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Research Article

Painlevé Analysis, Lie Symmetries and Abundant Wave Solutions for Family Fifth Order Kdv Equations

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Abstract. In this paper, we study integrability, similarity reduction and obtaining abundant solutions for the family fifth-order KdV equation. This equation expresses five different forms of the KdV equation, each of these equations has different applications in many fields, including fluid mechanics, ocean science and optics. We utilized Painlevé property for the governing equation to prove that the equation possesses Painlevé test. Then, the symmetry method is used to study the similarity reductions for the governing equation. Subsequently, we obtained a novel type of exact solutions for family KdV fifth-order by using (G'/G)-expansion method. The obtained solutions included hyperbolic and trigonometric functions. The solutions are also presented in 3D shapes to show their properties contained kink wave, singular wave, anti-kink wave, periodic wave and solitary wave solution.

Keywords. Painlevé analysis, Lie symmetries, (G'/G)-expansion method, Wave solutions, Fifth-order KdV equation

Mathematics Subject Classification (2020). 35A24, 35B10, 35C07

1. Introduction

In recent years, one of the most important issues in the research field of mathematics and physics is the construction of new exact solutions for *nonlinear partial differential equations* (NLPDEs) and study of their physical nature. This is due to a variety of NLPDEs equations have been formed to explain many physical phenomena, such as fluid dynamic, optics, astrophysics, oceans science, nanofluid, geophysics and among others which use the possibilities of NPDEs (Gaber and Bekir [6], and Gaber *et al.* [10]).

Many mathematical methods have been presented for solving NLPDEs such as $\tan(\phi/2)$-expansion method (Özkan and Yasar [15]), Kudryashov method (Gaber and Ahmad [8]), (G'/G)-expansion method (Biswas *et al.* [3]), reproducing kernal method (Ahmad *et al.* [2]), direct algebraic method (Lu *et al.* [12]), extended auxiliary equation mapping method (Seadawy *et al.* [17]) and others, but the most general is the symmetry method (Gaber and Wazwaz [7], Gaber and Shehata [9], and Zhao and He [19]). All traveling wave methods are special cases from symmertry method. The importance of symmetry method is due to its ability to give a large number of reductions to the same differential equation, which gives many physical explanations for this equations,

$$\frac{\partial u}{\partial t}+\frac{\partial^5 u}{\partial x^5}+h_1 u\,\frac{\partial^3 u}{\partial x^3}+h_2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+h_3 u^2\,\frac{\partial u}{\partial x}=0, \tag{1.1}$$

where $u=u(x,t)$. This equation has attracted the attention of many researchers (Ahmad *et al.* [1] and references therein), when $h_1=15$, $h_2=15$ and $h_3=45$, eq. (1.1) reduced to Sawada-Kotera (S-K) equation; $h_1=30$, $h_2=30$ and $h_3=180$, eq. (1.1) reduced to the Caudrey-Dodd-Gibbon (C-D-G) equation; $h_1=10$, $h_2=30$ and $h_3=30$, eq. (1.1) reduced to the Lax equation; $h_1=10$, $h_2=20$ and $h_3=30$, eq. (1.1) reduced to nonlinear fifth order KdV equation and $h_1=10$, $h_2=25$ and $h_3=20$, eq. (1.1) reduced to nonlinear fifth order Kaup–Keperschmidt (K-K) equation.

2. Painlevé Property

Studying the integrability of PDEs is one of the important aspects of obtaining exact solutions for these equations. Therefore, the Painleve test is considered one of the methods to test the integrability of PDEs. This method was developed by Weiss *et al.* [18].

