition 1.1 ([16]). Assume $S : M^3 \to [0,\infty)$ is a function, that satisfy the constraints, as here under, for every $h,i,j,k \in M$,

(i) $S(h,i,j) = 0$, iff $h = i = j$.

(ii) $S(h,i,j) \le S(h,h,k) + S(i,i,k) + S(j,j,k)$.

The couple (M,S) is named as S-metric space. Here M is non-empty set.

Example 1.1. Consider $M = R^n$, also $\|\cdot\|$ be norm on M. Then, S-metric spaces are:

(i) $S(h,i,j) = \|i + j - 2h\| + \|i - j\|$,

(ii) $S(h,i,j) = \|h - j\| + \|i - j\|$.

Definition 1.2 ([12]). Let M be a non-empty set. Then, the function $S : M^3 \to [0,\infty)$ is named as multiplicative S-metric on M, if and only if, the constraints below are true for all h,i,j and $x \in M$,

(i) $S(h,i,j) \ge 1$,

(ii) $S(h,i,j) = 1 \Longleftrightarrow h = i = j$,

(iii) $S(h,i,j) \le S(h,h,x)S(i,i,x)S(j,j,x)$, for $x \in M$.

Here (M,S) is named as multiplicative S-metric space.

Definition 1.3 ([12]). A sequence $\{h_n\}$ in multiplicative S-metric space (M,S) is multiplicative S-converges to some $h \in M$ iff for each $\varepsilon > 1$, there exists a $H \in \mathbb{N}$ such that

$$S(h_n,h_n,h) < \varepsilon, \quad \text{for all } n > H.$$

Definition 1.4 ([12]). The sequence $\{h_n\}$ in multiplicative S-metric space (M,S), is known as multiplicative S-Cauchy sequence in M if and only if, for every $\varepsilon > 1$, there occurs a $H \in \mathbb{N}$ such that $S(h_n,h_n,h_m) < \varepsilon$, for each $n,m > H$.

Definition 1.5 ([12]). The multiplicative S-metric space (M,S) is complete if and only if, each multiplicative S-Cauchy sequence in M is multiplicative S-convergent in M.

Definition 1.6 ([4]). Let M be a non-empty set. A pair $(h,i) \in M \times M$ is called as coupled fixed point of the mapping $T : M \times M \to M$ if $T(h,i) = h$ and $T(i,h) = i$.

Definition 1.7 ([14]). Let M be a nonempty set.

(i) A pair $(gh, gi) \in M \times M$ is known as coupled coincidence point of the functions $T : M \times M \to M$ and $g : M \to M$ if $T(h,i) = gh$ and $T(i,h) = gi$.

(ii) A pair $(h,i) \in M \times M$ is known as common coupled fixed point of functions $T : M \times M \to M$ and $g : M \to M$ if $T(h,i) = gh = h$ and $T(i,h) = gi = i$.

Definition 1.8 ([1]). Let M be a non-empty set. The functions $T : M \times M \to M$ and $g : M \to M$ are called

(i) ω-compatible if $g(T(h,i)) = T(gh, gi)$, whenever $T(h,i) = gh$ and $T(i,h) = gi$.

(ii) $\omega^{\#}$-compatible if $g(T(h,h)) = T(gh, gh)$, whenever $T(h,h) = gh$.

2. Main Result

Theorem 2.1. *Let (M,S) be a multiplicative S-metric space, $T : M \times M \to M$ and $g : M \to M$ satisfy the following constraints:*

(i) *$T(M \times M) \subset g(M), g(M)$ is multiplicative complete subspace of M,*

(ii) *$S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)] \le S^p(gh,gh,gj)S^p(gi,gi,gk)$, for $p \in (0,1)$,*

(iii) *T and g are $\omega^{\#}$-compatible mappings.*

Then, T and g have unique common coupled fixed point $(j,j) \in M \times M$.

Proof. Let $(h_0, i_0) \in M$. Since$T(M \times M) \subset g(M)$, we can choose $(h_1,i_1),(h_2,i_2) \in M$ such that $g(h_1) = T(h_0,i_0)$, $g(i_1) = T(i_0,h_0)$ and $g(h_2) = T(h_1,i_1)$, $g(i_2) = T(i_1,h_1)$.

