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Research Article

Δ^m-Statistical Convergence in Intuitionistic Fuzzy Metric Spaces

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Abstract. In this study, we propound statistical convergence and statistical Cauchy sequences for generalized difference sequences in intuitionistic fuzzy metric spaces and establish a Cauchy convergence criterion for this novel notion of convergence. Furthermore, we offer an exhaustive characterization of the mentioned notions of generalized difference sequences. Lastly, we discuss whether the phenomena should be further investigated.

Keywords. Statistical convergence, Difference sequence, Generalized difference sequence, Intuitionistic fuzzy metric space

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1. Introduction

Statistical convergence, originating from the density of natural numbers, was independently introduced by Steinhaus [19], and Fast [6]. Schoenberg [17] also proposed statistical convergence as a summability technique. Since its inception, statistical convergence has found applications in various domains, such as summability theory (Freedman and Sember [7]), locally convex sequence spaces (Maddox [13]), trigonometric series (Zygmund [22]), number theory (Erdös and Tenenbaum [3]), and measurement theory (Miller [14]). Šalát [16] characterized the statistical convergence of a sequence of real numbers as a generalization of classical convergence:

Let (ξ_n) be a sequence in $\mathbb{R}$ and $\xi_0 \in \mathbb{R}$. Then, a sequence (ξ_n) is called statistical convergent to ξ_0, if for all $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |\xi_n - \xi_0| \geq \varepsilon\}) = 0.$$

Kizmaz [11] defined $c_0(\Delta)$, $c(\Delta)$, and $l_\infty(\Delta)$ sequence spaces, $(\Delta\xi_n) = (\xi_n - \xi_{n+1})$ for (ξ_n) real number sequence, and showed that the considered spaces were Banach spaces according to the $\|x\|_\Delta = |\eta_1| + \|\Delta x\|_\infty$ norm. Et and Çolak [4] defined the generalized difference sequence spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$, and $c_0(\Delta^m)$ for a positive number m, formed by generalizing these sequence spaces to Δ^m-sequence spaces, l_∞, c, and c_0 being bounded, convergent and null convergent sequence spaces, respectively. Besides, Mikail and Nuray [5] introduced the notion of Δ^m-statistical convergence by combining the conception of generalized difference sequences with statistical convergence. For further details on Δ^m-statistical convergence, we refer Antal *et al*. [1], Karabacak and Or [10] and many others.

Definition 1 ([5])**.** Let $(\mathcal{X}, |\cdot|)$ be a metric space, $(\Delta^m \xi_n) = (\Delta^{m-1}\xi_n - \Delta^{m-1}\xi_{n+1})$, where $m \in \mathbb{N}$, be a generalized difference sequence in $\mathcal{X}$ and $\xi_0 \in \mathcal{X}$. Then, a sequence (ξ_n) is said to be Δ^m-statistical convergent to ξ_0 if, for all $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : |\Delta^m \xi_n - \xi_0| \geq \varepsilon\}) = 0.$$

Fuzzy sets were first introduced by Zadeh [21] and have since been utilized by many mathematicians in topology and analysis. *Fuzzy Metric Spaces* (FMSs) extend the notion of metric spaces by introducing degrees of membership or fuzziness of points. Kramosil and Michálek [12], and Kaleva and Seikkala [9] were among the first to investigate FMSs. Building on Kramosil and Michálek's [12] work, George and Veermani [8] redefined the concept of FMSs by utilizing a continuous triangular norm and obtained the Hausdorff topology of these spaces. Now, we recall some basic definitions such as triangular norm, triangular conorm, and others besides some related properties given by Schweizer and Sklar [18].

Definition 2 ([18])**.** Let $\odot : [0,1]^2 \to [0,1]$ be a binary operation. The $\odot$ is called a triangular norm if it satisfies the following axioms:

(i) $\odot$ is both associative and commutative,

(ii) $\omega \odot 1 = \omega$, for all $\omega \in [0,1]$,

(iii) whenever $\omega_1 \leq \omega_3$ and $\omega_2 \leq \omega_4$ for each $\omega_1, \omega_2, \omega_3, \omega_4 \in [0,1]$, $\omega_1 \odot \omega_2 \leq \omega_3 \odot \omega_4$ is satisfied.

