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Research Article

Fixed Point Theorems for α-ψ-Contractive Mappings of Integral Type in C^*-algebra Valued Metric Space

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Abstract. In this paper, we introduce two classes of generalized α-ψ-contractive type mappings of integral type and analyze the existence of fixed point for these mapping in C^*-algebra valued metric space. Our study extends and generalizes the result of several studies in the literature.

Keywords. Fixed point, α-ψ contractive mappings of integral type, C^*-algebra valued metric space

Mathematics Subject Classification (2020). 47H10, 46L07, 54H25

1. Introduction

In 1922, Banach [1] introduced the Banach contraction principle. Banach contraction principle plays an important role in modern analysis and is an important tool for solving existence problem in various field of science. It is the first principle to get a fixed point for a self-map on a complete metric space. Many researchers had generalized the Banach contraction principle.

In 2002, Branciari [3] introduced the concept of integral type contractive mapping and generalized the concept of Banach contraction Principle. In 2010, Khojasteh *et al.* [5] used the Branciari integral type contractive mapping for the cone metric space and proved some fixed-point theorems.

On the other hand, in 2012 Samet *et al.* [8] introduced a very interesting notion of α'-ψ-contractions via α'-admissible mapping in complete metric space. In 2014, Karapinar *et al.* [4] used the concept of α'-ψ-contractive type mappings of integral type and proved some fixed-point theorem of such type of mapping in complete metric space.

In 2014, Ma *et al.* [6] established the notion of C^*-algebra valued metric spaces and proved some fixed-point theorems for contractive and expansive type mapping.

Throughout this paper, we suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I_A. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succcurlyeq \theta$. Using positive elements, one can define a partial ordering $\preccurlyeq$ on $\mathbb{A}_h$ as follows: $x \preccurlyeq y$ if and only if $y - x \succcurlyeq \theta$, where θ means the zero element in $\mathbb{A}$. Now $\mathbb{A}_+ = \{x \in \mathbb{A} : x \succcurlyeq \theta\}$ and $|x| = (x^* x)^{\frac{1}{2}}$.

2. Preliminaries

In this section, we shall give some basic definitions which will be used in sequel.

Definition 2.1 ([6])**.** Let X be a non-empty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $\theta \preccurlyeq d(x,y)$, for all $x,y \in X$ and $d(x,y) = \theta \iff x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(iii) $d(x,y) \preccurlyeq d(x,z) + d(z,y)$, for all $x,y,z \in X$.

Then d is called C^*-algebra valued metric on X and $(X,\mathbb{A},d)$ is called C^*-algebra valued metric space.

Definition 2.2 ([6])**.** Let $(X,\mathbb{A},d)$ be a C^*-algebra valued metric space. Let $\{x_n\}$ be a sequence in X then

(i) $\{x_n\}$ is said to be Cauchy if for all $\theta \preccurlyeq c$, there is $N \in \mathbb{N}$ such that for all $n,m \geq N$, $d(x_n,x_m) \preccurlyeq c$.

(ii) $\{x_n\}$ is said to be converges to x if for all $\theta \preccurlyeq c$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n,x) \preccurlyeq c$.

(iii) $(X,\mathbb{A},d)$ is a complete C^*-algebra valued metric space if every Cauchy sequence is convergent in X.

Definition 2.3 ([6])**.** Suppose that $(X,\mathbb{A},d)$ is a complete C^*-algebra valued metric space. We call a mapping $f : X \to X$ is a C^*-algebra valued contractive mapping on X, if there exists an $a \in \mathbb{A}$ with $\|a\| < 1$ such that $d(fx,fy) \preccurlyeq a^* d(x,y) a$, for all $x,y \in X$.

Definition 2.4 ([8])**.** Let $f : X \to X$ be a self map and $\alpha : X \times X \to [0,\infty)$. Then f is called α'-admissible if

$$\alpha(x,y) \geq 1 \implies \alpha(fx,fy) \geq 1, \quad \text{for all } x,y \in X.$$

Definition 2.5 ([7])**.** Let X be a non-empty set and $\alpha_A : X \times X \to (A^+)'$ be a function. We say that the self map f is α_A-admissible if for all $(x,y) \in X \times X$, $\alpha_A(x,y) \succcurlyeq I_A \Longrightarrow \alpha_A(fx,fy) \succcurlyeq I_A$ where I_A is the unit of $\mathbb{A}$.