In this section, using Weiss's algorithm (Ahmad [5], Weiss *et al.* [18], and Ren *et al.* [16]) of singularity analysis to study the Painlevé property of eq. (1.1). For processing the Painlevé singularity analysis, we use

$$u(x,t)=\phi^s(x,t)\sum_{j=0}^{\infty}u_j(x,t)\phi^j(x,t), \tag{2.1}$$

where v_j $(j=0,1,2,\dots)$ and $\phi=\phi(x,t)$ are analytic function of the independent variable. Substituting $u\simeq u_0\phi^s$ in eq. (1.1) to obtain the leading order s and the expansion coefficients u_0, we get

$$\left.\begin{aligned}
&s=-2,\ u_0=\frac{-4}{3}\phi_x^2, \quad \text{for SK and KK equations,}\\
&s=-2,\ u_0=\frac{-2}{3}\phi_x^2, \quad \text{for C-D-G equation,}\\
&s=-2,\ u_0=\frac{-6}{5}\phi_x^2, \quad \text{for Lax equation,}\\
&s=-2,\ u_0=\frac{-3}{2}\phi_x^2, \quad \text{for fifth order KdV equation.}
\end{aligned}\right\} \tag{2.2}$$

As we know that the resonance $j=-2$ coincides with the singular manifold $\phi(x,t)=0$. then the earlier relation can be taken the expression

$$u(x,t)=u_0\phi^{-2}+\sum_{j=1}^{\infty}u_j\phi^{j-2}, \quad j\geq 1. \tag{2.3}$$

Substituting eq. (2.4) into eq. (1.1) and collecting terms of $v_j(x,y,z,t)$, we get

$$\begin{aligned}&[(j-2)(j-3)(j-4)(j-5)(j-6)+h_1(-24u_0+u_0(j-2)(j-3)(j-4))\\&\quad +h_2(6u_0(j-2)-2u_0(j-2)(j-3))]u_j\phi_x^5\\&\qquad =F(u_{j-1},\dots,u_{j,t},\dots).\end{aligned} \tag{2.4}$$

As a result, we obtain the resonance j, eq. (1.1) as follow

$$\left.\begin{array}{ll} j=-1,6, & \text{for S-K, K-K and C-D-G equations,}\\ j=-1,4,5,6,6, & \text{for Lax equation,}\\ j=-1,6,8, & \text{for fifth order KdV equation.}\end{array}\right\} \tag{2.5}$$

From studying the recursion relation we deduced that:

(i) The expansion coefficients u_0 of ϕ^{-7} is found for all cases of eq. (1.1) as we obtained before.

(ii) The expansion coefficients u_6 is arbitrary functions in S-K, K-K and C-D-G equations and it is not including in the equations of resonances $j=1,6$. By the same, u_4, u_5, u_6 are arbitrary functions for Lax equation at resonances $j=1,4,5,6$ and u_6, u_8 are arbitrary functions for fifth order KdV equation at resonances $j=1,6,8$.

(iii) From this discussion, that all cases of family fifth order has the Painlevé property.

3. Symmetry Analysis

In this section, we applied symmetry analysis on family equations.

Firstly, we shall derive the similarity solutions using symmetry analysis (Gaber *et al.* [11], and Olver [14]) under which eq. (1.1) is invariant in as follow:

Lie point symmetries

$$u^*=u+\varepsilon\phi(x,t,u)+O(\varepsilon^2), \quad x^*=x+\varepsilon\eta(x,t,u)+O(\varepsilon^2), \quad t^*=t+\varepsilon\zeta(x,t,u)+O(\varepsilon^2). \tag{3.1}$$

If we put

$$\Delta=\frac{\partial u}{\partial t}+\frac{\partial^5 u}{\partial x^5}+h_1u\frac{\partial^3 u}{\partial x^3}+h_2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+h_3u^2\frac{\partial u}{\partial x}. \tag{3.2}$$

Then, the invariance condition

$$\Gamma^{(3)}(\Delta)=0, \tag{3.3}$$

where $\Gamma^{(3)}$ can be written in the form

$$\Gamma^{(3)}=\chi+\phi^x\frac{\partial}{\partial u_x}+\phi^t\frac{\partial}{\partial u_t}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xxx}\frac{\partial}{\partial u_{xxx}}+\phi^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}, \tag{3.4}$$

where

$$\chi=\zeta\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial u}. \tag{3.5}$$