Definition 3 ([18])**.** Let $\diamond : [0,1]^2 \to [0,1]$ be a binary operation. The $\diamond$ is referred to as a triangular conorm if it satisfies the following axioms:

(i) $\diamond$ is both associative and commutative,

(ii) $\omega \diamond 0 = \omega$, for all $\omega \in [0,1]$,

(iii) whenever $\omega_1 \leq \omega_3$ and $\omega_2 \leq \omega_4$ for each $\omega_1, \omega_2, \omega_3, \omega_4 \in [0,1]$, $\omega_1 \diamond \omega_2 \leq \omega_3 \diamond \omega_4$ is satisfied.

For brevity, we shall often write TN and TC instead of "triangular norm" and "triangular conorm", respectively.

Example 1 ([18]). According to the previous two definition, the following operators are basic examples of TN and TC, respectively,

(i) $\omega_1 \odot \omega_2 = \omega_1\omega_2$,

(ii) $\omega_1 \odot \omega_2 = \min\{\omega_1,\omega_2\}$,

(iii) $\omega_1 \diamond \omega_2 = \max\{\omega_1,\omega_2\}$,

(iv) $\omega_1 \diamond \omega_2 = \min\{\omega_1+\omega_2,1\}$.

In 1986, Atanassov [2] extended the concept of fuzzy set. By adding the idea of not belonging to the degree of belonging to the fuzzy set, he defined the intuitionistic fuzzy set, which is a generalization of the fuzzy set notion. Intuitionistic fuzzy metric spaces, which are then generalizations of fuzzy metric spaces, were defined by Park [15] as follows:

Definition 4 ([15]). Let $\mathcal{X}$ be a non-empty set and $\odot,\diamond$ continuous t-norm and t-conorm, respectively. A five tuple $(\mathcal{X},\rho,\mu,\odot,\diamond)$ is known to be as an intuitionistic fuzzy metric space if ρ, μ are fuzzy sets on $\mathcal{X}^2\times(0,\infty)$ satisfy the following conditions:

(i) $\rho(\eta_1,\eta_2,u)+\mu(\eta_1,\eta_2,u)\le 1$,

(ii) $\rho(\eta_1,\eta_2,u)>0$,

(iii) $\rho(\eta_1,\eta_2,u)=1 \Leftrightarrow \eta_1=\eta_2$,

(iv) $\rho(\eta_1,\eta_2,u)=\rho(\eta_2,\eta_1,u)$,

(v) $\rho(\eta_1,\eta_3,u+s)\ge \rho(\eta_1,\eta_2,u)\odot\rho(\eta_2,\eta_3,s)$,

(vi) the function $\rho_{\eta_1\eta_2}:(0,\infty)\to(0,1]$, defined by $\rho_{\eta_1\eta_2}(u):=\rho(\eta_1,\eta_2,u)$ is continuous,

(vii) $\mu(\eta_1,\eta_2,u)>0$,

(viii) $\mu(\eta_1,\eta_2,u)=0 \Longleftrightarrow \eta_1=\eta_2$,

(ix) $\mu(\eta_1,\eta_2,u)=\mu(\eta_2,\eta_1,u)$,

(x) $\mu(\eta_1,\eta_3,u+s)\le \mu(\eta_1,\eta_2,u)\diamond\mu(\eta_2,\eta_3,s)$,

(xi) the function $\mu_{\eta_1\eta_2}:(0,\infty)\to(0,1]$, defined by $\mu_{\eta_1\eta_2}(u):=\mu(\eta_1,\eta_2,u)$ is continuous,

for all $\eta_1,\eta_2,\eta_3\in\mathcal{X}$ and $u,s>0$.

The $\rho(\eta_1,\eta_2,u)$ indicates the degree of nearness of the element η_1 to element η_2 according to the u parameter, while $\mu(\eta_1,\eta_2,u)$ indicates the degree of non-nearness.

We will refer to the intuitionistic fuzzy metric space $(\mathcal{X},\rho,\mu,\odot,\diamond)$ by $\mathcal{X}$ only, unless otherwise specified.

Example 2 ([15]). Let $\odot$, $\diamond$ t-norm and t-conorm, for all $\omega_1,\omega_2\in[0,1]$, such that $\omega_1\odot\omega_2=\omega_1\omega_2$ and $\omega_1\diamond\omega_2=\min\{\omega_1+\omega_2,1\}$, respectively. In this case, $(\mathcal{X},\rho,\mu,\odot,\diamond)$ is an IFMS, for all $\eta_1,\eta_2\in\mathcal{X}$

and $u > 0$, with

$$\rho(\eta_1,\eta_2,u) = \frac{u}{u+d(\eta_1,\eta_2)} \quad \text{and} \quad \mu(\eta_1,\eta_2,u) = \frac{d(\eta_1,\eta_2)}{u+d(\eta_1,\eta_2)},$$

where d is a metric on $\mathcal{X}$.