Continuing in this way, we can construct the sequences $\{gh_n\}$ and $\{gi_n\}$ in $g(M)$ such that $g(h_{n+1}) = T(h_n,i_n)$, $g(i_{n+1}) = T(i_n,h_n)$, for all $n \ge 0$.

$$
\begin{aligned}
&S(gh_n,gh_n,gh_{n+1})S(gi_n,gi_n,gi_{n+1})\\
&\quad = S(T(h_{n-1},i_{n-1}),T(h_{n-1},i_{n-1}),T(h_n,i_n))S(T(i_{n-1},h_{n-1}),T(i_{n-1},h_{n-1}),T(i_n,h_n))\\
&\quad \le S^p(gh_{n-1},gh_{n-1},gh_n)S^p(gi_{n-1},gi_{n-1},gi_n)\\
&\quad \le S^{p^2}(gh_{n-2},gh_{n-2},gh_{n-1})S^{p^2}(gi_{n-2},gi_{n-2},gi_{n-1})\\
&\quad \vdots\\
&\quad \le S^{p^n}(gh_0,gh_0,gh_1)S^{p^n}(gi_0,gi_0,gi_1), \quad \text{for all natural numbers } n.
\end{aligned}
$$

Now

$$
\begin{aligned}
&S(gh_m,gh_m,gh_n)S(gi_m,gi_m,gi_n)\\
&\quad \le S(gh_m,gh_m,gh_{m+1})S(gh_m,gh_m,gh_{m+1})S(gh_n,gh_n,gh_{m+1})\\
&\qquad \cdot S(gi_m,gi_m,gi_{m+1})S(gi_m,gi_m,gi_{m+1})S(gi_n,gi_n,gi_{m+1})\\
&\quad \le S^2(gh_m,gh_m,gh_{m+1})S^2(gi_m,gi_m,gi_{m+1})S(gh_n,gh_n,gh_{n+1})S(gh_n,gh_n,gh_{n+1})\\
&\qquad S(gh_{m+1},gh_{m+1},gh_{n+1})S(gi_n,gi_n,gi_{n+1})S(gi_n,gi_n,gi_{n+1})S(gi_{m+1},gi_{m+1},gi_{n+1})\\
&\quad = S^2(gh_m,gh_m,gh_{m+1})S^2(gi_m,gi_m,gi_{m+1})S^2(gh_n,gh_n,gh_{n+1})\\
&\qquad \cdot S^2(gi_n,gi_n,gi_{n+1})S(gh_{m+1},gh_{m+1},gh_{n+1})S(gi_{m+1},gi_{m+1},gi_{n+1})
\end{aligned}
$$

$$= S^2(gh_m,gh_m,gh_{m+1})S^2(gi_m,gi_m,gi_{m+1})S^2(gh_n,gh_n,gh_{n+1})$$
$$\cdot S^2(gi_n,gi_n,gi_{n+1})S(T(h_m,i_m),T(h_m,i_m),T(h_n,i_n))$$
$$\cdot S(T(i_m,h_m),T(i_m,h_m),T(i_n,h_n))$$
$$\le S^2(gh_m,gh_m,gh_{m+1})S^2(gi_m,gi_m,gi_{m+1})S^2(gh_n,gh_n,gh_{n+1})$$
$$\cdot S^2(gi_n,gi_n,gi_{n+1})S^p(gh_m,gh_m,gh_n)S^p(gi_m,gi_m,gi_n).$$

Therefore,

$$S^{1-p}(gh_m,gh_m,gh_n)S^{1-p}(gi_m,gi_m,gi_n)$$
$$\le S^{2p^m}(gh_0,gh_0,gh_1)S^{2p^m}(gi_0,gi_0,gi_1)S^{2p^n}(gh_0,gh_0,gh_1)S^{2p^n}(gi_0,gi_0,gi_1).$$

Therefore,

$$S(gh_m,gh_m,gh_n)S(gi_m,gi_m,gi_n)$$
$$\le S^{\frac{2p^m}{1-p}}(gh_0,gh_0,gh_1)S^{\frac{2p^m}{1-p}}(gi_0,gi_0,gi_1)S^{\frac{2p^n}{1-p}}(gh_0,gh_0,gh_1)S^{\frac{2p^n}{1-p}}(gi_0,gi_0,gi_1).$$

This implies

$$S(gh_m,gh_m,gh_n)S(gi_m,gi_m,gi_n)\to 1, \quad \text{as } n,m\to\infty$$

such that

$$S(gh_m,gh_m,gh_n)\to 1,\ S(gi_m,gi_m,gi_n)\to 1, \quad \text{as } n,m\to\infty.$$

Thus, the sequences $\{gh_n\}$ and $\{gi_n\}$are Cauchy sequences in $g(M)$.