Substituting eq. (3.2) into eq. (3.3), we obtained the following equation

$$\phi^t + \phi^{xxxxx} + h_1(\phi u_{xxx} + u\phi^{xxx}) + h_2(\phi^x u_{xx} + u_x\phi^{xx}) + h_3(2u\phi u_x + u^2\phi^x) = 0. \tag{3.6}$$

Since the components $\phi^x, \phi^{xx}, \dots$ take the form

$$\left.\begin{aligned} \phi^x &= D_x\phi - u_t D_x\zeta - u_x D_x\eta, \\ \phi^{xx} &= D_x\phi^x - u_{tx} D_x\zeta - u_{xx} D_x\eta, \\ \phi^{xt} &= D_t\phi^x - u_{tx} D_t\zeta - u_{xx} D_t\eta. \end{aligned}\right\} \tag{3.7}$$

Substituting the values of $\phi^x, \phi^{xx}, \dots$ from eq. (3.7) into eq. (3.6) and collecting the coefficient of derivatives of $u(x,t)$, and equaling it to zero. Solving these equations and obtaining the infinitessimal, as follow

$$\zeta = c_1 t + c_2, \quad \eta = \frac{1}{5}c_1 + c_3, \quad \phi = -\frac{2}{5}c_1 u. \tag{3.8}$$

In order to study the group theoretic structure, the vector field operator V is written as:

$$V = V_1(c_1) + V_2(c_2) + V_3(c_3) + V_4(c_4), \tag{3.9}$$

where

$$\left.\begin{aligned} V_1 &= t\frac{\partial}{\partial t} + \frac{1}{5}x\frac{\partial}{\partial x} - \frac{2}{5}u\frac{\partial}{\partial u}, \\ V_2 &= \frac{\partial}{\partial t}, \\ V_3 &= \frac{\partial}{\partial x}. \end{aligned}\right\} \tag{3.10}$$

The commutator relations are given in Table 1. It is clear from Table 1 that the vector field V in eq. (3.10) constitutes a finite dimensional Lie algebra.

Table 1

	V_1	V_2	V_3
V_1	0	$-V_2$	$-\frac{1}{5}V_3$
V_2	V_2	0	0
V_3	$\frac{1}{5}V_3$	0	0

Furthermore, we can compute the adjoint representations of the vector fields

$$Adj(\exp(\varepsilon V_i))Vi = Vi, \quad i = 1,2,3,\dots,$$

$$Adj(\exp(\varepsilon V_2))V_3 = V_3,$$

$$Adj(\exp(\varepsilon V_3))V_2 = V_2,$$

$$Adj(\exp(\varepsilon V_1))V_2 = e^{\frac{1}{5}\varepsilon}V_2,$$

$$Adj(\exp(\varepsilon V_2))V_1 = V_1 - \frac{1}{5}\varepsilon V_2,$$

$$Adj(\exp(\varepsilon V_1))V_3 = V_3,$$

$$Adj(\exp(\varepsilon V_3))V_1 = e^{\varepsilon}V_1.$$

Further, from the symmetries given in eq. (3.6) the following possibilities exist for the solution of eq. (1.1),

(I) $V_1 + V_2 + V_3$,

(II) $V_2 + V_3$.

4. Reductions and Exact Solutions

In order to obtain the invariant transformation in each of the above two cases we write the characteristic equation in the form

$$\frac{dt}{A(x,t,u)} = \frac{dx}{B(x,t,u)} = \frac{du}{\phi(x,t,u)}. \tag{4.1}$$

Case I: $V_1 + V_2 + V_3$.
We have the following invariant ζ and the form of u

$$\left.\begin{aligned} \xi &= \frac{x+m_2}{(t+m_1)^{\frac{1}{5}}}, \\ u &= (t+m_1)^{\frac{-2}{5}} F(\xi), \end{aligned}\right\} \tag{4.2}$$

where $m_1 = \frac{c_2}{c_1}$ and $m_2 = 3\frac{c_4}{c_1}$.