For brevity we shall often write "concerning (ρ,μ)" instead of "concerning intuitionistic fuzzy metric (ρ,μ)".

Definition 5 ([15]). Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is referred to as a convergent sequence to $\xi_0 \in \mathcal{X}$ concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\xi_n,\xi_0,u) > 1-\varepsilon \ \text{ and } \ \mu(\xi_n,\xi_0,u) < \varepsilon$$

whenever $n \geq n_\varepsilon$ or equivalently

$$\lim_{n\to\infty} \rho(\xi_n,\xi_0,u) = 1 \ \text{ and } \ \lim_{n\to\infty} \mu(\xi_n,\xi_0,u) = 0$$

and is denoted by $^{\rho}_{\mu}\text{-}\lim_{n\to\infty} \xi_n = \xi_0$ or $\xi_n \xrightarrow{\frac{\rho}{\mu}} \xi_0$ as $n \to \infty$.

Definition 6 ([15]). Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is called a Cauchy sequence concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\xi_n,\xi_N,u) > 1-\varepsilon \ \text{ and } \ \mu(\xi_n,\xi_N,u) < \varepsilon$$

whenever $n,N \geq n_\varepsilon$ or equivalently

$$\lim_{n,N\to\infty} \rho(\xi_n,\xi_N,u) = 1 \ \text{ and } \ \lim_{n,N\to\infty} \mu(\xi_n,\xi_N,u) = 0.$$

The concept of statistical convergence and Cauchy sequence in IFMS was expressed in 2022 by Varol [20] as follows.

Definition 7 ([20]). Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is called statistical convergent to $\xi_0 \in \mathcal{X}$ concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$,

$$\delta(\{n \in \mathbb{N} : \rho(\xi_n,\xi_0,u) \leq 1-\varepsilon \text{ or } \mu(\xi_n,\xi_0,u) \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{|\{n \in \mathbb{N} : \rho(\xi_n,\xi_0,u) \leq 1-\varepsilon \text{ or } \mu(\xi_n,\xi_0,u) \geq \varepsilon\}|}{n} = 0.$$

Definition 8 ([20]). Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is referred to as a statistical Cauchy sequence concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $N \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : \rho(\xi_n,\xi_N,u) \leq 1-\varepsilon \ \text{ or } \ \mu(\xi_n,\xi_N,u) \geq \varepsilon\}) = 0.$$

2. Main Result

In this section, we present exhaustive definitions and theorems with regard to Δ^m-statistical convergence and Δ^m-Cauchy sequence in IFMSs. In addition, we deal with the relations between these notions.

Definition 9. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is said to be Δ^m-convergent to $\xi_0 \in \mathcal{X}$ concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\Delta^m \xi_n, \xi_0, u) > 1-\varepsilon \quad \text{and} \quad \mu(\Delta^m \xi_n, \xi_0, u) < \varepsilon$$

whenever $n \geq n_\varepsilon$ or equivalently

$$\lim_{n\to\infty} \rho(\Delta^m \xi_n, \xi_0, u) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mu(\Delta^m \xi_n, \xi_0, u) = 0.$$

It is convenient to represent symbolically by ${}^{\rho}_{\mu} - \lim_{n\to\infty} \Delta^m \xi_n = \xi_0$ or $\Delta^m \xi_n \xrightarrow{\rho}_{\mu} \xi_0$ as $n \to \infty$.

Definition 10. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is called Δ^m-statistical convergent to $\xi_0 \in \mathcal{X}$ concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$,

$$\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \leq 1-\varepsilon \text{ or } \mu(\Delta^m \xi_n, \xi_0, u) \geq \varepsilon\}) = 0.$$

It is convenient to stand for symbolically by ${}^{\rho}_{\mu}st - \lim_{n\to\infty} \Delta^m \xi_n = \xi_0$ or $\Delta^m \xi_n \xrightarrow{{}^{\rho}_{\mu}st} \xi_0$ as $n \to \infty$.

