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Research Article

Properties of Modified Double Laplace Transforms and Special Functions

V. K. Borawake*[1] and A. P. Hiwarekar[2]

[1] *Department of Mathematics, Sinhgad Technical Education Society's Smt. Kashibai Navale College of Engineering (affiliated to Savitribai Phule Pune University), Vadgaon (Budruk), Pune 411041, Maharashtra, India*

[2] *Department of Mathematics, Vidya Pratishthan's Kamalnayan Bajaj Institute of Engineering and Technology (affiliated to Savitribai Phule Pune University), Baramati 413133, Pune, Maharashtra, India*

***Corresponding author:** vijayborawake@sinhgad.edu

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Abstract. This paper deals with new results on modified double Laplace transforms and special functions. Starting with basic definitions and results, we have obtained a modified double Laplace transform of unit step functions, and periodic functions and developed new theorems. Finally, we illustrate our results with examples.

Keywords. Laplace transform, Multiple integral transforms, Integral transform of special functions, Double Laplace transform, Convolution

Mathematics Subject Classification (2020). 44A10, 44A20, 44A30, 44A35

1. Introduction

Laplace transform is one of the most useful transform techniques to solve various problems in mathematics and other areas. It has applications in Science, Engineering, and Technology (Nyeo *et al*. [23], Nozhak and Paskar [22]), Finance (Kim *et al*. [20], Kim and Kim [19], Daci and Tola [7]), Population growth (Daci and Tola [8]). The Laplace transform approach is a practical method for engineers, Debnath and Bhatta [9], and is used to solve various differential equations, Borawake and Hiwarekar [3], Ali *et al*. [1]. It has a wide range of applications in many fields including mathematics, physics, statistics, Poularikas and Seely [24].

The Laplace transform of a function of one variable and its applications are found in the literature, Kokulan and Lai [21], Chiu and Li [5], Chung *et al*. [6], Schiff [27], Debnath and Bhatta [9], Eltayeb and Kiliçman [15], and Hiwarekar [18]. However, such few results are available on functions of two variables and there is a need to extend this theory. The function of two or more variables and its double Laplace transform plays an important role in solving many problems, Debnath [10], Dhunde and Waghmare [11], Dhunde *et al*. [12] to solve space time-fractional equation with initial and boundary conditions, further, its applications are found in Eltayeb and Kiliçman [13–15], Tsaur and Wang [28], Hiwarekar [16–18], Riekstyn's [25], and Viaggiu [29]. Generalization of the double Laplace transforms will play an important role in developing new theory and its applications. We have extended the theory of the double Laplace transform developed by Debnath [10], and Borawake and Hiwarekar [3, 4] by obtaining new results.

2. Notations, Definitions and Basic Results

Definition 2.1 (Modified Laplace Transform)**.** The modified Laplace transform of piece wise continuous and exponential order function $u(x)$ is

$$L_{1,a}[u(x)] = \bar{u}(p) = \int_0^\infty a^{-px} u(x)dx \qquad (Re(p) > 0, a \in (0,\infty)\setminus 1), \tag{2.1}$$

and $L_{1,a}[u(y)] = \bar{u}(q)$, provided that the integral exists (Saif *et al*. [26]), and the corresponding inverse transform is

$$L_{1,a}^{-1}[u(x)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} a^{px} U(p,a)dx \qquad (m \geq 0).$$

Definition 2.2 (Double Laplace Transform; Debnath [10])**.** The double Laplace transform is given by

$$L_2[u(x,y)] = L[L[u(x;y); x \to p]; y \to q] = \int_0^\infty \int_0^\infty e^{-(px+qy)} u(x,y)dxdy \tag{2.2}$$

and the corresponding inverse transform is $L_2^{-1}[\bar{\bar{u}}(p,q)] = u(x,y)$ is defined by

$$u(x,y) = L_2^{-1}[\bar{\bar{u}}(p,q)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} e^{px} dp \, \frac{1}{2\pi i}\int_{n-i\infty}^{n+i\infty} e^{qy}\, \bar{\bar{u}}(p,q)dq \qquad (m,n \geq 0).$$

