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Research Article

A Numerical Scheme for Solving Fourth-Order Convection–Reaction–Diffusion Problems with Boundary Layers

Charuka D. Wickramasinghe

Karmanos Cancer Institute, Department of Oncology, School of Medicine, Wayne State University, Detroit, USA
gi6036@wayne.edu

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Abstract. This paper presents a numerical approach for decoupling singularly perturbed boundary value problems involving fourth-order ordinary differential equations, characterized by a small positive parameter ϵ multiplying the highest derivative. Such equations arise in various engineering and physics applications, including the modeling of diffusing chemical species, viscous flows with convection and diffusion, and heat transfer in electronic chips or microfluidic channels. We focus on problems with Lidstone boundary conditions and demonstrate how the fourth-order equation can be decomposed into a system of two second-order problems—one independent of ϵ, and the other singularly perturbed with ϵ multiplying the highest derivative. These problems often exhibit boundary layers, where the solution undergoes rapid changes near the domain boundaries. Numerical solutions to such higher-order problems are typically more challenging than those for lower-order ones. To address this, we propose a linear finite element method combined with a Shishkin mesh to accurately resolve boundary layers. We prove that the solution obtained from the decoupled second-order system is equivalent to that of the original fourth-order problem. The proposed method is direct and highly accurate, with computational time increasing linearly with the number of grid points.

Keywords. Shishkin mesh, Finite element algorithm, Boundary layers, Convection-diffusion problems

Mathematics Subject Classification (2020). 65L10, 65L11, 65L50, 65L60

1. Introduction

In this paper, we consider the following stationary state fourth order singularly perturbed differential equation with Lidstone boundary conditions:

$$\left.\begin{array}{l} L_\epsilon = -\varepsilon u^{iv}(x) - a(x)u'''(x) + b(x)u''(x) = -f(x), \\ u(0)=0,\ u(1)=0,\ u''(0)=0,\ u''(1)=0. \end{array}\right\} \tag{1.1}$$

The function $f(x)$, $a(x)$, and $b(x)$ are all smooth and satisfy $a(x) \geq \alpha > 0$ and $b(x) \geq \beta > 0$. The parameter ϵ is assumed to be a small positive value, such that $0 < \epsilon \leq 1$. While it may be easy to analytically solve this problem in some cases, finding the solution u with analytical techniques can be difficult or even impossible for a general function f. The well-posedness of problem (1.1) has been discussed in more deeply by Ehme *et al.* [3], and Sun and Styne [16].

The convection-diffusion-reaction equation is used in three processes: convection, which involves the movement of materials from one region to another; diffusion, which involves the movement of materials from an area of high concentration to an area of low concentration; and reaction, which involves decay, adsorption, and the reaction of substances with other components. Singularly perturbed problems have many applications in engineering and applied mathematics, including chemicals and nuclear engineering , linearized Navier-Stokes equation at high Reynolds number, elasticity, aerodynamics, oceanography, meteorology, modeling of semiconductor devices, control theory, and oil extraction from underground reservoirs, and in many other fields (Kreiss and Lorenz [8], and Polak *et al.* [11]). However, solving these problems numerically presents major computational difficulties due to boundary layers, where the solution changes rapidly. The study of second-order singularly perturbed differential equations are quite large, as seen in the extensive literature and references cited therein (Farrell [5], Kadalbajoo and Reddy [6], and Natesan and Ramanujam [10]). However, very few studies have addressed singularly perturbed fourth-order boundary value problems in the literature.

The following paragraph presents a brief overview of analytical and numerical methods used for solving singularly perturbed fourth-order differential equations. El-Zahar [4] introduced the Differential Transform Method (DTM) as an alternative to existing methods for solving higher-order singularly perturbed boundary value problems (SPBVPs). Vrabel [17] provided a detailed analysis of the boundary layer phenomenon subject to the Lidstone boundary conditions by analyzing the integral equation associated with the SPBVPs. Sun and Stynes [15] developed piecewise polynomial Galerkin finite-element methods on a Shishkin mesh, which achieved almost optimal convergence results in various norms. Shanthi and Ramanujam [13] transformed the SVBVP into a system of weakly coupled system of two second-order ODEs and then used the fitted operator method (FOM), fitted mesh method (FMM), and boundary value technique (BVT) to approximate the solution.