Example 3. Let $\mathcal{X} = \mathbb{R}$, $\omega_1 \odot \omega_2 = \omega_1\omega_2$, and $\omega_1 \diamond \omega_2 = \min\{\omega_1 + \omega_2, 1\}$, for all $\omega_1, \omega_2 \in [0,1]$. Define ρ and μ by

$$\rho(\eta_1,\eta_2,u) = \frac{u}{u+|\eta_1-\eta_2|} \quad \text{and} \quad \mu(\eta_1,\eta_2,u) = \frac{|\eta_1-\eta_2|}{u+|\eta_1-\eta_2|},$$

for all $\eta_1, \eta_2 \in \mathbb{R}$ and $u > 0$. Then, $(\mathbb{R}, \rho, \mu, \odot, \diamond)$ is an IFMS. Now define a sequence

$$\Delta^m \xi_n := \begin{cases} n, & n \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $\varepsilon \in (0,1)$ and for any $u > 0$, let

$$K = \{n \in \mathbb{N} : \rho(\Delta^m \xi_n, 0, u) \leq 1-\varepsilon \text{ or } \mu(\Delta^m \xi_n, 0, u) \geq \varepsilon\}.$$

Thus,

$$\begin{aligned} K &= \left\{ n \in \mathbb{N} : \frac{u}{u+|\Delta^m \xi_n|} \leq 1-\varepsilon \text{ or } \frac{|\Delta^m \xi_n|}{u+|\Delta^m \xi_n|} \geq \varepsilon \right\} \\ &= \{n \in \mathbb{N} : \Delta^m \xi_n = 1\} \\ &= \{n \in \mathbb{N} : n \text{ are squares}\} \end{aligned}$$

and we obtain

$$\frac{|K|}{n} \leq \frac{|\{n \in \mathbb{N} : n \text{ are squares}\}|}{n} \leq \frac{\sqrt{n}}{n} \to 0 \text{ as } n \to \infty.$$

Consequently, (ξ_n) is Δ^m-statistical convergent to 0 concerning (ρ,μ).

Lemma 1. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$, and $\xi_0 \in \mathcal{X}$. Then, for all $\varepsilon \in (0,1)$ and $u > 0$, the following are equivalent:

(i) ${}^{\rho}_{\mu}st - \lim_{n\to\infty} \Delta^m \xi_n = \xi_0$;

(ii) $\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) > 1-\varepsilon\}) = \delta(\{n \in \mathbb{N} : \mu(\Delta^m \xi_n, \xi_0, u) < \varepsilon\}) = 1$;

(iii) $\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \leq 1-\varepsilon\}) = \delta(\{n \in \mathbb{N} : \mu(\Delta^m \xi_n, \xi_0, u) \geq \varepsilon\}) = 0$.

Proof. Using Definition 10 and properties of density, the proof is trivial. □

Theorem 1. If a sequence (ξ_n) in an IFMS $\mathcal{X}$ is Δ^m-statistical convergence concerning (ρ,μ), then the Δ^m-statistical limit is unique.

Proof. Suppose that (ξ_n) statistical converges in $\mathcal{X}$ and has two distinct statistical limits, η_1 and η_2. Let us take $r>0$ such that $(1-r)\odot(1-r)>1-\varepsilon$ and $r\diamond r<\varepsilon$, for each $\varepsilon\in(0,1)$. Then, we define the following sets for any $u>0$

$$K_1(r,u):=\{n\in\mathbb{N}:\rho(\Delta^m\xi_n,\eta_1,u)\le 1-r\},$$

$$K_2(r,u):=\{n\in\mathbb{N}:\rho(\Delta^m\xi_n,\eta_2,u)\le 1-r\},$$

$$T_1(r,u):=\{n\in\mathbb{N}:\mu(\Delta^m\xi_n,\eta_1,u)\ge r\},$$

$$T_2(r,u):=\{n\in\mathbb{N}:\mu(\Delta^m\xi_n,\eta_2,u)\ge r\}.$$

Since ${}^{\rho}_{\mu}st-\lim_{n\to\infty}\Delta^m\xi_n=\eta_1$ and Lemma 1, then

$$\delta(K_1(r,u))=0 \text{ and } \delta(T_1(r,u))=0.$$

Similarly, since ${}^{\rho}_{\mu}st-\lim_{n\to\infty}\Delta^m\xi_n=\eta_2$ and Lemma 1,

$$\delta(K_2(r,u))=0 \text{ and } \delta(T_2(r,u))=0.$$

Let

$$A(r,u)=K_{\rho\mu}(r,u):=\{K_1(r,u)\cup K_2(r,u)\}\cap\{T_1(r,u)\cup T_2(r,u)\}.$$