Definition 2.3 (Modified Double Laplace Transform; Borawake and Hiwarekar [3, 4])**.** The modified double Laplace transform of a function $u(x,y)$ is defined by

$$\bar{\bar{u}}(p,q) = L_{2,a}[u(x,y)] = L_a[L_a[u(x,y);\ x \to p];\ y \to q] = L_{2,a}[\bar{u}(p,y);\ y \to q]$$

and

$$\bar{\bar{u}}(p,q) = \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x,y)dx\,dy \qquad (a > 0). \tag{2.3}$$

The modified double Laplace transform of $u(x,y)$ exists for all p and q, where $\text{Re}(p) > c$ and $\text{Re}(q) > d$ and $u(x,y)$ is a piece-wise continuous and of exponential order defined in finite intervals $(X,0)$ and $(0,Y)$.

The corresponding inverse transform is

$$L_{2,a}^{-1}[\bar{\bar{u}}(p,q)] = \frac{1}{2\pi i}\int_{m-i\infty}^{m+i\infty} a^{px}\, dp \, \frac{1}{2\pi i}\int_{n-i\infty}^{n+i\infty} a^{qy}\, \bar{\bar{u}}(p,q)\, dq \qquad (m,n \geq 0,\ a > 0).$$

Definition 2.4 (Heaviside Unit Step Function)**.** The $H(x,y)$ is Heaviside unit step function given by

$$H(x-h,y-k)=\begin{cases}1, & \text{if } x\ge h,\ y\ge k,\\ 0, & \text{if } x<h,\ y<k.\end{cases} \tag{2.4}$$

Definition 2.5 (Periodic Function; Amberkhane *et al.* [2])**.** The function $\mathbb{U}(x,y)$ is a periodic function of periods T and S given by

$$\mathbb{U}(x+T,y+S)=\mathbb{U}(x,y),\quad \text{for all } x \text{ and } y, \tag{2.5}$$

where T and S are non-zero constants and independent of x and y, respectively.

In this work, we used all terms, definitions, and standard results developed in [3,4]. We also used the following results.

Theorem 2.1 (Shifting Property)**.**

$$L_{2,a}[e^{-\alpha x-\beta y}u(x,y)]=L_{2,e}[p\log a+\alpha,q\log a+\beta].\quad ([7]) \tag{2.6}$$

Theorem 2.2 (Change of Scale Property)**.** *If* $L_{2,a}[u(x,y)]=\bar{\bar{u}}(p,q)$*, then*

$$L_{2,a}[u(\alpha x,\beta y)]=\frac{1}{\alpha\beta}L_{2,a}\left[u\left(\frac{x}{\alpha},\frac{y}{\beta}\right)\right].\quad ([4]) \tag{2.7}$$

Here we developed the following properties of modified double Laplace transform.

3. Properties of Modified Double Laplace Transforms

In continuation with results in [3], [4] and [10], in this paper, we developed some new results on the modified double Laplace transform which are included in this section.

We consider $u(x,y)$ to be an exponentially ordered and piece-wise continuous function.

Theorem 3.1.

(i) $$L_{2,a}[u(x)]=\frac{1}{q\log a}[\bar{u}(p)], \tag{3.1}$$

(ii) $$L_{2,a}[u(y)]=\frac{1}{p\log a}[\bar{u}(q)]. \tag{3.2}$$

Proof. By Definition 2.3, we have

$$\begin{aligned}L_{2,a}[u(x)]&=\int_0^\infty\int_0^\infty a^{-(px+qy)}u(x)dxdy\\ &=\left[\int_0^\infty a^{-qy}dy\right]\left[\int_0^\infty a^{-px}u(x)dx\right]\\ &=\frac{1}{q\log a}[\bar{u}(p)].\end{aligned}$$