The problem being studied involves rapidly changing solutions in very thin regions near the boundary. Traditional numerical methods often fail to accurately capture these changes, which can result in errors across the entire domain. To address this issue, various methods such as Bakhavalov and Gartland meshes have been developed (Bakhvalov [1], Miller *et al.* [9], Roos *et al.* [12], and Wickramasinghe and Ahire [19]). In this study, we analyze a standard finite element method combined with the Shishkin mesh, which is a type of local refinement strategy introduced by a Russian mathematician Grigorii Ivanovich Shishkin in 1988 [14]

(Kopteva and Riordan [7]). Finite element methods on Shishkin meshes in 1D were first studied 1995 by Sun and Styne [16]. The analysis for second order problems was published in the two books by Miller *et al.* [9], and Roos *et al.* [12]. The goal is to propose a new finite element algorithm that is reliable, effective, and easy to implement, and can be used to solve (1.1) and even higher order singularly perturbed differential equations. The main advantage of decoupling system is that it reduces both memory space and time requirements.

It is noted that, for simplicity, the current paper focuses on analyzing a one-dimensional problem where the coefficients $a(x)$ and $b(x)$ are both constant. However, the analysis could be extended to two dimensions and variable coefficients, although this may present some challenges. Additionally, the problem could be expanded to include non-homogeneous boundary conditions through a simple linear transformation.

The rest of the article is organized as follows: In Section 2, we introduce the decouple formulation of (1.1). In Section 3, we present Shishkin mesh method and the finite element algorithm. We also present error estimate results of the decouple formulations. In Section 4, we present numerical results to validate out theoretical results. Throughout the following text, the generic positive constants C, a and b may take different values in different formulas but is always independent of the mesh and the small positive parameter ϵ.

2. The Decouple Formulation

Hereafter, we will consider the following problem by setting $a(x) = a$ = constant and $b(x) = b$ = constant in equation (1.1):

$$\left.\begin{array}{l} L_\epsilon = -\varepsilon u^{iv}(x) - a u'''(x) + b u''(x) = -f(x), \\ u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0. \end{array}\right\} \tag{2.1}$$

The function $f(x)$, $a(x)$, and $b(x)$ are assumed to be sufficiently smooth for $0 \le x \le 1$, where

$$\left.\begin{array}{l} a \ge \alpha > 0, \\ b \ge \beta > 0, \\ a + \frac{1}{2}b' > c > 0, \quad \text{for all } x \in [0,1]. \end{array}\right\} \tag{2.2}$$

Under the conditions in (2.2) the problem (2.1) is well posed (Sun and Styne [16]). Let $(\cdot,\cdot)$ denote the usual $L^2(0,1)$ inner product. We define the bilinear form of equation (2.1) as follows:

$$A_\varepsilon(u,v) = (-\varepsilon u'', v'') + (a u'', v') - (b u', v') = -(f, v), \tag{2.3}$$

for all $u, v \in H_0^2(0,1)$.

The weighted energy norm is given by

$$|||v||| = \{\varepsilon |v|_2^2 + \|v\|_1^2\}^{\frac{1}{2}}, \quad \text{for all } v \in H_0^2(0,1).$$

Assume (2.2) holds. Then there exist positive constants C_1, and C_2 such that for all $u, v \in H_0^2(0,1)$ (Sun and Stynes [15]),

$$|A_\varepsilon(u,v)| \le C_1 \varepsilon^{-\frac{1}{2}} |||u||| \cdot |||v||| \tag{2.4}$$

and

$$C_2 |||u|||^2 \le |A_\varepsilon(u,u)|. \tag{2.5}$$

The weak formulation of (2.1) is to find $u \in H_0^2(0,1)$ such that

$$A_\varepsilon(u,v) = (-f,v), \quad \text{for all } v \in H_0^2(0,1). \tag{2.6}$$

By the Lax-Milgram lemma, equation (2.6) has a unique solution u in $H_0^2(0,1)$.

We are now able to present our decoupled formulation. It is widely recognized that numerical solutions of higher order problems, such as (2.1), are significantly more challenging than those of lower order problems. To address this issue, we decouple (2.1) into a system of lower order differential equations, as follows:

$$\left.\begin{aligned} &-w''(x) = f(x), \quad \text{for } x \in (0,1),\\ &w(0)=0, \;\; w(1)=0, \end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned} &-\varepsilon u''(x) - au'(x) + bu(x) = w(x),\\ &u(0)=0, \;\; u(1)=0. \end{aligned}\right\} \tag{2.8}$$

The equation represented by equation (2.7) is a standard Poison equation, which has the same source term $f(x)$ as equation (2.1). Assuming that $f(x)$ belongs to $L^2(0,1)$ and the given boundary conditions for equation (2.7) are met, the problem defined by equation (2.7) is well-posed, according to Ciarlet [2]. These kind of finite element decouple formulations can be found from the literature for some particular problems for two dimensions (Wickramasinghe [18], and Li *et al.* [20]).