Let Ψ_A be the set of positive functions $\psi_A : A^+ \to A^+$ satisfying the following conditions:

(i) $\psi_A(a)$ is continuous and non-decreasing,

(ii) $\psi_A(a) = 0$ iff $a = 0$, and

(iii) $\sum_{n=1}^{\infty} \psi_A^n(a) < \infty$, $\lim_{n\to\infty} \psi_A^n(a) = 0$, for each $a \succ 0$ where ψ_A^n is the nth-iterate of ψ_A.

Definition 2.6 ([2]). Let $N \in \mathbb{N}$. We say that α is N-transitive (on X) if

$x_0, x_1, \dots, x_{N+1} \in X: \quad \alpha(x_i, x_{i+1}) \succcurlyeq I_A, \quad \text{for all } i \in \{0,1,\dots,N\} \Longrightarrow \alpha(x_0, x_{N+1}) \succcurlyeq I_A.$

In particular, we say that α is transitive if it is 1-transitive, that is,

$x, y, z \in X: \qquad \alpha(x,y) \succcurlyeq I_A \text{ and } \alpha(y,z) \succcurlyeq I_A \Longrightarrow \alpha(x,z) \succcurlyeq I_A.$

3. Main Results

In this section, we shall prove some fixed-point results for the generalized integral type contractions in C^*-algebra valued metric space.

Definition 3.1. Let $(X, \mathbb{A}, d_A)$ be a C^*-algebra valued-metric space and $f : X \to X$ be a given mapping. We say that f is an α_A-ψ_A-contractive mapping of integral type 1 if there exist two functions $\alpha_A : X \times X \to (A^+)'$ and $\psi_A \in \Psi_A$ such that for each $x, y \in X$ and $\|a\| < I_A$,

$$\alpha_A(x,y) \int^{d_A(fx,fy)_0} \varphi(t)dt \preccurlyeq a^* \psi_A \left(\int_0^{M(x,y)} \varphi(t)dt \right) a, \tag{3.1}$$

where $\varphi \in \Phi$ and

$$M(x,y) = \max\left\{ d_A(x,y), d_A(x,fx), d_A(y,fy), \frac{1}{2}[d_A(x,fy) + d_A(y,fx)] \right\}.$$

Theorem 3.2. *Let $(X, \mathbb{A}, d_A)$ be a complete C^*-algebra valued metric space and $\alpha_A : X \times X \to (A^+)'$ be a transitive mapping. Suppose that $f : X \to X$ generalized α_A-ψ_A-contractive mapping of integral type 1 and satisfies the following conditions:*

(i) *f is α_A-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha_A(x_0, fx_0) \succcurlyeq I_A$;*

(iii) *f is continuous.*

Then f has a fixed point.

Proof. Let x_0 be an arbitrary point of X such that $\alpha_A(x_0, fx_0) \succcurlyeq I_A$. We construct an iterative sequence $\{x_n\}$ in X in the following way:

$$x_{n+1} = fx_n, \quad \text{for all } n \geq 0.$$

Due to fact that f is α_A-admissible, we find that

$$\alpha_A(x_0, x_1) = \alpha_A(x_0, fx_0) \succcurlyeq I_A \implies \alpha_A(fx_0, fx_1) = \alpha_A(x_1, x_2) \succcurlyeq I_A. \tag{3.2}$$

Iteratively, we obtain that

$$\alpha_A(x_n, x_{n+1}) \succcurlyeq I_A.$$

By applying inequality (3.1) for

$$x = x_{n-1} \text{ and } y = x_n,$$

$$\alpha_A(x_{n-1}, x_n) \int_0^{d_A(fx_{n-1}, fx_n)} \varphi(t)dt \preccurlyeq a^* \psi_A \left(\int_0^{M(x_{n-1}, x_n)} \varphi(t)dt \right) a,$$

where

$$M(x_{n-1},x_n)=\max\left\{d_A(x_{n-1},x_n),d_A(x_{n-1},x_n),d_A(x_n,x_{n+1}),\frac{d_A(x_{n-1},x_{n+1})+d_A(x_n,x_n)}{2}\right\}$$
$$\leq \max\{d_A(x_{n-1},x_n),d_A(x_n,x_{n+1})\}. \tag{3.3}$$

By using (3.3) and regarding the properties of the function ψ_A,

$$\begin{aligned}\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt&=\int_0^{d_A(fx_{n-1},fx_n)}\varphi(t)dt\\&\preccurlyeq \alpha_A(x_{n-1},x_n)\int_0^{d_A(fx_{n-1},fx_n)}\varphi(t)dt\\&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_{n-1},x_n)}\varphi(t)dt,\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right)a.\end{aligned} \tag{3.4}$$

Case 1: If $M(x_{n-1},x_n)=d_A(x_n,x_{n+1})$, then we have

$$\begin{aligned}\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right)a\\&\prec a^*\left(\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right)a.\end{aligned}$$

Applying norm on both sides, we get

$$\left\|\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right\|<\left\|\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right\|,$$

a contradiction.