Similarly, we have proof of (ii) part of Theorem 3.1. □

Theorem 3.2.

$$L_{2,a}[u(x+y)]=\frac{1}{(p-q)\log a}[\bar{u}(q)-\bar{u}(p)]. \tag{3.3}$$

Proof. By Definition 2.3, we have

$$L_{2,a}[u(x+y)] = \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x+y)dxdy$$

(Put $x+y=t$, $dy=dt$, when $y=0$, $t=x$ and $y=\infty$, $t=\infty$)

$$= \int_0^\infty a^{-(p-q)x} \left[\int_x^\infty a^{-qt} u(t)dt\right] dx.$$

Using the change of order of integration, we have

$$= \int_0^\infty \left[\int_0^t a^{-(p-q)x} dx\right] a^{-qt} u(t)dt$$

$$= \frac{1}{(p-q)\log a} \int_0^\infty [1-a^{-(p-q)t}] a^{-qt} u(t)dt$$

$$= \frac{1}{(p-q)\log a} [\bar{u}(q) - \bar{u}(p)]. \qquad \square$$

Theorem 3.3.

$$L_{2,a}[u(x-y)] = \frac{1}{(p+q)\log a}[\bar{u}(p)+\bar{u}(q)], \ \textit{when } u \textit{ is even.} \tag{3.4}$$

$$= \frac{1}{(p+q)\log a}[\bar{u}(p)-\bar{u}(q)], \ \textit{when } u \textit{ is odd.} \tag{3.5}$$

Proof. By Definition 2.3, we have

$$L_{2,a}[u(x-y)] = \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x-y)dxdy$$

(Put $x-y=t$, $dy=-dt$, when $y=0$, $t=x$ and $y=\infty$, $t=-\infty$)

$$= \int_0^\infty a^{-px} \left[\int_{-\infty}^x a^{-q(x-t)} u(t)dt\right] dx.$$

Using the change of order of integration, we have

$$= \int_0^\infty a^{-(p+q)x} \left[\int_{-\infty}^0 a^{qt} u(t)dt\right] dx + \int_t^\infty a^{-(p+q)x} \left[\int_0^{-\infty} a^{qt} u(t)dt\right] dx$$

$$= \frac{1}{(p+q)\log a} \left[\int_{-\infty}^0 a^{qt} u(t)dt + \int_0^\infty a^{-pt} u(t)dt\right]$$

Put $t=-\theta$, $dt=-d\theta$, in the first integral

$$= \frac{1}{(p+q)\log a} \left[\int_\infty^0 a^{q(-\theta)} u(-\theta)(-d\theta) + \int_0^\infty a^{-pt} u(t)dt\right]$$

$$= \frac{1}{(p+q)\log a} \left[\int_0^\infty a^{q(-\theta)} u(-\theta)d\theta + \int_0^\infty a^{-pt} u(t)dt\right];$$

$$L_{2,a}[u(x-y)] = \frac{1}{(p+q)\log a}[\bar{u}(p)+\bar{u}(q)], \ \text{when } u \text{ is even.}$$

$$= \frac{1}{(p+q)\log a}[\bar{u}(p)-\bar{u}(q)], \ \text{when } u \text{ is odd.} \qquad \square$$

4. Modified Double Laplace Transform of Special Functions

Here we obtained the modified double Laplace transform of special functions included in the following.

Theorem 4.1 (Unit Step Function)**.**

$$L_{2,a}[H(x-\alpha,y-\beta)] = \frac{1}{pq(\log a)^2}\, a^{-(p\alpha+q\beta)}, \tag{4.1}$$

where $H(x,y)$ *is defined by equation* (2.4) *and* $pq(\log a)^2 > 0$.

