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On the Sum and Product of Covering Numbers of Graphs and their Line Graphs

Research Article

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Abstract. The bounds on the sum and product of chromatic numbers of a graph and its complement are known as Nordhaus-Gaddum inequalities. In this paper, we study the bounds on the sum and product of the covering numbers of graphs and their line graphs. We also provide a new characterization of the certain graph classes.

Keywords. Covering number; Independence number; Matching number; Line graph.

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1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [10]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

Many problems in extremal graph theory seek the extreme values of graph parameters on families of graphs. The classic paper of Nordhaus and Gaddum [6] study the extreme values of the sum (or product) of a parameter on a graph and its complement, following solving these problems for the chromatic number on n-vertex graphs. In this paper, we study such problems for some graphs and their associated graphs.

Definition 1.1 ([5]). A *Walk*, $W = v_0e_1v_1e_2v_2 \dots v_{k-1}e_kv_k$, in a graph G is a finite sequence whose terms are alternately vertices and edges such that, for $1 \le i \le k$, the edge e_i has ends $v_{i-1}$and v_i.

Definition 1.2 ([5]). If the vertices $v_0, v_1, \dots, v_k$ of a walk W are distinct then W is called a *Path*. A path with n vertices will be denoted by P_n. P_n has length $n-1$.

Definition 1.3 ([12]). The Covering number of a graph G is the size of a minimum vertex cover in a graph G, known as the vertex cover number of G, denoted by $\beta(G)$.

Definition 1.4 ([4]). Two vertices that are not adjacent in a graph G are said to be *independent*. A set S of vertices is independent if any two vertices of S are independent. The *vertex independence number* or simply the *independence number*, of a graph G, denoted by $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of G.

Definition 1.5 ([2]). A subset M of the edge set of G, is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M, both the end vertices of e are different from the end vertices of f.

Definition 1.6 ([2]). A *perfect matching* of a graph G is a matching of G containing $n/2$ edges, the largest possible, meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching sometimes called a *complete matching* or 1-*factor*.

Definition 1.7 ([2]). The matching number of a graph G, denoted by $\nu(G)$, is the size of a maximal independent edge set. It is also known as *edge independence number*. The matching number $\nu(G)$ satisfies the inequality $\nu(G) \le \lfloor \frac{n}{2} \rfloor$.
Equality occurs only for a perfect matching and graph G has a perfect matching if and only if $|G| = 2\nu(G)$, where $|G| = n$ is the vertex count of G.

Definition 1.8 ([2]). A *maximal independent set* in a line graph corresponds to maximal matching in the original graph.

In this paper, we discussed the sum and product of the covering numbers of certain class of graphs and their line graphs.

2. New Results

Definition 2.1 ([13]). The line graph $L(G)$ of a simple graph G is the graph whose vertices are in one-one correspondence with the edges of G, two vertices of $L(G)$ being adjacent if and only if the corresponding edges of G are adjacent.

Theorem 2.2 ([12]). *The independence number $\alpha(G)$ of a graph G and vertex cover number $\beta(G)$ are related by $\alpha(G) + \beta(G) = |G|$, where $|G| = n$, the vertex count of G.*

Theorem 2.3 ([12]). *The independence number of the line graph of a graph G is equal to the matching number of G.*

Proposition 2.4. *For a complete graph K_n, $n \geq 3$,*

$$\beta(K_n)+\beta(L(K_n)) = \begin{cases} \frac{n^2-2}{2}\ ; & \textit{if } n \textit{ is even} \\ \frac{n^2-1}{2}\ ; & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$\beta(K_n)\cdot\beta(L(K_n)) = \begin{cases} \frac{n(n-1)(n-2)}{2}\ ; & \textit{if } n \textit{ is even} \\ \frac{(n-1)^3}{2}\ ; & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. The independence number of a complete graph K_n on n vertices is 1, since each vertex is joined with every other vertex of the graph G. By Theorem 2.2, the covering number, $\beta(K_n)$ of $K_n = n-1$. By [11], $\alpha(L(K_n)) = \lfloor\frac{n}{2}\rfloor$. Then, since there are $\frac{n(n-1)}{2}$ vertices in $L(K_n)$, by Theorem 2.2, $\beta(L(K_n)) = \frac{n(n-1)}{2} - \lfloor\frac{n}{2}\rfloor = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}$, when n is even and $\frac{n(n-1)}{2} - (\frac{n-1}{2}) = \frac{(n-1)^2}{2}$, when n is odd.