Due to the completeness of $g(M)$, there exists $g(h),g(i)\in g(M)$ such that $\{gh_n\}$ and $\{gi_n\}$ converges to $g(h)$ and $g(i)$ correspondingly.

Now, we prove that $T(h,i)=g(h)$, $T(i,h)=g(i)$,

$$S(T(h,i),T(h,i),gh)S(T(b,h),T(i,h),gi)$$
$$\le S(T(h,i),T(h,i),gh_{n+1})S(T(h,i),T(h,i),gh_{n+1})S(gh,gh,gh_{n+1})$$
$$\cdot S(T(i,h),T(i,h),gi_{n+1})S(T(i,h),T(i,h),gi_{n+1})S(gi,gi,gi_{n+1})$$
$$\le S^2(T(h,i),T(h,i),gh_{n+1})S^2(T(i,h),T(i,h),gi_{n+1})S(gh,gh,gh_{n+1})S(gi,gi,gi_{n+1})$$
$$= S^2(T(h,i),T(h,i),T(h_n,i_n))S^2(T(i,h),T(i,h),T(i_n,h_n)S(gh,gh,gh_{n+1})S(gi,gi,gi_{n+1})$$
$$\le S^{2p}(gh,gh,gh)S^{2p}(gi,gi,gi_n)S(gh,gh,gh_{n+1})S(gi,gi,gi_{n+1})$$
$$= S^{2p}(gh,gh,gh)S^{2p}(gi,gi,gi)S(gh,gh,gh)S(gi,gi,gi)=1, \quad \text{as } n\to\infty.$$

Thus

$$S(T(h,i),T(h,i),gh)S(T(i,h),T(i,h,gi)=1$$
$$\Longrightarrow \quad T(h,i)=gh,T(i,h)=gi$$

Thus (gh,gi) is coupled coincidence point of mappings T and g.

To prove uniqueness, consider $(h^*,i^*)\in M\times M$ such that (gh^*,gi^*) is coupled coincidence point of functions T and g.

$$S(gh,gh,gh^*)S(gi,gi,gi^*)=S(T(h,i),T(h,i),T(h^*,i^*))S(T(i,h),T(i,h),T(i^*,h^*))$$
$$\le S^p(gh,gh,gh^*)S^p(gi,gi,gi^*)$$

$\implies \quad S^{1-p}(gh,gh,gh^*)S^{1-p}(gi,gi,gi^*)=1$

$\implies \quad S^{1-p}(gh,gh,gh^*)=1,\ S^{1-p}(gi,gi,gi^*)=1$

$\implies \quad gh=gh^*,\ gi=gi^*.$

Thus (gh,gi) is unique coupled coincidence point of functions T and g.

$$\begin{aligned}S(gh,gh,gi)S(gi,gi,gh)&=S(T(h,i),T(h,i),T(i,h))S(T(i,h),T(i,h),T(h,i))\\&\le S^p(gh,gh,gi)S^p(gi,gi,gh)\end{aligned}$$

$\implies \quad S(gh,gh,gi)S(gigi,gh)=1$

$\implies \quad S(gh,gh,gi)=1,\ S(gi,gi,gh)=1$

$\implies \quad gh=gi.$

Therefore, (gh,gh) is unique coupled coincidence point of T and g.

To show that unique common coupled fixed point.

Let $gh\neq h$ and $gh=j$, then

$$j=gh=T(h,i),\ j=gi=T(i,h)$$

$\implies \quad j=gi=gj=T(i,j)=T(j,i)$

$\implies \quad i=j$

Therefore, $gh=T(h,h)=j$. Since T and g are compatible, we have

$$gj=g(gh)=gT(h,h)=T(gh,gh)=T(j,j).$$

By uniqueness of coupled coincidence point, we have $gi=gh$.

Thus $j=gj=T(j,j)$.

Therefore, $(j,j)=gj=T(j,j)$.

Hence (j,j) is unique common coupled fixed point of T and g. □

3. Example

Let $M=[0,1]$, $S(h,i,j)=2^{|2h-i-j|}$, $T(h,i)=\frac{h+i+1}{4}$, $g(h)=5h-2$, for all $h,i\in M$, $p=\frac{1}{2}$.