Substituting eq. (4.2) into eq. (1.1), we have the following equation:

$$\frac{2}{5}F + \frac{1}{5}\zeta F' - F''''' - h_1 F F'' - h_2 F' F'' - h_3 F^2 F' = 0. \tag{4.3}$$

The solution of this equations can be written in the form

$$F = a_0 + a_1\xi + a_2\xi^2 + b_1\xi^{-1} + b_2\xi^{-2}. \tag{4.4}$$

Substituting eq. (4.4) into eq. (4.3), the solution of eq. (4.3) is expressed as:

$$F(\zeta) = \frac{b_2}{\xi^2}, \quad b_2 = \frac{3}{h_3}\left(-h_2 - 2h_1 + \sqrt{h_2^2 + 4h_2h_1 + 4h_1^2 - 40h_3}\right). \tag{4.5}$$

Case II: $V_2 + V_3$.
Corresponding to this case the associated similarity variable and similarity solutions are given as $\xi = x - wt$, $u = F(\xi)$, where $w = \frac{c_2}{c_3}$.

The reduced system of ordinary differential equations is

$$wF - F''''' - h_1 F F'' - h_2 F' F'' - h_3 F^2 F' = 0 = 0. \tag{4.6}$$

By using $\frac{G'}{G}$-expansion method [13], for simplicity the solution takes the form [4]:

$$F(\zeta) = a_0 + a_1\frac{G'(\xi)}{G(\xi)} + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2, \tag{4.7}$$

where $G(\xi)$ is satisfied second order auxiliary equations

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \tag{4.8}$$

Substituting eq. (4.7) into eq. (4.6), equating the coefficients of all powers of $\frac{G'}{G}$ to zero yields a set of algebraic equations for a_0, a_1, a_2. By aid MAPLE software we solve these algebraic equations, we get:

Result 4.1.

$$\left.\begin{aligned} a_0 &= \frac{1}{12}Qk^2(\lambda^2+8\mu), \;\; a_1 = Qk^2\lambda^2, \;\; a_2 = Qk^2, \\ Q &= \frac{3}{h_3}\left(-2h_1-h_2+\sqrt{4h_1^2+4h_1h_2+h_2^2-40h_3}\right), \\ w &= \frac{k^5}{24}(\lambda^4 Qh_2+35\lambda^4-8\lambda^2\mu Qh_2-288\lambda^2\mu+16\mu^2Qh_2+566\mu^2). \end{aligned}\right\} \tag{4.9}$$

Substituting eq. (4.9) into eq. (4.7) and eq. (4.5) with based eq. (4.8), we obtained the general travelling wave solutions of eq. (1.1), as follow:

Case 1: The exact solitary of Caudrey-Dodd-Gibbon equation when $h_1 = 30$, $h_2 = 30$, $h_3 = 180$, $\lambda^2 < 4\mu$,

$$\begin{aligned} u(x,t) &= \frac{1}{12}Qk^2(\lambda^2+8\mu)+\frac{1}{2}Qk^2\lambda^2\sqrt{\lambda^2-4\mu}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) \\ &\quad +\frac{1}{2}Qk^2\sqrt{\lambda^2-4\mu}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2, \end{aligned} \tag{4.10}$$

where $\xi = x - wt$.

Case 2: The wave solution of Lax equation where $h_1 = 10$, $h_2 = 30$, $h_3 = 30$, and $\lambda^2 > 4\mu$,

$$\begin{aligned} u(x,t) &= \frac{1}{12}Qk^2(\lambda^2+8\mu)+\frac{1}{2}Qk^2\lambda^2\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+B\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right) \\ &\quad +Qk^2\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+B\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi+B\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2. \end{aligned} \tag{4.11}$$