Equation (2.8) is a second-order problem that involves a convection-reaction-diffusion process with a singular perturbation. The source term for this equation is represented by $w(x)$, which is the solution to the problem defined by equation (2.7). Under following assumptions,

$$\left.\begin{aligned} &a \ge \beta > 0,\\ &b \ge 0,\\ &b + \tfrac{a'}{2} > 0, \quad \text{for all } x \in [0,1], \end{aligned}\right\} \tag{2.9}$$

the problem defined by equation (2.8) is well-posed.

In order to establish a connection between the solution u obtained from equations (2.7) and (2.8) and the fourth-order problem represented by equation (2.1), we present the following lemma. Let us define $H^m(0,1)$ as the Sobolev space comprising functions whose ith derivative, $0 \le i \le m$, is square-integrable.

Lemma 2.1. *The solution $u \in H^4(0,1)$ obtained through* (2.7) *and* (2.8) *satisfies the following fourth order differential equation:*

$$\left.\begin{aligned} &-\varepsilon u^{iv}(x) - au'''(x) + bu''(x) = -f(x),\\ &u(0)=0, \;\; u(1)=0, \;\; u''(0)=0, \;\; u''(1)=0. \end{aligned}\right\} \tag{2.10}$$

Proof. We first apply the differential operator $L = d^2/dx^2$ to both sides of equation (2.8), which gives us:

$$-\varepsilon u^{iv}(x) - au'''(x) + bu''(x) = w''(x). \tag{2.11}$$

As we know from equation (2.7), $-w''(x) = f(x)$. Therefore, the equation:

$$-\varepsilon u^{iv}(x) - au'''(x) + bu''(x) = -f(x)$$

holds.

To verify the boundary conditions, we apply the differential operator L to the boundary conditions $u(0)=0$ and $u(1)=0$ in equation (2.8). This yields the boundary conditions $u''(0)=0$ and $u''(1)=0$. Thus, the conclusion of Lemma 2.1 holds. □

3. The Finite Element Method on Shishkin Mesh

In this section, we present a linear finite element method to solve the singularly perturbed boundary value problem (2.1) based on the results obtained in the previous section. We utilize the Shishkin mesh to present error estimates for the singularly perturbed convection reaction diffusion problem (2.8), and we adopt the same definitions and notations as in [16].

3.1 Layer Adapted Shishkin Mesh

Given an even positive integer N, the Shishkin mesh X_s^N is constructed by dividing the interval $[0,1]$ into two subintervals. We shall consider a mesh $X_s^N : 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ that is equidistant in $[\tau,1]$ but graded in $[0,\tau]$, where we choose the transition point τ as Shishkin does:

$$\tau = \min\{1/2, (s+1)\alpha^{-1}\varepsilon N\}$$

which depends on ε and N, where s is the order of the highest derivative.

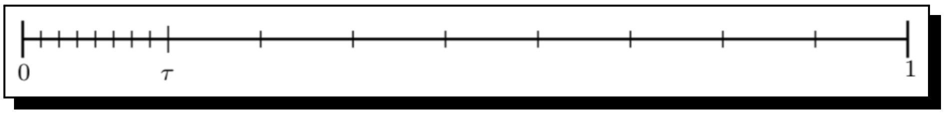

Figure 1. 1D Shishkin mesh with transition point τ

Assumption 3.1. *In this study, we will assume that*

$$\epsilon \le CN^{-1} \tag{3.1}$$

as is generally the case for discretizations of convection-dominated problems.

3.2 The Finite Element Algorithm

Let V be a Hilbert space with norm $\|\cdot\|_v$ (but we shall often omit the subscript v to simplify the notation) and scalar product. In the discretization of second-order differential equations with domain I, one generally chooses V as a subset of the Sobolev space $H^1(I)$. Let $I=[0,L]$ be an interval and let the $n+1$ node points $\{x_i\}_{i=0}^n$ define a partition. $I : 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = L$ of I into n sub-intervals of length $h_i = x_i - x_{i-1}$ for $i = 1,\cdot,\cdot,\cdot,n$, and $H = \max_i h_i$. On the mesh I we define the space $V_n \subseteq V$ of continuous piecewise linear functions by $V_n = \{v : v \in C^0(I), v|_{I_i} \in P_I(I_i)\}$, where $C^0(I)$ denotes the space of continuous functions on I, and $P_I(I_i)$ denotes the space of linear functions on I_i.