Proof. By Definition 2.3, we have

$$L_{2,a}[H(x-\alpha,y-\beta)] = \int_\alpha^\infty \int_\beta^\infty a^{-(px+qy)} H(x-\alpha,y-\beta)dxdy.$$

By Definition 2.4, we have

$$= \int_\alpha^\infty \int_\beta^\infty a^{-(px+qy)}\, 1\, dxdy$$

$$= \left(\int_\alpha^\infty a^{-px}dx\right)\left(\int_\beta^\infty a^{-qy}dy\right)$$

$$= \frac{1}{pq(\log a)^2}\, a^{-(p\alpha+q\beta)}.$$ □

Theorem 4.2.

$$L_{2,a}[u(x)H(x-y)] = \frac{1}{q\log a}[\bar{u}(p) - \bar{u}(p+q)]. \tag{4.2}$$

Proof. By Definition 2.3, we have

$$L_{2,a}[u(x)H(x-y)] = \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x)H(x-y)dxdy$$

$$= \int_0^\infty a^{-qy}\left[\int_0^\infty a^{-px}u(x)H(x-y)dx\right]dy$$

$$= \int_0^\infty a^{-qy}\left[\int_y^\infty a^{-px}u(x)dx\right]dy.$$

Using the change of order of integration, we have

$$= \int_0^\infty a^{-px}u(x)\left[\int_0^x a^{-qy}dy\right]dx$$

$$= \frac{1}{q\log a}\int_0^\infty a^{-px}u(x)[1-a^{-qx}dy]dx$$

$$= \frac{1}{q\log a}\left[\int_0^\infty a^{-px}u(x)dx - \int_0^\infty a^{-(p+q)x}u(x)dx\right]$$

$$= \frac{1}{q\log a}[\bar{u}(p) - \bar{u}(p+q)].$$ □

Theorem 4.3.

$$L_{2,a}[u(x)H(y-x)] = \frac{1}{q\log a}[\bar{u}(p+q)]. \tag{4.3}$$

Theorem 4.4.

$$L_{2,a}[u(x)H(x+y)] = \frac{1}{q\log a}[\bar{u}(p)]. \tag{4.4}$$

Proof. By Definition 2.3, we have

$$\begin{aligned}
L_{2,a}[u(x)H(x+y)] &= \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x)H(x+y)dxdy \\
&= \int_0^\infty \int_0^\infty a^{-(px+qy)} u(x)dxdy \\
&= \int_0^\infty a^{-qy}dy \left[\int_0^\infty a^{-px}u(x)dx\right] \\
&= \frac{1}{q\log a}[\bar{u}(p)].
\end{aligned}$$

□

Theorem 4.5.

$$L_{2,a}[H(x-y)] = \frac{1}{p(p+q)(\log a)^2}. \tag{4.5}$$

Proof. By Definition 2.3, we have

$$\begin{aligned}
L_{2,a}[H(x-y)] &= \int_0^\infty \int_0^\infty a^{-(px+qy)} H(x-y)dxdy \\
&= \int_0^\infty \int_0^\infty a^{-(px+qy)} dxdy.
\end{aligned}$$

Using the change of order of integration, we have

$$\begin{aligned}
&= \int_0^\infty \left[\int_0^x a^{-px-qy}dy\right]dx \\
&= \frac{1}{q\log a}\int_0^\infty a^{-px}[1-a^{-qx}]dx \\
&= \frac{1}{p(p+q)(\log a)^2}.
\end{aligned}$$

□

Theorem 4.6. *If* $\mathbb{U}(x,y)$ *be a periodic function of periods* T *and* S (Definition 2.5)*, and* $L_{2,a}[\mathbb{U}(x,y)]$ *exists, then*

$$L_{2,a}[\mathbb{U}(x,y)] = \frac{1}{[1-a^{-(pT+qS)}]}\int_0^T \int_0^S a^{-(px+qy)}\mathbb{U}(x,y)dxdy, \tag{4.6}$$

where $(1-a^{-(pT+qS)}) > 0$.