The below mentioned conditions are fulfilled:

(i) $S(h,i,j)\ge 1$,

(ii) $S(h,i,j)=1\iff h=i=j$,

(iii) $S(h,i,j)\le S(h,h,x)S(i,i,x)S(j,j,x)$, for $x\in M$.

R.H.S. $=S(h,h,x)S(i,i,x)S(j,j,x)=2^{|2h-h-x|}2^{|2i-i-x|}2^{|2j-j-x|}=2^{|h-x|+|i-x|+|j-x|}$.

L.H.S. $=S(h,i,j)=2^{|2h-i-j|}=2^{|2(h-x)-(i-x)-(j-x)|}$.

Obviously, L.H.S. < R.H.S.

$$S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)]\le S^p(gh,gh,gj)S^p(gi,gi,gk)$$

$$\begin{aligned}\text{L.H.S.}&=S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)]\\&=S\left(\frac{h+i+1}{4},\frac{h+i+1}{4},\frac{j+k+1}{4}\right)S\left(\frac{i+h+1}{4},\frac{i+h+1}{4},\frac{k+j+1}{4}\right)\end{aligned}$$

$$= 2^{\left|\frac{2(h+i+1)}{4}-\frac{h+i+1}{4}-\frac{j+k+1}{4}\right|}2^{\left|\frac{2(i+h+1)}{4}-\frac{i+h+1}{4}-\frac{k+j+1}{4}\right|}$$

$$= 2^{\left|\frac{h+i-j-k}{2}\right|},$$

$$\text{R.H.S.} = S^p(gh,gh,gj)S^p(gi,gi,gk)$$

$$= S^{\frac{1}{2}}(5h-2,5h-2,5j-2)S^{\frac{1}{2}}(5i-2,5i-2,5k-2)$$

$$= 2^{5\left|\frac{h+i-j-k}{2}\right|}.$$

Therefore, L.H.S. < R.H.S. Hence contraction condition is satisfied.

The function g is continuous. T and g are compatible functions.

Now $T\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\frac{1}{2}+\frac{1}{2}+1}{4} = \frac{1}{2}$ and $g\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)-2 = \frac{1}{2}$.

Hence $T\left(\frac{1}{2},\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = \frac{1}{2}$, i.e., T and g have unique common coupled fixed point $\left(\frac{1}{2},\frac{1}{2}\right)$.

4. Applications

In this section, we will discuss the presence and uniqueness of solution for a class of non-linear integral equations using the results obtained. We consider M is set of continuous functions defined on $[0,1]$.

Theorem 4.1. *Let $S: M^3 \to R^+$ by $S(h,i,j) = \sup_{\lambda\in(0,1)} 2^{|2h(\lambda)-i(\lambda)-j(\lambda)|}$. Then, (M,S) is a multiplicative S-metric space. The non-linear integral equations as here under, have unique solution in $CF[0,1]$,*

$$h(\lambda) = e(\lambda) + \int_0^1 l(\lambda,\mu)\{f_1(\mu,h(\mu)) + f_2(\mu,i(\mu))\}d\mu,$$

$$i(\lambda) = e(\lambda) + \int_0^1 l(\lambda,\mu)\{f_1(\mu,i(\mu)) + f_2(\mu,h(\mu))\}d\mu,$$

where $e:[0,1]\to R$, $l:[0,1]\times[0,1]\to R^+$, $f_1,f_2:[0,1]\times R\to[0,1]$, under the following conditions:

(i) *f_1,f_2,e,l are continuous functions,*

(ii) *the constants $\tau_1,\tau_2>0$ such that*

$$|f_1(\lambda,h)-f_2(\lambda,i)| \le \tau_1|h-i|,\ |f_2(\lambda,h)-f_2(\lambda,i)| \le \tau_2|h-i|, \quad \textit{for all } \lambda\in[0,1],\ h,i\in R,$$

(iii) *$0<\lambda = 2\max(\tau_1,\tau_1)\|l\|_\infty$, where $\|l\|_\infty = \sup\{l(\lambda,\mu):\lambda,\mu\in[0,1]\}$.*