Let $A_\varepsilon^n(u,v)$ be the discrete bilinear form of $A_\varepsilon(u,v)$ in equation (2.3). There exists a positive constant h_0 (independent of ε) such that for $H \le h_0$ (Sun and Stynes [15]), we have

$$C_1 |||v|||^2 \le A_\varepsilon^n(v,v), \quad \text{for all } v \in H_0^2. \tag{3.2}$$

Then, the Galerkin discretization of problem (2.1) is to find $u_n \in V_n$ such that

$$A_\varepsilon^n(u_n,v) = (f,v), \quad \text{for all } v \in V_n. \tag{3.3}$$

We will use the following notations:

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx, \qquad (f,v) = \int_0^1 f(x)v(x)\,dx, \qquad b(u,v) = \varepsilon a(u,v) + (u',v) + (u,v),$$

for all $u,v \in V := H_0^1(0,1)$.

The variational formulation of (2.7) is to find $w \in V := H_0^1(0,1)$ such that

$$a(w,v) = (f,v), \quad \text{for all } v \in V. \tag{3.4}$$

The variational formulation of (2.8) is to find $u \in V := H_0^1(0,1)$ such that

$$b(u,v) = (f,v), \quad \text{for all } v \in V. \tag{3.5}$$

Algorithm 1 summarizes the basic steps in computing the finite element solution w_n for the two point boundary value problem (2.7).

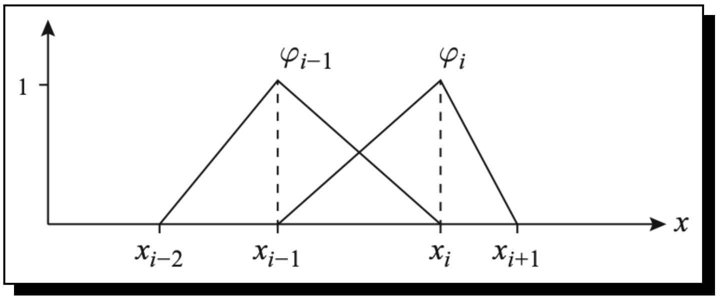

Figure 2. The linear hat basis functions in 1D

Algorithm 1

Step 1: Create a mesh $X_s^N : 0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ and define the corresponding space of continuous piecewise linear functions $V_{n,0} = \{v \in V_n : v(0) = v(1) = 0\}$ with has basis functions:

$$\varphi_j(x_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \text{ for } i,j = 0,1,2,\cdots,n. \end{cases}$$

Step 2: Compute the $(n-1)\times(n-1)$ matrix A and the $(n-1)\times 1$ vector b, with entries

$$A_{i,j} = \int_0^1 \varphi_j'\varphi_i'\,dx, \quad b_i = \int_0^1 f\varphi_i\,dx. \tag{3.6}$$

Step 3: Solve the linear system

$$A\xi = b. \tag{3.7}$$

Step 4: Set

$$w_n = \sum_{j=1}^{n-1} \xi_j \varphi_j. \tag{3.8}$$

Next, we introduce Algorithm 2 for solving the fourth-order singularly perturbed convection reaction diffusion equation (2.10). To obtain the finite element solution of the singularly perturbed boundary value problem (2.10), we utilize the decomposition method described through equations (2.7) and (2.8).

Algorithm 2

Let k be the order of the interpolation polynomial. For, any $f \in H^{-1}(0,1)$ and $k \geq 1$, we consider the following steps:

Step 1: Find $w_n \in V_n^k$ in the weak formulation of the Poisson equation (2.7) on a Shishkin mesh X_s^N

$$a(w_n, v) = (f, v), \quad \text{for all } v \in V. \tag{3.9}$$

Step 2: Set

$$-w_n = f_\epsilon, \tag{3.10}$$

where f_ϵ is the source terms of the equation singularly purtubed second order differential equation (2.8).