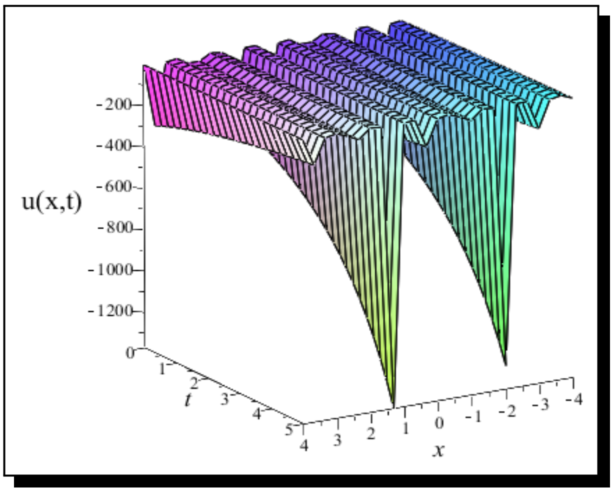

Figure 1. Solitary wave solution of eq. (4.10) for C-D-G equation

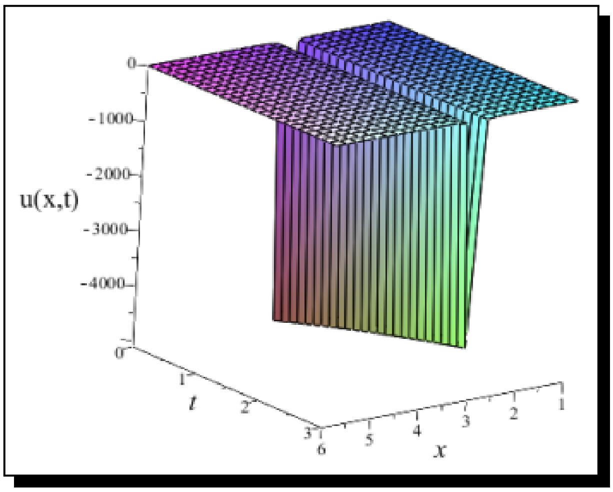

Figure 2. Singular wave solution of eq. (4.11) for Lax equation

Case 3: The single wave solution for nonlinear fifth order KdV equation at $h_1 = 10$, $h_2 = 20$, $h_3 = 30$, $A \neq 0$, $B = 0$ and $\lambda^2 > 4\mu$,

$$u(x,t) = \frac{1}{12}Qk^2(\lambda^2+8\mu) + Qk^2\lambda^2\left(\lambda\tanh\frac{\lambda}{2}\xi + d - \lambda\right) + Qk^2\left(\lambda\tanh\frac{\lambda}{2}\xi + d - \lambda\right)^2, \tag{4.12}$$

where $\xi = x - wt$.

Case 4: The travelling wave solution of Kaup–Keperschmidt equation where $h_1 = 10$, $h_2 = 25$, $h_3 = 20$, $A \neq 0$, $B = 0$ and $\lambda^2 < 4\mu$,

$$u(x,t) = \frac{1}{12}Qk^2(\lambda^2+8\mu) + Qk^2\lambda^2\left(\lambda\coth\frac{\lambda}{2}\xi + d - \lambda\right) + Qk^2\left(\lambda\coth\frac{\lambda}{2}\xi + d - \lambda\right)^2. \tag{4.13}$$