Proof. Let $T: M\times M\to M$ and $g: M\to M$ are defined as

$$T(h,i)(\lambda) = e(\lambda) + \int_0^1 l(\lambda,\mu)\{f_1(\mu,h(\mu)) + f_2(\mu,i(\mu))\}d\mu, \quad \text{for all } h,i\in M,$$

$$gh(\lambda) = h(\lambda), \quad \text{for all } h\in M.$$

Then

$$G(M\times M)\subset g(M).$$

Now

$$
\begin{aligned}
&S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)]\\
&\quad \le S^p(gh,gh,gj)S^p(gi,gi,gk)\\
&\quad = \sup_{\lambda\in[0,1]} 2^{|2T(h,i)(\lambda)-T(h,i)(\lambda)-T(j,k)(\lambda)|} \sup_{\lambda\in[0,1]} 2^{|2T(i,h)(\lambda)-T(i,h)(\lambda)-T(k,j)(\lambda)|}\\
&\quad = \sup_{\lambda\in[0,1]} 2^{|T(h,i)(\lambda)-T(j,k)(\lambda)|} \sup_{\lambda\in[0,1]} 2^{|T(i,h)(\lambda)-T(k,j)(\lambda)|}\\
&\quad = \sup_{\lambda\in(0,1)} 2^{\left|e(\lambda)+\int_0^1 l(\lambda,\mu)\{f_1(\mu,h(\mu))+f_2(\mu,i(\mu))\}d\mu-e(\lambda)-\int_0^1 l(\lambda,\mu)\{f_1(\mu j(\mu))+f_2(\mu,k(\mu))\}d\mu\right|}\\
&\qquad \cdot \sup_{\lambda\in(0,1)} 2^{\left|e(\lambda)+\int_0^1 l(\lambda,\mu)\{f_1(\mu,i(\mu))+f_2(\mu,h(\mu))\}d\mu-e(\lambda)-\int_0^1 l(\lambda,\mu)\{f_1(\mu,k(\mu))+f_2(\mu j(\mu))\}d\mu\right|}\\
&\quad = \sup_{\lambda\in(0,1)} 2^{\left|\int_0^1 l(\lambda,\mu)\{f_1(\mu,h(\mu))-f_1(\mu,j(\mu))+f_2(\mu,i(\mu))-f_2(\mu,k(\mu))\}d\mu\right|}\\
&\qquad \cdot \sup_{\lambda\in(0,1)} 2^{\left|\int_0^1 l(\lambda,\mu)\{f_1(\mu,i(\mu))-f_1(\mu,k(\mu))+f_2(\mu,h(\mu))-f_2(\mu,j(\mu))\}d\mu\right|}\\
&\quad \le \sup_{\lambda\in(0,1)} 2^{\int_0^1 |l(\lambda,\mu)[f_1(\mu,h(\mu))-f_1(\mu,j(\mu))|+|f_2(\mu,i(\mu))-f_2(\mu,k(\mu))|]d\mu}\\
&\qquad \cdot \sup_{\lambda\in(0,1)} 2^{\int_0^1 |l(\lambda,\mu)[f_1(\mu,i(\mu))-f_1(\mu,k(\mu))|+|f_2(\mu,h(\mu))-f_2(\mu,j(\mu))|]d\mu}.
\end{aligned}
$$

From (ii),

$$
\begin{aligned}
&|f_1(\mu,h(\mu))-f_1(\mu,i(\mu))| \le \tau_1|h(\mu)-i(\mu)|,\\
&|f_2(\mu,h(\mu))-f_2(\mu,i(\mu))| \le \tau_2|h(\mu)-i(\mu)|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)]\\
&\quad \le \sup_{\lambda\in(0,1)} 2^{\int_0^1 l(\lambda,\mu)[\tau_1|h(\mu)-j(\mu)|+\tau_2|i(\mu)-j(\mu)|]d\mu} \sup_{\lambda\in(0,1)} 2^{\int_0^1 l(\lambda,\mu)[\tau_1|i(\mu)-j(\mu)|+\tau_2|h(\mu)-j(\mu)|]d\mu}\\
&\quad \le \left(\sup_{\lambda\in(0,1)} 2^{\int_0^1 l(\lambda,\mu)[|h(\mu)-j(\mu)|+|i(\mu)-k(\mu)|]d\mu}\right)^{\max(\tau_1,\tau_2)}\\
&\qquad \cdot \left(\sup_{\lambda\in(0,1)} 2^{\int_0^1 l(\lambda,\mu)[|i(\mu)-k(\mu)|+|h(\mu)-j(\mu)|]d\mu}\right)^{\max(\tau_1,\tau_2)}.
\end{aligned}
$$