Step 3: Then find $u_n \in V_n^k$ in the weak formulation of equation (2.8) on the Shishkin mesh X_s^N for sufficiently large N (an even positive integer) independent of ϵ,

$$b(u_n, v) = (f_\epsilon, v), \quad \text{for all } v \in V. \tag{3.11}$$

The discretized linear systems corresponding to the stiffness matrix S, convection matrix C, and mass matrix M are obtained as shown below. Equation (3.12) represents the discretized linear system of (3.9), while equation (3.13) corresponds to the discretized linear system of (3.11),

$$\left(\frac{\epsilon}{h} S\right) W = F_1, \tag{3.12}$$

$$\left(\frac{\epsilon}{h} S - C + \frac{h}{6} M\right) U = F_2, \tag{3.13}$$

where $U, F \in \mathbb{R}^{n-1}$ and $S, C, M \in \mathbb{R}^{n-1 \times n-1}$ with:

$$W := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{bmatrix}, \quad F_1 := \begin{bmatrix} (f, \phi_1) \\ (f, \phi_2) \\ \vdots \\ (f, \phi_{n-1}) \end{bmatrix}, \quad U := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix}, \quad F_2 := \begin{bmatrix} (f_\epsilon, \phi_1) \\ (f_\epsilon, \phi_2) \\ \vdots \\ (f_\epsilon, \phi_{n-1}) \end{bmatrix},$$

$$S := \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad C := \frac{1}{2} \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}, \quad M := \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{bmatrix}.$$

3.3 The Error Estimates

In this section, we present maximum norm error estimate result for our model problem (2.1) using linear finite elements. As a special case to our model problem we present maximum norm error estimate result with $\varepsilon = 1$ for linear finite elements. In the case of $\varepsilon = 1$ there is no boundary layer and thus the Shishkin mesh can be replaced by an uniformly refined mesh (equidistant). However, we present numerical results for different values of ε.

Lemma 3.1. *Suppose that we use a sufficiently accurate quadrature rule, namely, that* $k+1 \geq 2$. *Let* u *be the exact solution to equation* (2.1) *with* $\varepsilon = 1$ *and* u_n *be the finite element approximation to the weak formulation of equation* (2.8) *with* $\varepsilon = 1$, *on a uniform mesh. Then, we have*

$$\|u - u_n\|_\infty \leq CN^{-2}. \tag{3.14}$$

Proof. On a uniformly refined mesh it is well known that one has

$$|||u - u_n||| \leq CN^{-2}. \tag{3.15}$$

It is easy to see that

$$\|u - u_n\|_\infty \leq \|u - u_n\|_1 \leq |||u - u_n|||. \tag{3.16}$$

Combining equations (3.15) and (3.16) the conclusion holds. □

Remark 3.1. Let u be the solution of problem (2.1). Let u_n be the solution of (3.3) on the Shishkin mesh X_s^N and k be the order of the interpolation polynomial. Then, for N sufficiently large (independently of ε) as shown in [16, Corollary 5.1], we have

$$\|u - u_n\|_\infty \leq C(N^{-1} \ln N)^{\min(2,k+1)}. \tag{3.17}$$

4. Numerical Results

In this section, we present a few numerical experiments to illustrate the computational method discussed in this paper. The numerical experiments are performed on a laptop computer with MATLAB R2022a in MACBOOK AIR with M1 chip. The linear finite elements are used to solve our model problem. We use the following numerical convergence rate to validate the theoretical convergence rates:

$$\mathcal{R} = \frac{\ln \|u - u_n\|_\infty - \ln \|u - u_{n-1}\|_\infty}{\ln 2},$$

where u_n is the finite element approximation after n mesh refinements and u is the exact solution at the same mesh level as the finite element approximation u_n is calculated. We use the following model problem to validate theoretical results over the following three examples:

$$-\epsilon u^{iv}(x) - u'''(x) + u''(x) = -f(x), \quad \text{for } x \in (0,1), \tag{4.1}$$

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0, \tag{4.2}$$

where $f(x)$ is chosen so that the exact solution is

$$u(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} - \frac{x^2 + x + 1}{2} - \epsilon, \tag{4.3}$$

where

$$c_1 = -c_2 + \left(\frac{1}{2} + \epsilon\right),$$

$$c_2 = \frac{e^{r_1}\left(\frac{1}{2}+\epsilon\right)-\left(\frac{3}{2}+\epsilon\right)}{e^{r_1}-e^{r_2}},$$
$$r_1 = \frac{2}{1+\sqrt{(1+4\epsilon)}},$$
$$r_2 = \frac{2}{1-\sqrt{(1+4\epsilon)}}.$$

Example 4.1. In this example, we compared the finite element approximation to equation (4.1) with its exact solution (4.3). Table 1 and Table 2 summarize the maximum norm errors for different ε values for both uniform and Shishkin meshes with various mesh sizes N. Both tables show that when the problem has high singularly perturbed features, our numerical approximation captures the true solution under the Shishkin mesh, while the uniform mesh fails to capture the solution. This observation strongly agrees with existing theoretical results.