Hence, $\delta(A(r,u))=0$. If $n\in\mathbb{N}\backslash A(r,u)$, then we have two statements:

$$n\in\mathbb{N}\backslash\{K_1(r,u)\cup K_2(r,u)\} \text{ or } n\in\mathbb{N}\backslash\{T_1(r,u)\cup T_2(r,u)\}.$$

Let us consider $n\in\mathbb{N}\backslash\{K_1(r,u)\cup K_2(r,u)\}$. Then, we obtain

$$\rho(\eta_1,\eta_2,u)\ge\rho\left(\eta_1,\Delta^m\xi_n,\frac{u}{2}\right)\odot\rho\left(\Delta^m\xi_n,\eta_2,\frac{u}{2}\right)>(1-r)\odot(1-r)>1-\varepsilon.$$

Since $\varepsilon\in(0,1)$ is arbitrary, then we obtain $\rho(\eta_1,\eta_2,u)=1$, for all $u>0$, which implies $\eta_1=\eta_2$. Now, let us consider $n\in\mathbb{N}\setminus\{T_1(r,u)\cup T_2(r,u)\}$. Then,

$$\mu(\eta_1,\eta_2,u)\le\mu\left(\eta_1,\Delta^m\xi_n,\frac{u}{2}\right)\diamond\mu\left(\Delta^m\xi_n,\eta_2,\frac{u}{2}\right)<r\diamond r<\varepsilon.$$

Since $\varepsilon\in(0,1)$ is arbitrary, then we obtain $\mu(\eta_1,\eta_2,u)=0$, for all $u>0$, which implies $\eta_1=\eta_2$. □

Theorem 2. Let $\mathcal{X}$ be an IFMS and (ξ_n) a sequence in $\mathcal{X}$. If (ξ_n) is Δ^m-convergent to $\xi_0\in\mathcal{X}$ concerning (ρ,μ), then it is Δ^m-statistical convergent to ξ_0 concerning (ρ,μ).

Proof. Let (ξ_n) be Δ^m-convergent to ξ_0 concerning (ρ,μ). Then, for all $\varepsilon\in(0,1)$ and $u>0$, there exists $n_0\in\mathbb{N}$ such that

$$\rho(\Delta^m\xi_n,\xi_0,u)>1-\varepsilon \text{ and } \mu(\Delta^m\xi_n,\xi_0,u)<\varepsilon$$

whenever $n\ge n_0$. Hence, the set

$$\{n\in\mathbb{N}:\rho(\Delta^m\xi_n,\xi_0,u)\le 1-\varepsilon \text{ or } \mu(\Delta^m\xi_n,\xi_0,u)\ge\varepsilon\},$$

has a finite number of terms. Therefore,

$$\delta(\{n\in\mathbb{N}:\rho(\Delta^m\xi_n,\xi_0,u)\le 1-\varepsilon \text{ or } \mu(\Delta^m\xi_n,\xi_0,u)\ge\varepsilon\})=0.$$

Consequently, (ξ_n) is Δ^m-statistical convergent to ξ_0 concerning (ρ,μ). □

The converse of this theorem does not always hold, as shown in Example (4).

Example 4. Let $(\mathbb{R},|\cdot|)$ denote the space of all real numbers with the usual metric and $\omega_1 \odot \omega_2 = \omega_1\omega_2$, $\omega_1 \diamond \omega_2 = \min\{\omega_1+\omega_2,1\}$, for all $\omega_1,\omega_2 \in [0,1]$. Then, $(\mathbb{R},\rho,\mu,\odot,\diamond)$ is an IFMS such that

$$\rho(x,y,u) := \frac{u}{u+|x-y|} \quad \text{and} \quad \mu(x,y,u) := \frac{|x-y|}{u+|x-y|},$$

for all $x,y \in \mathbb{R}$ and $u>0$. Now, we define a sequence

$$\Delta^m \xi_n := \begin{cases} n, & n \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases}$$