Therefore

$$\beta(K_n)+\beta(L(K_n)) = \begin{cases} \frac{n^2-2}{2}\ ; & ;\text{ if } n \text{ is even} \\ \frac{n^2-1}{2}\ ; & ;\text{ if } n \text{ is odd} \end{cases}$$

and

$$\beta(K_n)\cdot\beta(L(K_n)) = \begin{cases} \frac{n(n-1)(n-2)}{2}\ ; & \text{if } n \text{ is even} \\ \frac{(n-1)^3}{2}\ ; & \text{if } n \text{ is odd}. \end{cases}$$ □

Proposition 2.5. *For a complete bipartite graph $K_{m,n}$,*

$$\beta(K_{m,n})+\beta(L(K_{m,n})) = mn \quad \textit{and} \quad \beta(K_{m,n})\cdot\beta(L(K_{m,n})) = m^2(n-1).$$

Proof. Without the loss of generality, let $m<n$. The independence number of a complete bipartite graph, $\alpha(K_{m,n}) = \max(m,n) = n$. Since a complete bipartite graph consists of $m+n$ number of vertices, by Theorem 2.2, $\beta(K_{m,n}) = m+n-n = m$. The number of vertices in $L(K_{m,n})$ is mn and $\alpha(L(K_{m,n}))$ = matching number of $K_{m,n} = \nu(K_{m,n}) = \min(m,n) = m$. Then by Theorem 2.2, $\beta(L(K_{m,n})) = mn-m = m(n-1)$.
Therefore,

$$\beta(K_{m,n})+\beta(L(K_{m,n})) = mn \quad \text{and} \quad \beta(K_{m,n})\cdot\beta(L(K_{m,n})) = m^2(n-1).$$ □

Definition 2.6. [10] For $n \geq 3$, a *wheel graph* W_{n+1} is the graph K_1+C_n. A wheel graph W_{n+1} has $n+1$ vertices and $2n$ edges.

Theorem 2.7. *For $n \geq 3$,*

$$\beta(W_{n+1})+\beta(L(W_{n+1})) = \begin{cases} 2n+1\ ; & \textit{if } n \textit{ is even} \\ 2(n+1)\ ; & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$\beta(W_{n+1})\cdot\beta(L(W_{n+1})) = \begin{cases} \frac{3n(n+2)}{4}\ ; & \textit{if } n \textit{ is even} \\ \frac{(n+3)(3n+1)}{4}\ ; & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. By [11], the independence number of a wheel graph W_{n+1} is $\lfloor \frac{n}{2} \rfloor$. The number of vertices in W_{n+1} is $n+1$. Then by theorem 2.2, $\beta(W_{n+1}) = (n+1) - \lfloor \frac{n}{2} \rfloor = (n+1) - \frac{n}{2} = \frac{(n+2)}{2}$, if n is even and $(n+1) - \frac{(n-1)}{2} = \frac{(n+3)}{2}$, if n is odd.