Table 1. $\|u-u_n\|_\infty$ for $\varepsilon = 10^{-10}$, 10^{-8} and 10^{-6} for uniform and Shishkin meshes

N	$\epsilon = 10^{-10}$		$\epsilon = 10^{-8}$		$\epsilon = 10^{-6}$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
128	0.0300	5.2171e-05	0.0300	6.4006e-05	0.0311	6.3812e-05
256	0.0301	1.5676e-05	0.0301	1.5991e-05	0.0342	1.5830e-05
512	0.0302	5.3830e-06	0.0303	4.8273e-06	0.0429	4.7611e-06
1024	0.0302	1.8953e-06	0.0309	1.4789e-06	0.0512	1.4328e-06
2048	0.0302	7.0754e-07	0.0329	4.4459e-07	0.0514	4.2862e-07
4096	0.0303	2.8955e-07	0.0395	1.3130e-07	0.0510	1.2852e-07
8192	0.0307	4.0176e-08	0.0500	3.8094e-08	0.0501	3.8150e-08

Table 2. $\|u-u_n\|_\infty$ for $\varepsilon = 10^{-4}$, 10^{-2} and 1 for uniform and Shishkin meshes

N	$\epsilon = 10^{-4}$		$\epsilon = 10^{-2}$		$\epsilon = 1$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
128	0.0491	5.2171e-05	0.0010	2.8929e-04	6.8864e-07	6.7792e-07
256	0.0467	1.5676e-05	2.4386e-04	1.5611e-04	1.7081e-07	1.6948e-07
512	0.0422	5.3830e-06	6.0942e-05	8.2720e-05	4.2536e-08	4.2370e-08
1024	0.0342	1.8953e-06	1.5157e-05	4.3196e-05	1.0613e-08	1.0593e-08
2048	0.0221	7.0754e-07	3.7826e-06	2.2282e-05	2.6510e-09	2.6486e-09
4096	0.0097	2.8955e-07	9.4508e-07	1.1371e-05	6.6242e-10	6.6407e-10
8192	0.0027	1.3217e-07	2.3621e-07	5.7469e-06	1.6553e-10	1.7331e-10

Remark 4.1. All of the N and ε values in Table 1 satisfy Assumption 3.1. However, for some values of N and ε in Table 2, we have $\varepsilon \geq CN^{-1}$.

Figures 3 and 4 compare the exact solution with the numerical approximation under a uniformly refined meshes and Shishkin meshes, respectively. When the singularly perturbed effect is high, it can be observed from Figures 3 that the numerical solution does not converge to the true solution under a uniform mesh due to the boundary layer occurring at the point $x = 0$. However, from Figures 4 it can be seen that our algorithm converges to the true solution under a Shishkin mesh.

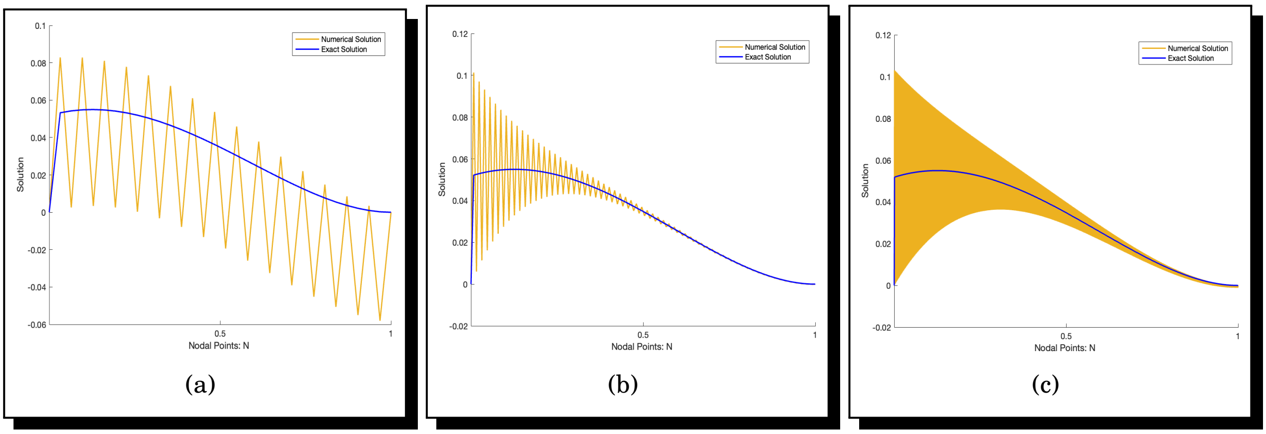

Figure 3. Uniform mesh: (a) $N = 32$, $\varepsilon = 10^{-8}$; (b) $N = 128$, $\varepsilon = 10^{-4}$; (c) $N = 1024$, $\varepsilon = 10^{-6}$