Proof. By Definition 2.3, we have

$$\begin{aligned}
L_{2,a}[\mathbb{U}(x,y)] &= \int_0^\infty \int_0^\infty a^{-(px+qy)}\mathbb{U}(x,y)dxdy \\
&= \int_0^T \int_0^S a^{-(px+qy)}\mathbb{U}(x,y)\,dxdy + \int_T^{2T}\int_S^{2S} a^{-(px+qy)}\mathbb{U}(x,y)\,dxdy \\
&\quad + \int_{2T}^{3T}\int_{2S}^{3S} a^{-(px+qy)}\mathbb{U}(x,y)\,dxdy + \dots.
\end{aligned}$$

Put $x = t+T$, $y = s+S$ in the second integral $x = t+2T$, $y = s+2S$ in the third integral, we have

$$L_{2,a}[\mathbb{U}(x,y)] = \int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy + \int_0^T\int_0^S a^{-p(t+T)-q(s+S)}\mathbb{U}(t+T,s+S)\,dtds + \int_0^T\int_0^S a^{-p(t+2T)-q(s+2S)}\mathbb{U}(t+2T,s+2S)\,dtds + \dots.$$

Since

$$\begin{aligned}\mathbb{U}(t,s) &= \mathbb{U}(t+T,s+S) = \mathbb{U}(t+2T,s+2S) = \dots\\
&= \int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy + a^{-(pT+qS)}\int_0^T\int_0^S a^{-pt-qs}\mathbb{U}(t,s)\,dtds\\
&\quad + a^{-2(pT+qS)}\int_0^T\int_0^S a^{-pt-qs}\mathbb{U}(t,s)\,dtds + \dots\\
&= \int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy + a^{-(pT+qS)}\int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy\\
&\quad + a^{-2(pT+qS)}\int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy + \dots\\
&= [1 + a^{-(pT+qS)} + a^{-2(pT+qS)} + \dots]\int_0^T\int_0^S a^{-px-qy}\mathbb{U}(x,y)\,dxdy\\
&= \frac{1}{[1-a^{-(pT+qS)}]}\int_0^T\int_0^S a^{-(px+qy)}\mathbb{U}(x,y)dxdy.\end{aligned}$$ □

Remark 4.1. Results of Debnath [10], see equations (36), (37), (38), (39), (40), (41), (42), (43) and Theorem 3.2 are special cases of our results Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.5 and Theorem 4.6 respectively with $a = e$.

Remark 4.2. If we put $u(x) = 1$, in Theorem 4.5, which is a special case of Theorem 4.2.

5. Illustrative Examples

Now we illustrate our results through the following examples.

5.1: Using Theorem 3.1, equation (3.1), we have

$$L_{2,a}[\cos 2x] = \frac{p}{q(p^2+4)\log a}. \tag{5.1}$$

5.2: Using Theorem 3.1, equation (3.2), we have

$$L_{2,a}[\sinh(3y)] = \frac{3}{p\log a[q^2(\log a)^2-9]}. \tag{5.2}$$

5.3: Using Theorem 4.2, equation (4.2), we have

$$L_{2,a}[e^{2x}H(x-y)] = \frac{1}{(p\log a-2)[(p+q)\log a-2]}. \tag{5.3}$$

5.4: Using Theorem 4.3, equation (4.3) and Theorem 2.1, we have

$$L_{2,a}[e^{-3x}H(y-x)] = -\frac{1}{q(p+q+9)(\log a)^2}. \tag{5.4}$$

5.5: Using Theorem 4.4, equation (4.4), we have

$$L_{2,a}[x^2H(x+y)] = \frac{2}{p^3 q \log a}. \tag{5.5}$$

5.6: Using Theorem 4.4, equation (4.4) and Theorem 2.2, we have

$$L_{2,a}[\sin 3x H(x+y)] = \frac{3}{(q\log a)[p^2(\log a)^2+9]}. \tag{5.6}$$

6. Concluding Remark

We developed new properties, theorems on modified double Laplace transform with suitable examples. There is lot of scope to extend the theory further and its corresponding applications.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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