Case 2: If $M(x_{n-1},x_n)=d_A(x_{n-1},x_n)$, then we have

$$\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_{n-1},x_n)}\varphi(t)dt\right)a. \tag{3.5}$$

By using mathematical induction, we get

$$\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\preccurlyeq (a^*\psi_A)^n\left(\int_0^{d_A(x_0,x_1)}\varphi(t)dt\right)a^n. \tag{3.6}$$

Letting $n\to\infty$ in above inequality and taking the property of ψ_A on the account, we find that

$$\begin{aligned}&\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt=\theta\\ \Longrightarrow\quad&\left\|\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt\right\|\to 0\\ \Longrightarrow\quad& d_A(x_n,x_{n+1})\to\theta \text{ or } \|d_A(x_n,x_{n+1})\|\to 0 \text{ as } n\to\infty\end{aligned} \tag{3.7}$$

For $n>m$ and by triangular inequality and sub additive property in C^*-algebra valued metric space, we get

$$\begin{aligned}\int_0^{d_A(fx_n,fx_m)}\varphi(t)dt&\preccurlyeq \alpha_A(x_n,x_m)\int_0^{d_A(fx_n,fx_m)}\varphi(t)dt\\&\preccurlyeq a^*\psi_A\left(\int_0^{M(x_n,x_m)}\varphi(t)dt\right)a,\end{aligned}$$

where

$$M(x_n,x_m)=\max\left\{d_A(x_n,x_m),d_A(x_n,x_{n+1}),d_A(x_m,x_{m+1}),\frac{d_A(x_n,x_{m+1})+d_A(x_m,x_{n+1})}{2}\right\}$$

$$\preccurlyeq \max\left\{d_A(x_n,x_m), \frac{d_A(x_n,x_{m+1})+d_A(x_m,x_{n+1})}{2}\right\}. \tag{3.8}$$

Now

$$\frac{d_A(x_n,x_{m+1})+d_A(x_m,x_{n+1})}{2} \preccurlyeq d_A(x_n,x_m),$$

thus

$$M(x_n,x_m) = d_A(x_n,x_m).$$

Therefore, we have

$$\begin{aligned}
\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt &\preccurlyeq \alpha_A(x_n,x_m)\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt \\
&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_m)} \varphi(t)dt\right)a \\
&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_{n+1})+d_A(x_{n+1},x_{n+2})+\cdots+d_A(x_{m-1},x_m)} \varphi(t)dt\right)a \\
&\preccurlyeq a^*\psi_A\left[\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt + \cdots + \int_0^{d_A(x_{m-1},x_m)} \varphi(t)dt\right]a \\
&\preccurlyeq \{(a^*\psi_A)^n a^n + (a^*\psi_A)^{n+1}a^{n+1} + \cdots + (a^*\psi_A)^m a^m\}\int_0^{d_A(x_0,x_1)} \varphi(t)dt \\
&\preccurlyeq \sum\sum_{i=n}^{m} |a^i|^2(\psi_A)^i \int_0^{d_A(x_0,x_1)} \varphi(t)dt \\
&\preccurlyeq \left\|\sum_{i=n}^{m} |a^i|^2(\psi_A)^i \int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \left\|\sum_{i=n}^{m} |a^i|^2(\psi_A)^i\right\| \left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \sum_{i=n}^{m} \|a\|^{2i} \left\|\sum_{i=n}^{m} \psi_A^i\right\| \left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \frac{\|a\|^{2m}}{1-\|a\|} \left\|\sum_{i=n}^{m} \psi_A^i\right\| \left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A.
\end{aligned}$$

Thus

$$\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt \to \theta \ \text{ as } \ m,n\to\infty \tag{3.9}$$

which implies that

$$\lim_{m,n\to\infty} \|d_A(fx_n,fx_m)\| = 0. \tag{3.10}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is Complete. Hence $\{x_n\}$ converges to $x\in X$.

$$\lim_{n\to\infty} x_n = x. \tag{3.11}$$

From the continuity of f, it follows that $x_{n+1} = fx_n \to fx$ as $n\to\infty$.