Now, using Cauchy-Schwartz inequality

$$
\begin{aligned}
2^{\int_0^1 l(\lambda,\mu)[|h(\mu)-j(\mu)|+|i(\mu)-k(\mu)|]d\mu} &\le 2^{(\int_0^1 l^2(\lambda,\mu)d\mu)^{\frac{1}{2}}(\int_0^1(|h(\mu)-j(\mu)|+|i(\mu)-k(\mu)|)^2 d\mu)^{\frac{1}{2}}}\\
&\le \left(2^{\sup_{\lambda\in(0,1)}|h(\mu)-j(\mu)|+\sup_{\lambda\in(0,1)}|i(\mu)-k(\mu)|}\right)^{\|l\|_\infty}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
2^{\int_0^1 l(\lambda,\mu)[|i(\mu)-k(\mu)|+|h(\mu)-j(\mu)|]d\mu} &\le 2^{(\int_0^1 l^2(\lambda,\mu)d\mu)^{\frac{1}{2}}(\int_0^1(|i(\mu)-k(\mu)|+|h(\mu)-j(\mu)|)^2 d\mu)^{\frac{1}{2}}}\\
&\le \left(2^{\sup_{\lambda\in(0,1)}|i(\mu)-k(\mu)|+\sup_{\lambda\in(0,1)}|h(\mu)-j(\mu)|}\right)^{\|l\|_\infty}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&S[T(h,i),T(h,i),T(j,k)]S[T(i,h),T(i,h),T(k,j)]\\
&\le \left(\sup_{\lambda\in(0,1)} 2^{\sup\limits_{\lambda\in(0,1)}|h(\mu)-j(\mu)|+\sup\limits_{\lambda\in(0,1)}|i(\mu)-k(\mu)|}\right)^{\max(\tau_1,\tau_2)\|l\|_\infty}\\
&\quad\cdot\left(\sup_{\lambda\in(0,1)} 2^{\sup\limits_{\lambda\in(0,1)}|i(\mu)-k(\mu)|+\sup\limits_{\lambda\in(0,1)}|h(\mu)-j(\mu)|}\right)^{\max(\tau_1,\tau_2)\|l\|_\infty}\\
&\le \left(\sup_{\lambda\in(0,1)} 2^{|h(\mu)-j(\mu)|+|i(\mu)-k(\mu)|}\right)^{\max(\tau_1,\tau_2)\|l\|_\infty}\left(\sup_{\lambda\in(0,1)} 2^{|i(\mu)-k(\mu)|+|h(\mu)-j(\mu)|}\right)^{\max(\tau_1,\tau_2)\|l\|_\infty}\\
&\le \left(\sup_{\lambda\in(0,1)} 2^{|h(\mu)-j(\mu)|}\sup_{\lambda\in(0,1)} 2^{|i(\mu)-k(\mu)|}\right)^{2\max(\tau_1,\tau_2)\|l\|_\infty}\\
&\le \left(\sup_{\lambda\in(0,1)} 2^{|gh(\mu)-gj(\mu)|}\sup_{\lambda\in(0,1)} 2^{|gi(\mu)-gk(\mu)|}\right)^{2\max(\tau_1,\tau_2)\|l\|_\infty}\\
&= \left(\sup_{\lambda\in(0,1)} 2^{|2gh(\mu)-gh(\mu)-gj(\mu)|}\sup_{\lambda\in(0,1)} 2^{|2gi(\mu)-gi(\mu)-gk(\mu)|}\right)^{2\max(\tau_1,\tau_2)\|l\|_\infty}\\
&= \left(\sup_{\lambda\in(0,1)} 2^{|2gh(\mu)-gh(\mu)-gj(\mu)|}\sup_{\lambda\in(0,1)} 2^{|2gb(\mu)-gb(\mu)-gk(\mu)|}\right)^{2\max(\tau_1,\tau_2)\|l\|_\infty}\\
&\quad\cdot S^{2\max(\tau_1,\tau_2)\|l\|_\infty}(gh,gh,gj)S^{2\max(\tau_1,\tau_2)\|l\|_\infty}(gi,gi,gk).
\end{aligned}$$

Thus contraction condition is satisfied.

Thus T and g fulfill all conditions of Theorem 2.1. Therefore, T and g possesses unique common coupled fixed point (j,j).