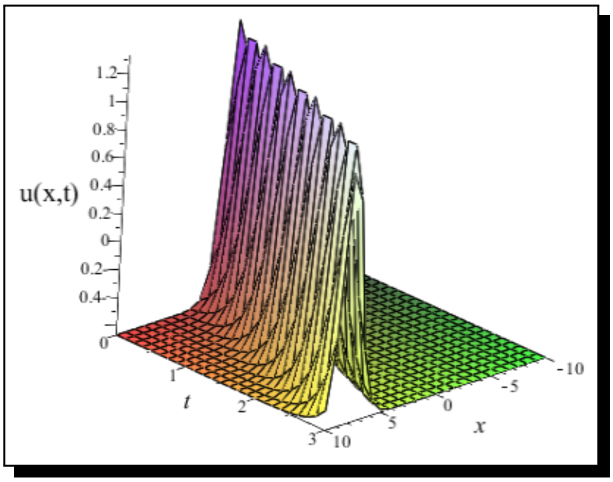

Figure 3. Single wave solution of eq. (4.12) for Lax equation

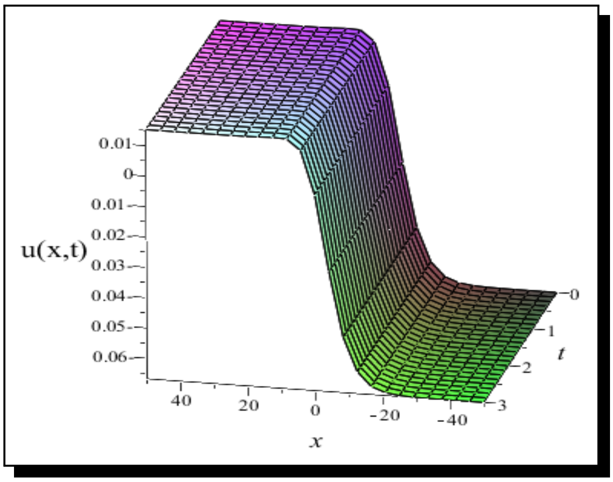

Figure 4. Kink wave solution of eq. (4.13) for K-K equation

Result 4.2.

$$\left.\begin{aligned} a_0 &= \frac{1}{kh_1(h_1+h_2)}\left(\sqrt{\begin{bmatrix} k^6(15h_1^2\lambda^4 - 120h_1^2\lambda^2\mu + 240\mu^2h_1^2 - 10h_1h_2\lambda^4 \\ +80h_1\mu h_2\lambda^2 - 160h_1\mu^2h_2) - k(10h_1^2w - 10h_1wh_2) \end{bmatrix}} \right. \\ &\qquad \left. -5k^3h_1\lambda^2 - 40k^3\mu h_1\right), \\ a_1 &= \frac{-60k^2\lambda}{h_1+h_2},\quad a_2 = \frac{-60k^2}{h_1+h_2},\quad h_3 = \frac{1}{10}h_1(h_1+h_2). \end{aligned}\right\} \tag{4.14}$$

Substituting eq. (4.14) into eq. (4.7) and eq. (4.5) with based eq. (4.8), the general travelling wave solutions of eq. (1.1), are created.

Case 1: The periodic solution of nonlinear fifth order KdV equation at $h_1 = 10$, $h_2 = 20$, $h_3 = 30$ and $\lambda^2 < 4\mu$,

$$\begin{aligned} u(x,t) &= a_0 + \frac{1}{2}\frac{-60k^2\lambda}{h_1+h_2}\sqrt{\lambda^2-4\mu}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + B\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + B\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right) \\ &\quad + \frac{1}{2}\frac{-60k^2}{h_1+h_2}\sqrt{\lambda^2-4\mu}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + B\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}{A\cos\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi + B\sin\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\right)\xi}\right)^2, \end{aligned} \tag{4.15}$$

where $\xi = x - wt$.

Case 2: The anti-kink wave solution of Caudrey-Dodd-Gibbon equation when $h_1 = 30$, $h_2 = 30$, $h_3 = 180$, $A = 0$, $B \neq 0$ and $\lambda^2 > 4\mu$,

$$u(x,t) = a_0 + \frac{-60k^2\lambda}{h_1+h_2}\left(\lambda \tanh\frac{\lambda}{2}\xi + d - \lambda\right) + \frac{-60k^2}{h_1+h_2}\left(\lambda \tanh\frac{\lambda}{2}\xi + d - \lambda\right)^2, \tag{4.16}$$

where $\xi = x - wt$.