Now, consider the line graph of the wheel graph W_{n+1}. By [11], the independence number, $\alpha(L(W_{n+1})) = \lfloor \frac{n}{2} \rfloor$. Since W_{n+1} consists of $2n$ edges, the line graph $L(W_{n+1})$ have exactly $2n$ vertices, and by theorem 2.2, $\beta(L(W_{n+1})) = 2n - \lfloor \frac{n}{2} \rfloor = 2n - \frac{n}{2} = \frac{3n}{2}$, if n is even and $2n - \frac{(n-1)}{2} = \frac{3n+1}{2}$, if n is odd.
Therefore,

$$\beta(W_{n+1}) + \beta(L(W_{n+1})) = \begin{cases} 2n+1\,; & \text{if } n \text{ is even} \\ 2(n+1)\,; & \text{if } n \text{ is odd} \end{cases}$$

and

$$\beta(W_{n+1}) \cdot \beta(L(W_{n+1})) = \begin{cases} \frac{3n(n+2)}{4}\,; & \text{if } n \text{ is even} \\ \frac{(n+3)(3n+1)}{4}\,; & \text{if } n \text{ is odd}. \end{cases}$$ □

Definition 2.8 ([9]). *Helm graphs* are graphs obtained from a wheel by attaching one pendant edge to each vertex of the cycle.

Theorem 2.9. *For a helm graph H_n, $n \geq 3$, $\beta(H_n) + \beta(L(H_n)) = 3n$ and $\beta(H_n) \cdot \beta(L(H_n)) = 2n^2$.*

Proof. A helm graph (H_n) consists of $2n+1$ vertices and $3n$ edges. By [11], the independence number of a helm graph, $\alpha(H_n) = n+1$. Since, the number of vertices of (H_n) is $2n+1$, by Theorem 2.2, $\beta(H_n) = (2n+1) - (n+1) = n$.

Now consider the line graph of the helm graph H_n. By theorem 2.3, the independence number of $L(H_n)$ is equal to the matching number of $H_n = n$. Since, the number of vertices of $L((H_n)) = 3n$, by Theorem 2.2, $\beta(L(H_n)) = 3n - n = 2n$.

Therefore, $\beta(H_n) + \beta(L(H_n)) = 3n$ and $\beta(H_n) \cdot \beta(L(H_n)) = 2n^2$. □

Definition 2.10 ([12]). Given a vertex x and a set U of vertices, an x, $U-$fan is a set of paths from x to U such that any two of them share only the vertex x. A $U-$fan is denoted by $F_{1,n}$.

Theorem 2.11. *For a fan graph $F_{1,n}$,*

$$\beta(F_{1,n}) + \beta(L(F_{1,n})) = \begin{cases} 2n\,; & \text{if } n \text{ is even} \\ 2n-1\,; & \text{if } n \text{ is odd} \end{cases}$$

and

$$\beta(F_{1,n}) \cdot \beta(L(F_{1,n})) = \begin{cases} \frac{(n+2)(3n-2)}{4}\,; & \text{if } n \text{ is even} \\ \frac{(n+1)(3n-3)}{4}\,; & \text{if } n \text{ is odd}. \end{cases}$$

Proof. A fan graph $F_{1,n}$ is defined to be a graph $K_1 + P_n$. By [11], the independence number of a fan graph $F_{1,n}$ is either $\frac{n}{2}$ or $\frac{n+1}{2}$, depending on n is even or odd. Since the number of vertices of

$F_{1,n}$ is $n+1$, by Theorem 2.2, $\beta(F_{1,n}) = (n+1) - \frac{n}{2} = \frac{n+2}{2}$, if n is even and $(n+1) - \frac{(n+1)}{2} = \frac{n+1}{2}$, if n is odd. Now consider the line graph of the fan graph $F_{1,n}$. By Theorem 2.3, the independence number of $L(F_{1,n})$, $\alpha(L(F_{1,n})) = \nu(F_{1,n})$ is either $\frac{n}{2}$ or $\frac{n+1}{2}$ depending on n is even or odd. Since the number of vertices in $F_{1,n}$ is $2n-1$, by Theorem 2.2, $\beta(L(F_{1,n})) = (2n-1) - \frac{n}{2} = \frac{(3n-2)}{2}$, if n is even and $(2n-1) - \frac{(n+1)}{2} = \frac{(3n-3)}{2}$, if n is odd.
Therefore For a fan graph $F_{1,n}$,