Therefore,

$$T(j,j)=gj=j.$$

Hence (j,j) is unique solution of integral equation. □

5. Conclusion

We established a new common coupled fixed point theorem for pair of mappings in multiplicative S-metric space with an example and an Application.

Acknowledgement

The authors express their thankfulness to the college authorities for providing the facilities necessary to complete this research paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, M. A. Khan and S. Redenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, *Applied Mathematics and Computation* **217**(1) (2010), 195 – 202, DOI: 10.1016/j.amc.2010.05.042.

[2] H. Aydi, B. Samet and C. Vetro, Coupled fixed point results in cone metric spaces for $\tilde{w}$-compatible mappings, *Fixed Point Theory and Applications* **2011** (2011), Article number: 27, DOI: 10.1186/1687-1812-2011-27.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3**(1) (1922), 133 – 181, URL: https://eudml.org/doc/213289.

[4] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods & Applications* **65**(7) (2006), 1379 – 1393, DOI: 10.1016/j.na.2005.10.017.

[5] C. Chifu and G. Petrusel, New results on coupled fixed point theory in metric spaces endowed with a directed graph, *Fixed Point Theory and Applications* **2014** (2014), Article number: 151, DOI: 10.1186/1687-1812-2014-151.

[6] Y. J. Cho, M. H. Shah and N. Hussain, Coupled fixed points of weakly F-contractive mappings in topological spaces, *Applied Mathematics Letters* **24**(7) (2011), 1185 – 1190, DOI: 10.1016/j.aml.2011.02.004.

[7] D. Đorić, Z. Kadelburg and S. Radenović, Coupled fixed point results for mappings without mixed monotone property, *Applied Mathematics Letters* **25**(11) (2012), 1803 – 1804, DOI: 10.1016/j.aml.2012.02.022.

[8] M. Gandhi and A. Aserkar, Coupled fixed point theorem in quasi metric space, *Italian Journal of Pure and Applied Mathematics* **47** (2022), 449 – 457, URL: https://ijpam.uniud.it/online_issue/202247/30%20Gandhi-Aserkar.pdf.

[9] F. Gu, Some new common coupled fixed point results in two generalized metric spaces, *Fixed Point Theory and Applications* **2013** (2013), Article number: 181, DOI: 10.1186/1687-1812-2013-181.

[10] X. He, M. Song and D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, *Fixed Point Theory and Applications* **2014** (2014), Article number: 48, DOI: 10.1186/1687-1812-2014-48.

[11] M. Jain, K. Tas, S. Kumar and N. Gupta, Coupled common fixed point results involving a (φ,ψ)-contractive condition for mixed g-monotone operators in partially ordered metric spaces, *Journal of Inequalities and Applications* **2012** (2012), Article number: 285, DOI: 10.1186/1029-242X-2012-285.

[12] P. Kanchanapally and V. N. Raju, Common fixed point theorem in a multiplicative S-metric space with an application, *Communications in Mathematics and Applications* **12**(2) (2021), 303 – 314, DOI: 10.26713/cma.v12i2.1495.

[13] E. Karapınar, Couple fixed point theorems for nonlinear contractions in cone metric spaces, *Computers & Mathematics with Applications* **59**(12) (2010), 3656 – 3668, DOI: 10.1016/j.camwa.2010.03.062.

[14] V. Lakshmikantham and Lj. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* **70**(12) (2009), 4341 – 4349, DOI: 10.1016/j.na.2008.09.020.

[15] M. Mohammad, R. Jamal, J. Mishra, Q. A. Kabir and R. Bhardwaj, Coupled fixed point theorem and dislocated quasi-metric space, *International Journal of Pure and Applied Mathematics* **119**(10) (2018), 1249 – 1260.

[16] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorem in S-metric spaces, *Matematički Vesnik* **64**(3) (2012), 258 – 266, URL: http://www.vesnik.math.rs/landing.php?p=mv123.cap&name=mv12309.

[17] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, *Mathematical and Computer Modelling* **55**(3-4) (2012), 680 – 687, DOI: 10.1016/j.mcm.2011.08.042.

[18] N. Singh, D. Singh, A. Badal and V. Joshi, Fixed point theorems in complex valued metric spaces, *Journal of the Egyptian Mathematical Society* **24**(3) (2016), 402 – 409, DOI: 10.1016/j.joems.2015.04.005.