Case 3: The Solitary wave solution of Sawada-Kotera equation where $h_1 = 5$, $h_2 = 15$, $h_3 = 45$, $A \neq 0$, $B = 0$ and $\lambda^2 < 4\mu$,

$$u(x,t) = \frac{1}{12}Qk^2(\lambda^2 + 8\mu) + Qk^2\lambda^2\left(\lambda \coth\frac{\lambda}{2}\xi + d - \lambda\right) + Qk^2\left(\lambda \coth\frac{\lambda}{2}\xi + d - \lambda\right)^2. \tag{4.17}$$

where $\xi = x - wt$.

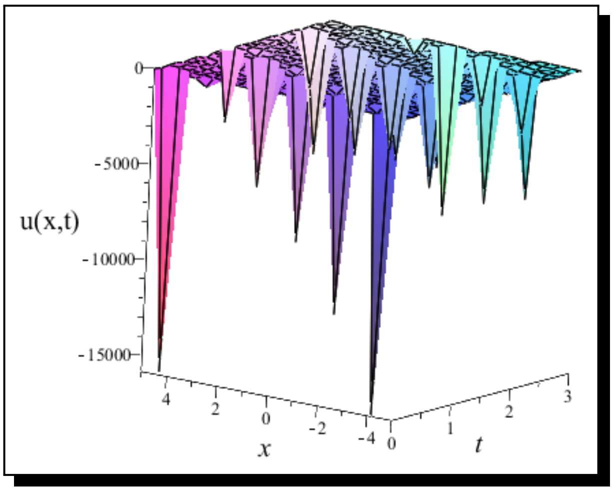

Figure 5. Single wave solution of eq. (4.12) for Lax equation

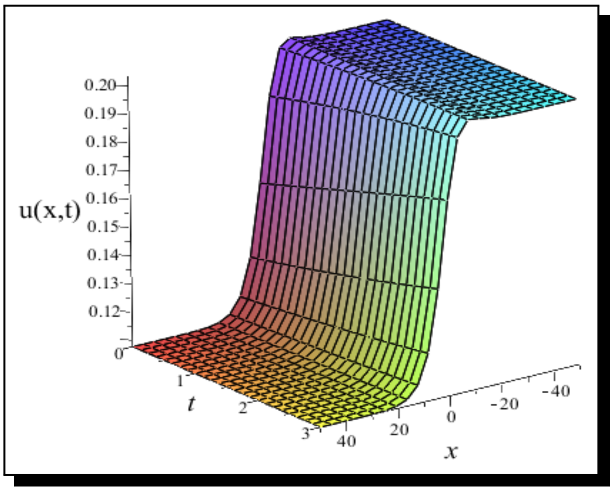

Figure 6. Kink wave solution of eq. (4.13) for K-K equation

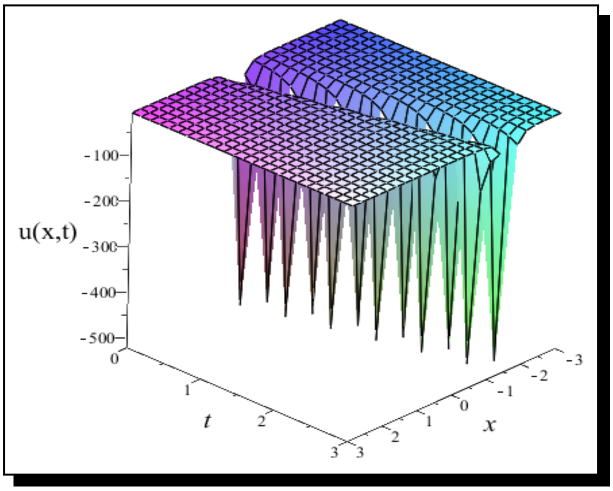

Figure 7. Kink wave solution of eq. (4.13) for K-K equation

5. Conclusion

In this investigation, we study the integrability for fifth order KdV equations and obtained the integrability condition for each of them at different resonance. The governing equations have passed Painleve property. On the other hand, we have used the symmetry method to reduce the governing partial differential equations to two various types of ordinary differential equations. Finally, by using the G'/G-expansion method, we have found a novel wave solutions for the fifth order KdV equations. The behavior of the solutions are shown through 3-D garphs.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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