From Example 3 (ξ_n) is Δ^m-statistical convergent to 0 concerning (ρ,μ). On the other hand, (ξ_n) is not Δ^m-convergent to 0 concerning (ρ,μ) since

$$\begin{aligned} \rho(\Delta^m\xi_n,0,u) &= \frac{u}{u+|\Delta^m\xi_n|} \\ &= \begin{cases} \frac{u}{u+n}, & n \text{ are squares,} \\ 0, & \text{otherwise} \end{cases} \\ &\le 1 \end{aligned}$$

and

$$\begin{aligned} \mu(\Delta^m\xi_n,0,u) &= \frac{|\Delta^m\xi_n|}{u+|\Delta^m\xi_n|} \\ &= \begin{cases} \frac{n}{u+n}, & n \text{ are squares,} \\ 0, & \text{otherwise} \end{cases} \\ &\ge 0. \end{aligned}$$

Theorem 3. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is Δ^m-statistical convergent to ξ_0 if and only if there exists a subset

$$K = \{j_k : j_1 < j_2 < \dots\}$$

such that $\delta(K)=1$ and ${}^{\rho}_{\mu}-\lim_{j_k\to\infty} \Delta^m \xi_{j_k} = \xi_0$.

Proof. $(\Rightarrow)$: Let ${}^{\rho}_{\mu}st-\lim_{n\to\infty} \Delta^m\xi_n = \xi_0$. Assume that

$$M(r,u) = \left\{ n\in\mathbb{N} : \rho(\Delta^m\xi_n,\xi_0,u) > 1-\frac{1}{r} \text{ and } \mu(\Delta^m\xi_n,\xi_0,u) < \frac{1}{r} \right\}$$

and

$$K(r,u) = \left\{ n\in\mathbb{N} : \rho(\Delta^m\xi_n,\xi_0,u) \le 1-\frac{1}{r} \text{ or } \mu(\Delta^m\xi_n,\xi_0,u) \ge \frac{1}{r} \right\},$$

for all $r=1,2,\dots$ and $u>0$. Then,

$$\delta(K(r,u)) = 0 \quad \text{and} \quad \delta(M(r,u)) = 1 \tag{2.1}$$

and

$$M(r+1,u) \subset M(r,u) \tag{2.2}$$

for all $r = 1,2,\ldots$ and $u > 0$. It suffices to prove that

$$^{\rho}_{\mu}- \lim_{\substack{n\to\infty \\ n\in M(r,u)}} \Delta^m \xi_n = \xi_0.$$

Suppose that $(\Delta^m \xi_n)$ is not convergent to ξ_0 concerning (ρ,μ). Therefore, there exists a $\lambda > 0$ such that

$$\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \le 1-\lambda \text{ or } \mu(\Delta^m \xi_n, \xi_0, u) \ge \lambda\}$$

for infinitely many terms $\Delta^m \xi_n$. Let

$$M(\lambda,u) = \{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) > 1-\lambda \text{ and } \mu(\Delta^m \xi_n, \xi_0, u) < \lambda\}$$

and $\lambda > \frac{1}{r}$. Then, $\delta(M(\lambda,u)) = 0$ and by (2.2), $M(r,u) \subset M(\lambda,u)$. Therefore, $\delta(M(r,u)) = 0$ which contradicts (2.1). Consequently, the proof is complete.

($\Leftarrow$): Suppose that there exists a set $K = \{j_k : j_1 < j_2 < \ldots\}$ such that $\delta(K) = 1$ and $^{\rho}_{\mu}- \lim_{\substack{n\to\infty \\ n\in K}} \Delta^m \xi_n = \xi_0$. From Definition 10, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\Delta^m \xi_n, \xi_0, u) > 1-\varepsilon \quad\text{and}\quad \mu(\Delta^m \xi_n, \xi_0, u) < \varepsilon$$

whenever $n \ge n_\varepsilon$. Hence,

$$A(\varepsilon,u) := \{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \le 1-\varepsilon \text{ or } \mu(\Delta^m \xi_n, \xi_0, u) \ge \varepsilon\}$$

such that

$$A(\varepsilon,u) \subseteq \mathbb{N} \setminus \{(j_{n_\varepsilon+1}, k_{n_\varepsilon+1}), (j_{n_\varepsilon+2}, k_{n_\varepsilon+2}), \ldots\}.$$

Therefore, $\delta(A(\varepsilon,u)) = 0$. Consequently, $^{\rho}_{\mu}st - \lim_{n\to\infty} \Delta^m \xi_n = \xi_0$. □