$$\beta(F_{1,n}) + \beta(L(F_{1,n})) = \begin{cases} 2n\,; & \text{if } n \text{ is even} \\ 2n-1\,; & \text{if } n \text{ is odd} \end{cases}$$

and

$$\beta(F_{1,n}) \cdot \beta(L(F_{1,n})) = \begin{cases} \frac{(n+2)(3n-2)}{4}\,; & \text{if } n \text{ is even} \\ \frac{(n+1)(3n-3)}{4}\,; & \text{if } n \text{ is odd.} \end{cases}$$ □

Definition 2.12 ([1, 14])**.** An $n-$sun or a *trampoline*, denoted by S_n, is a chordal graph on $2n$ vertices, where $n \geq 3$, whose vertex set can be partitioned into two sets $U = \{u_1, u_2, u_3, \dots, u_n\}$ and $W = \{w_1, w_2, w_3, \dots, w_n\}$ such that U is an independent set of G and u_i is adjacent to w_j if and only if $j = i$ or $j = i+1 (\mathrm{mod}\, n)$. A *complete sun* is a sun G where the induced subgraph $\langle U \rangle$ is complete.

Theorem 2.13. *For a complete sun graph S_n, $n \geq 3$, $\beta(S_n) + \beta(L(S_n)) = \frac{n(n+3)}{2}$ and $\beta(S_n) \cdot \beta(L(S_n)) = \frac{n^2(n+1)}{2}$.*

Proof. Let S_n be a complete sun graph on $2n$ vertices. By [11], the independence number of S_n, $\alpha(S_n) = n$. Since S_n consists of $2n$ vertices, by theorem 2.2, $\beta(S_n) = 2n - n = n$. Now consider the line graph $L(S_n)$ of S_n. By [11], the independence number of $L(S_n)$, $\alpha L(S_n) = n$. The number of vertices of $L(S_n)$ is the number of edges of $S_n = \frac{n(n-1)}{2} + 2n = \frac{n(n+3)}{2}$. Then by theorem 2.2, $\beta(L(S_n)) = [\frac{n(n+3)}{2}] - n = \frac{n(n+1)}{2}$. Therefore, $n \geq 3$, $\beta(S_n) + \beta(L(S_n)) = \frac{n(n+3)}{2}$ and $\beta(S_n) \cdot \beta(L(S_n)) = \frac{n^2(n+1)}{2}$. □

Definition 2.14 ([14])**.** The $n-$sunlet graph is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n and is denoted by L_n.

Theorem 2.15. *For a sunlet graph L_n on $2n$ vertices, $n \geq 3$, $\beta(L_n) + \beta(L(L_n)) = 2n$ and $\beta(L_n) \cdot \beta(L(L_n)) = n^2$.*

Proof. Let L_n be a sunlet graph on $2n$ vertices. By [11], the independence number of L_n, $\alpha(L_n) = n$. Since L_n consists of $2n$ vertices, by Theorem 2.2, $\beta(L_n) = 2n - n = n$. Now consider the line graph $L(L_n)$ of L_n. The number of vertices of $L(L_n)$ is the number of edges of $L_n = 2n$. Then by theorem 2.2, $\beta(L(L_n)) = 2n - n = n$.
Therefore, $\beta(L_n) + \beta(L(L_n)) = 2n$ and $\beta(L_n) \cdot \beta(L(L_n)) = n^2$. □

Definition 2.16 ([10])**.** The *armed crown* is a graph G obtained by adjoining a path P_m to every vertex of a cycle C_n.