By continuity of this limit, we have $fx = x$, that is, x is a fixed point of f. □

Definition 3.3. Let $(X,\mathbb{A},d_A)$ be a C^*-algebra valued metric space and $f:X\to X$ be a given mapping. We say that f is an α_A-ψ_A-contractive mapping of integral type 2 if there exist two functions $\alpha_A: X\times X\to (A^+)'$ and $\psi_A\in\Psi_A$ such that for each $x,y\in X$ and $\|a\|\prec I_A$,

$$\alpha_A(x,y)\int_0^{d_A(fx,fy)}\varphi(t)dt \preccurlyeq a^*\psi_A\left(\int_0^{M(x,y)}\varphi(t)dt\right)a, \tag{3.12}$$

where $\varphi\in\Phi$ and

$$M(x,y)=\max\left\{d_A(x,y),\frac{1}{2}[d_A(x,fx)+d_A(y,fy)],\frac{1}{2}[d_A(x,fy)+d_A(y,fx)]\right\}.$$

Theorem 3.4. *Let $(X,\mathbb{A},d_A)$ be a complete C^*-algebra valued metric space and $\alpha_A: X\times X\to (A^+)'$ be a transitive mapping. Suppose that $f:X\to X$ generalized α_A-ψ_A-contractive mapping of integral type 2 and satisfies the following conditions:*

(i) *f is α_A-admissible;*

(ii) *there exists $x_0\in X$ such that $\alpha_A(x_0,fx_0)\succcurlyeq I_A$;*

(iii) *f is continuous.*

Then f has a fixed point.

Proof. Let x_0 be an arbitrary point of X such that $\alpha_A(x_0,fx_0)\succcurlyeq I_A$. We construct an iterative sequence $\{x_n\}$ in X in the following way:

$$x_{n+1}=fx_n,\quad \text{for all } n\ge 0.$$

Due to fact that f is α_A-admissible, we find that

$$\alpha_A(x_0,x_1)=\alpha_A(x_0,fx_0)\succcurlyeq I_A \implies \alpha_A(fx_0,fx_1)=\alpha_A(x_1,x_2)\succcurlyeq I_A. \tag{3.13}$$

Iteratively, we obtain that

$$\alpha_A(x_n,x_{n+1})\succcurlyeq I_A.$$

By applying inequality (3.12) with

$$x=x_{n-1}\ \text{ and }\ y=x_n,$$

$$\alpha_A(x_{n-1},x_n)\int_0^{d_A(fx_{n-1},fx_n)}\varphi(t)dt \preccurlyeq a^*\psi_A\left(\int_0^{M(x_{n-1},x_n)}\varphi(t)dt\right)a, \tag{3.14}$$

where

$$\begin{aligned}M(x_{n-1},x_n)&=\max\left\{d_A(x_{n-1},x_n),\frac{d_A(x_{n-1},x_n)+d_A(x_n,x_{n+1})}{2},\frac{d_A(x_{n-1},x_{n+1})+d_A(x_n,x_n)}{2}\right\}\\&\le \max\{d_A(x_{n-1},x_n),d_A(x_n,x_{n+1})\}.\end{aligned}$$

Now

$$\begin{aligned}\frac{d_A(x_{n-1},x_n)+d_A(x_n,x_{n+1})}{2}&\le\frac{d_A(x_{n-1},x_{n+1})+d_A(x_{n+1},x_n)+d_A(x_n,x_{n+1})}{2}\\&=\frac{d_A(x_{n-1},x_{n+1})+2d_A(x_n,x_{n+1})}{2}\\&\le d_A(x_n,x_{n+1}).\end{aligned}$$

By using (3.14) and regarding the properties of the function ψ_A,

$$\int_0^{d_A(x_n,x_{n+1})}\varphi(t)dt=\int_0^{d_A(fx_{n-1},fx_n)}\varphi(t)dt$$

$$\preccurlyeq \alpha_A(x_{n-1},x_n)\int_0^{d_A(fx_{n-1},fx_n)} \varphi(t)dt$$
$$\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right)a. \tag{3.15}$$

Case 1: If $M(x_{n-1},x_n) = d_A(x_n,x_{n+1})$, then we have

$$\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt \preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right)a$$
$$\prec a^*\left(\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right)a.$$

Applying norm on both sides, we get

$$\left\|\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right\| < \left\|\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right\|$$

a contradiction.