Definition 11. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is said to be Δ^m-Cauchy sequence in $\mathcal{X}$ concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\rho(\Delta^m \xi_n, \Delta^m \xi_N, u) > 1-\varepsilon \quad\text{and}\quad \mu(\Delta^m \xi_n, \Delta^m \xi_N, u) < \varepsilon,$$

whenever $n, N \ge n_\varepsilon$ or equivalently

$$\lim_{n,N\to\infty} \rho(\Delta^m \xi_n, \Delta^m \xi_N, u) = 1 \quad\text{and}\quad \lim_{n,N\to\infty} \mu(\Delta^m \xi_n, \Delta^m \xi_N, u) = 0.$$

Definition 12. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, (ξ_n) is called Δ^m-statistically Cauchy sequence concerning (ρ,μ) if, for all $\varepsilon \in (0,1)$ and $u > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \Delta^m \xi_N, u) \le 1-\varepsilon \text{ or } \mu(\Delta^m \xi_n, \Delta^m \xi_N, u) \ge \varepsilon\}) = 0,$$

whenever $n, N \ge n_\varepsilon$ or equivalently

$$\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \Delta^m \xi_N, u) > 1-\varepsilon \text{ and } \mu(\Delta^m \xi_n, \Delta^m \xi_N, u) < \varepsilon\}) = 1.$$

Theorem 4. Each Δ^m-statistical convergent sequences with respect to (ρ,μ) in an IFMS $\mathcal{X}$ is a Δ^m-statistical Cauchy sequence.

Proof. Let (ξ_n) be a sequence in $\mathcal{X}$ and it is Δ^m-statistical convergent to ξ_0 concerning (ρ,μ). Let $r \in (0,1)$ be given and we take $\varepsilon \in (0,1)$ such that

$$(1-r)\odot(1-r) > 1-\varepsilon \text{ and } r\diamond r < \varepsilon.$$

Then, for all $u > 0$, we have

$$\delta(A(r,u)) = \delta\left(\left\{n \in \mathbb{N} : \rho\left(\Delta^m \xi_n, \xi_0, \frac{u}{2}\right) > 1-r \text{ and } \mu\left(\Delta^m \xi_n, \xi_0, \frac{u}{2}\right) < r\right\}\right) = 1.$$

Let $N \in A(r,u)$. Then,

$$\rho\left(\Delta^m \xi_N, \xi_0, \frac{u}{2}\right) > 1-r \text{ and } \mu\left(\Delta^m \xi_N, \xi_0, \frac{u}{2}\right) < r.$$

Hence,

$$\begin{aligned}\rho\left(\Delta^m \xi_n, \Delta^m \xi_N, u\right) &\geq \rho\left(\Delta^m \xi_n, \xi_0, \frac{u}{2}\right) \odot \rho\left(\xi_0, \Delta^m \xi_N, \frac{u}{2}\right)\\ &> (1-r)\odot(1-r) > \varepsilon\end{aligned}$$

and

$$\begin{aligned}\mu\left(\Delta^m \xi_n, \Delta^m \xi_N, u\right) &\leq \mu\left(\Delta^m \xi_n, \xi_0, \frac{u}{2}\right) \diamond \mu\left(\xi_0, \Delta^m \xi_N, \frac{u}{2}\right)\\ &< r\diamond r < \varepsilon.\end{aligned}$$

Therefore, $N \in B(r,u)$ where

$$B(r,u) = \{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \Delta^m \xi_N, u) > 1-r \text{ and } \mu(\Delta^m \xi_n, \Delta^m \xi_N, u) < r\}.$$

Consequently, (ξ_n) be Δ^m-statistical Cauchy sequence concerning (ρ,μ). □

Theorem 5. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, the following situations are equivalent:

(i) (ξ_n) is a Δ^m-statistical Cauchy sequence concerning (ρ,μ).

(ii) There exists a subset $K = \{j_k : j_1 < j_2 < \dots\}$ of the set $\mathbb{N}$ such that $\delta(K) = 1$ and the subsequence (ξ_{j_k}) of the sequence (ξ_n) is a Δ^m-Cauchy sequence concerning (ρ,μ).