Theorem 2.17. *For an armed crown graph G with a path P_m and a cycle C_n,*

$$\beta(G)+\beta(L(G))=\begin{cases} mn+1\,; & \textit{if } m,\, n \textit{ are odd} \\ mn\,; & \textit{otherwise} \end{cases}$$

and

$$\beta(G)\cdot\beta(L(G))=\begin{cases} \frac{(mn+1)^2}{4}\,; & \textit{if } m,\, n \textit{ are odd} \\ \frac{m^2n^2}{4}\,; & \textit{otherwise}. \end{cases}$$

Proof. Note that the number of vertices of P_m is m. By [11], $\alpha(G)=\frac{mn}{2}$, except for m and n are odd. The number of vertices of an armed crown graph is mn. By Theorem 2.2, $\beta(G)=mn-\frac{mn}{2}=\frac{mn}{2}$. Now consider the line graph, $L(G)$ of G. From [11], the independence number of the line graph of G is $\frac{mn}{2}$. The number of vertices of $L(G)$ is the number of edges in G and is equal to $n(m-1)+n=mn$. Then by Theorem 2.2, $\beta(G)=mn-\frac{mn}{2}=\frac{mn}{2}$.
Therefore, $\beta(G)+\beta(L(G))=\frac{mn}{2}+\frac{mn}{2}=mn$ and $\beta(G).\beta(L(G))=\frac{mn}{2}\cdot\frac{mn}{2}=\frac{m^2n^2}{4}$.
When m and n are odd, by [11], the independence number of G, $\alpha(G)=\lfloor\frac{n}{2}\rfloor[\frac{m+1}{2}]+\lceil\frac{n}{2}\rceil[\frac{m-1}{2}]$. By Theorem 2.2,

$$\begin{aligned}\beta(G)&=mn-\left[\left\lfloor\frac{n}{2}\right\rfloor\left[\frac{m+1}{2}\right]+\left\lceil\frac{n}{2}\right\rceil\left[\frac{m-1}{2}\right]\right]\\ &=mn-\left[\left(\frac{n-1}{2}\right)\left(\frac{m+1}{2}\right)+\left(\frac{n+1}{2}\right)\left(\frac{m-1}{2}\right)\right]\\ &=\frac{mn+1}{2}.\end{aligned}$$

Now consider the line graph, $L(G)$ of G. From [11], $\alpha(L(G))=n(\frac{m-1}{2})+\lfloor\frac{n}{2}\rfloor$. The number of vertices of $L(G)$ is the number of edges in G and is equal to $n(m-1)+n=mn$. Then by theorem 2.2,

$$\begin{aligned}\beta(L(G))&=mn-\left[n\left(\frac{m-1}{2}\right)+\left\lfloor\frac{n}{2}\right\rfloor\right]\\ &=mn-\left[n\left(\frac{m-1}{2}\right)+\frac{n-1}{2}\right]\\ &=\frac{mn+1}{2}.\end{aligned}$$

That is, $\beta(G)+\beta(L(G))=\frac{mn+1}{2}+\frac{mn+1}{2}=mn+1$ and $\beta(G)\cdot\beta(L(G))=\frac{mn+1}{2}\cdot\frac{mn+1}{2}=\frac{(mn+1)^2}{4}$.
Therefore, for an armed crown graph,

$$\beta(G)+\beta(L(G))=\begin{cases} mn+1\,; & \text{if } m,\, n \text{ are odd} \\ mn & \text{otherwise} \end{cases}$$

and

$$\beta(G)\cdot\beta(L(G))=\begin{cases} \frac{(mn+1)^2}{4}\,; & \text{if } m,\, n \text{ are odd} \\ \frac{m^2n^2}{4}\,; & \text{otherwise}. \end{cases}$$ □

3. Conclusion

The theoretical results obtained in this research may provide a better insight into the problems involving covering number and independence number by improving the known lower and upper bounds on sums and products of independence numbers of a graph G and an associated graph of G. More properties and characteristics of operations on covering number and also other graph parameters are yet to be investigated. The problems of establishing the inequalities on sums and products of covering numbers for various graphs and graph classes still remain unsettled. All these facts highlight a wide scope for further studies in this area.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both the authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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