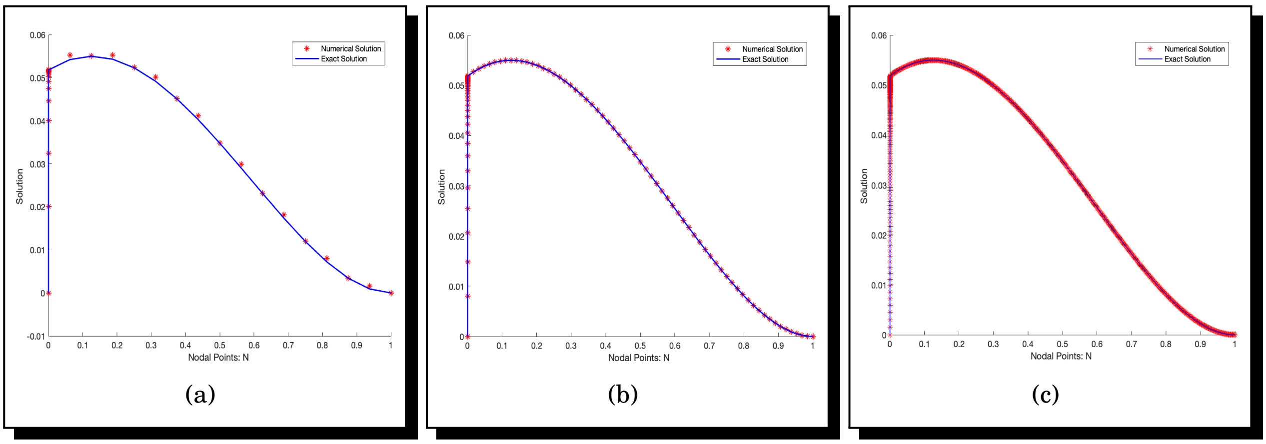

Figure 4. Shishkin mesh: (a) $N = 32$, $\varepsilon = 10^{-8}$; (b) $N = 128$, $\varepsilon = 10^{-4}$; (c) $N = 1024$, $\varepsilon = 10^{-6}$

Example 4.2. In this example we validated the results in Lemma 3.1 and the Remark 3.1. As can be seen from Tables 3 and 4, as we decrease the singular perturbed properties of the problem, the uniform mesh starts showing convergent rates. For example, in Table 4, when $\varepsilon = 1$, the convergent rate $\mathcal{R} = 2$ is observed under a uniformly refined mesh, which is in strong agreement with Lemma 3.1. The convergent rates under a Shishkin meshes are also in strong agreement with Remark 3.1. This concludes that our proposed algorithm works well on fourth-order singularly perturbed problems like (2.1). Figure 5 supports to the same conclusions in Tables 3 and 4. It shows that there is no convergent rate for uniformly refined meshes with high singularly perturbed properties of the problem. However, with a Shishkin mesh, we can observe the expected convergent rates as explained by Remark 3.1.

Table 3. Convergent rate $\mathcal{R}$ for $\varepsilon = 10^{-10}$, 10^{-8} and 10^{-6} for uniform and Shishkin meshes

N	$\epsilon = 10^{-10}$		$\epsilon = 10^{-8}$		$\epsilon = 10^{-6}$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
128	0.0080	1.9975	0.0084	1.9976	0.0457	2.0006
256	0.0043	2.0008	0.0059	2.0009	0.1372	2.0111
512	0.0023	1.7275	0.0086	1.7280	0.3273	1.7333
1024	0.0014	1.7051	0.0260	1.7067	0.2550	1.7325
2048	0.0016	1.7288	0.0919	1.7340	0.0059	1.7410
4096	0.0044	1.7425	0.2631	1.7596	0.0117	1.7377
8192	0.0163	1.7330	0.3392	1.7853	0.0236	1.7522

Table 4. Convergent rate $\mathcal{R}$ for $\varepsilon = 10^{-4}$, 10^{-2} and 1 for uniform and Shishkin meshes