Proof. The proof is the same as the proof of Theorem 3. □

Theorem 6. Let (ξ_n) be a sequence in an IFMS $\mathcal{X}$. Then, ${}^{\rho}_{\mu}st-\lim\limits_{n\to\infty} \Delta^m \xi_n = \xi_0$ if and only if there is a sequence (y_n) such that

$${}^{\rho}_{\mu}-\lim_{n\to\infty} \Delta^m y_n = \xi_0 \text{ and } \delta(\{n \in \mathbb{N} : \Delta^m \xi_n = \Delta^m y_n\}) = 1.$$

Proof. Let ${}^{\rho}_{\mu}st-\lim_{n\to\infty} \Delta^m \xi_n = \xi_0$. By Theorem 3 we get a set $J = \{k_p : p = 1,2,3,\dots\} \subseteq \mathbb{N}$ with $\delta(J) = 1$ and ${}^{\rho}_{\mu}-\lim\limits_{k_p\to\infty} \Delta^m \xi_{k_p} = \xi_0$. Consider the sequence $(\Delta^m y_n)$ such that $\Delta^m y_n = \Delta^m \xi_n$ for $n \in J$ and $\Delta^m y_n = \xi_0$ for $n \notin J$, i.e.,

$$\Delta^m y_n = \begin{cases}\Delta^m \xi_n, & n \in J,\\ \xi_0, & \text{otherwise.}\end{cases}$$

In this case, ${}^{\rho}_{\mu}-\lim\limits_{n\to\infty} \Delta^m y_n = \xi_0$ and $\delta(\{n \in \mathbb{N} : \Delta^m \xi_n = \Delta^m y_n\}) = 1$ are hold.

Conversely, let $\xi = (\xi_n)$, $y = (y_n)$ be sequences in $\mathcal{X}$, and

$${}^{\rho}_{\mu}-\lim_{n\to\infty} \Delta^m y_n = \xi_0 \text{ and } \delta(\{n \in \mathbb{N} : \Delta^m y_n = \Delta^m \xi_n\}) = 1.$$

Then, for all $\epsilon > 0$ and $u > 0$,

$$\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \leq 1-\epsilon \text{ or } \mu(\Delta^m \xi_n, \xi_0, u) \geq \epsilon\} \subseteq A \cup B,$$

where

$$A = \left\{n \in \mathbb{N} : \rho\left(\Delta^m y_n, \xi_0, \frac{u}{2}\right) \leq 1-\epsilon \text{ or } \mu\left(\Delta^m y_n, \xi_0, \frac{u}{2}\right) \geq \epsilon\right\}$$

and

$$B = \{n \in \mathbb{N} : (\Delta^m y_n \neq \Delta^m \xi_n)\}.$$

For $k \in A^c \cap B^c$, we have

$$\begin{aligned}\rho(\Delta^m \xi_k, \xi_0, u) &\geq \rho\left(\Delta^m \xi_k, \Delta^m y_k, \frac{u}{2}\right) \odot \rho\left(\Delta^m \xi_k, \xi_0, \frac{u}{2}\right) \\ &> 1 \odot (1-\epsilon) > 1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\mu(\Delta^m \xi_k, \xi_0, u) &\leq \mu\left(\Delta^m \xi_k, \Delta^m y_k, \frac{u}{2}\right) \diamond \mu\left(\Delta^m \xi_k, \xi_0, \frac{u}{2}\right) \\ &< 0 \diamond \epsilon < \epsilon.\end{aligned}$$

Since ${}^{\rho}_{\mu} - \lim_{n\to\infty} \Delta^m y_n = \xi_0$, then the set A contains at most finitely many terms. Moreover, since $\delta(B^c) = 1$, $\delta(B) = 0$. Therefore,

$$\delta(\{n \in \mathbb{N} : \rho(\Delta^m \xi_n, \xi_0, u) \leq 1-\epsilon \text{ or } \mu(\Delta^m \xi_n, \xi_0, u) \geq \epsilon\}) = 0.$$

We obtain ${}^{\rho}_{\mu}st - \lim_{n\to\infty} \Delta^m \xi_n = \xi_0$. □

3. Conclusion

Intuitionistic Fuzzy Metric Spaces (IFMSs) are a mathematical framework that generalizes traditional metric spaces to accommodate uncertainty and ambiguity through intuitionistic fuzzy sets. In this context, the concept of convergence plays a crucial role in understanding the behaviour of sequences. Convergence in IFMS refers to the conduct of sequences of intuitionistic fuzzy points as they approach a limiting value, much like convergence in classical metric spaces. In this current study, we introduced the concepts of statistical convergence and statistical Cauchy sequences for generalized difference sequences in intuitionistic fuzzy metric spaces. Additionally, we established a Cauchy convergence criterion for this innovative form of convergence. This study can shed future studies on deferred statistical convergence of generalized difference sequences and their hybrid versions in intuitionistic fuzzy metric spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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