Case 2: If $M(x_{n-1},x_n) = d_A(x_{n-1},x_n)$, then we have

$$\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt \preccurlyeq a * \psi_A\left(\int_0^{d_A(x_{n-1},x_n)} \varphi(t)dt\right)a. \tag{3.16}$$

By using mathematical induction, we get

$$\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt \preccurlyeq (a^*\psi_A)^n\left(\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right)a^n. \tag{3.17}$$

Letting $n \to \infty$ in above inequality and taking the property of ψ_A on the account, we find that

$$\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt = \theta$$
$$\implies \left\|\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt\right\| \to 0 \tag{3.18}$$
$$\implies d_A(x_n,x_{n+1}) \to 0 \text{ as } n \to \infty \tag{3.19}$$

For $n > m$ and by triangular inequality and sub additive property in C^*-algebra valued metric space, we get

$$\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt \preccurlyeq \alpha_A(x_n,x_m)\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt$$
$$\preccurlyeq a^*\psi_A\left(\int_0^{M(x_n,x_m)} \varphi(t)dt\right)a,$$

where

$$M(x_n,x_m) = \max\left\{d_A(x_n,x_m), d_A(x_n,x_{n+1}), d_A(x_m,x_{m+1}), \frac{d_A(x_n,x_{m+1}) + d_A(x_m,x_{n+1})}{2}\right\}$$
$$\le \max\left\{d_A(x_n,x_m), \frac{d_A(x_n,x_{m+1}) + d_A(x_m,x_{n+1})}{2}\right\} \tag{3.20}$$

Now

$$\frac{d_A(x_n,x_{m+1}) + d_A(x_m,x_{n+1})}{2} \le d_A(x_n,x_m),$$

thus

$$M(x_n,x_m) = d_A(x_n,x_m).$$

Therefore, we have

$$\begin{aligned}
\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt &\preccurlyeq \alpha_A(x_n,x_m)\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt \\
&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_m)} \varphi(t)dt\right)a \\
&\preccurlyeq a^*\psi_A\left(\int_0^{d_A(x_n,x_{n+1})+d_A(x_{n+1},x_{n+2})+\cdots+d_A(x_{m-1},x_m)} \varphi(t)dt\right)a \\
&\preccurlyeq a^*\psi_A\left[\int_0^{d_A(x_n,x_{n+1})} \varphi(t)dt+\cdots+\int_0^{d_A(x_{m-1},x_m)} \varphi(t)dt\right]a \\
&\preccurlyeq \{(a^*\psi_A)^n a^n+(a^*\psi_A)^{n+1}a^{n+1}+\cdots+(a^*\psi_A)^m a^m\}\int_0^{d_A(x_0,x_1)} \varphi(t)dt \\
&\preccurlyeq \sum_{i=n}^m |a^i|^2(\psi_A)^i\int_0^{d_A(x_0,x_1)} \varphi(t)dt \\
&\preccurlyeq \left\|\sum_{i=n}^m |a^i|^2(\psi_A)^i\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \sum_{i=n}^m |a^i|^2(\psi_A)^i\left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \sum_{i=n}^m \|a\|^{2i}\left\|\sum_{i=n}^m \psi_A^i\right\|\left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A \\
&\preccurlyeq \frac{\|a\|^{2m}}{1-\|a\|}\left\|\sum_{i=n}^m \psi_A^i\right\|\left\|\int_0^{d_A(x_0,x_1)} \varphi(t)dt\right\| I_A. \qquad (3.21)
\end{aligned}$$

Thus,

$$\int_0^{d_A(fx_n,fx_m)} \varphi(t)dt \to \theta \text{ as } m,n\to\infty,$$

which implies that

$$\lim_{m,n\to\infty}\|d_A(fx_n,fx_m)\| = 0. \qquad (3.22)$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is Complete. Hence $\{x_n\}$ converges to $x\in X$. Therefore,

$$\lim_{n\to\infty} x_n = x. \qquad (3.23)$$

From the continuity of f, it follows that $x_{n+1} = fx_n \to fx$ as $n\to\infty$.
By continuity of this limit, we have $fx = x$, that is, x is a fixed point of f. □

Corollary 3.5. *Let $(X,\mathbb{A},d_A)$ be a complete C^*-algebra valued metric space and $\alpha_A : X\times X\to (A^+)'$ be a transitive mapping. Suppose that $f : X\to X$ generalized α_A-ψ_A-contractive mapping of integral type and satisfies the following conditions:*

(i) *f is α_A-admissible;*

(ii) *there exists $x_0\in X$ such that $\alpha_A(x_0,fx_0)\succcurlyeq I_A$;*

(iii) *f is continuous.*

Then f has a fixed point.

Proof. By putting $M(x,y) = d_A(x,y)$ in Theorem 3.2, we get the required result. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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