N	$\epsilon = 10^{-4}$		$\epsilon = 10^{-2}$		$\epsilon = 1$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
128	0.1240	2.1640	2.1549	0.8570	2.0227	2.0000
256	0.0711	1.7347	2.0872	0.8899	2.0113	2.0000
512	0.1478	1.5421	2.0005	0.9163	2.0057	2.0000
1024	0.3023	1.5060	2.0074	0.9373	2.0028	2.0000
2048	0.6301	1.4215	2.0026	0.9551	2.0012	1.9998
4096	1.1936	1.2890	2.0009	0.9705	2.0007	1.9958
8192	1.8130	1.1314	2.0004	0.9845	2.0006	1.9380

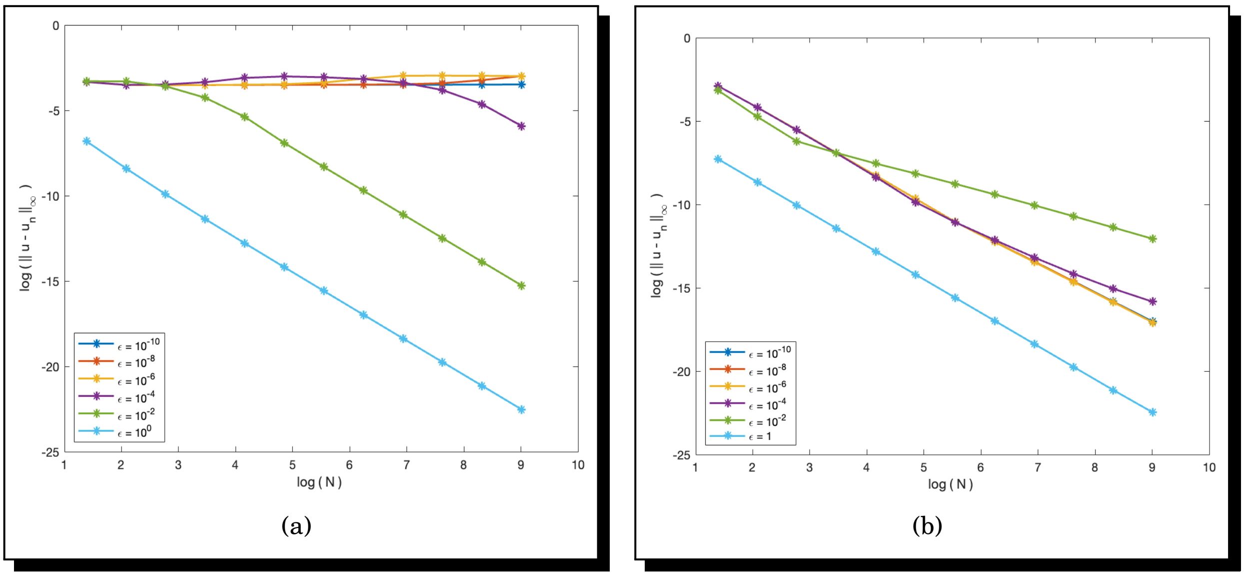

Figure 5. (a) Convergent rate plots under uniform mesh; (b) Convergent rate plots under Shishkin rate

Example 4.3. In this example, we presented the CPU time required by our finite element algorithm to solve problem (2.1) on uniform and Shishkin meshes. The results are presented in Table 5 and Table 6. From the tables, we can observe that the algorithm converges to the true solution faster under the Shishkin mesh compared to the uniform mesh.

Table 5. The CPU time for $\varepsilon = 10^{-10}$, 10^{-8} and 10^{-6}

N	$\epsilon = 10^{-10}$		$\epsilon = 10^{-8}$		$\epsilon = 10^{-6}$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
4096	1.43	0.76	1.42	0.78	1.24	0.81
8192	5.21	2.64	4.90	2.88	4.12	2.92
16384	20.06	13.27	18.69	14.05	17.69	13.52

Table 6. The CPU time for $\varepsilon = 10^{-4}$, 10^{-2} and 1

N	$\epsilon = 10^{-4}$		$\epsilon = 10^{-2}$		$\epsilon = 1$	
	Uniform	Shishkin	Uniform	Shishkin	Uniform	Shishkin
4096	1.15	0.72	1.13	0.78	0.91	0.85
8192	4.00	2.75	3.86	2.73	3.78	2.88
16384	17.34	14.11	17.27	13.04	17.00	14.23

5. Conclusion

It is worth noting that the proposed method can be applied to a wide range of singularly perturbed problems, not only the specific type of problem studied in this work and can be used as a benchmark for comparing the performance of other numerical methods. Furthermore, there are promising avenues for extending this approach to two-dimensional and three-dimensional problems through the development of innovative mesh algorithms. Overall, the proposed method provides a promising approach for solving singularly perturbed problems efficiently and accurately for future advancements in numerical analysis.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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