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Research Article

Coefficient Problems on Bi-Univalent Functions With (*p*,*q*)-Gegenbauer Polynomials

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Abstract. In this paper our main aim is to study a new subclass of bi-univalent functions and to obtain initial coefficient bounds of starlike and convex bi-univalent functions involving (p,q)-Gegenbauer polynomials. Also, we aim at obtaining sharp bound for Fekete-Szegö functional.

Keywords. Taylor-Maclaurin series, Starlike functions, Convex functions, Biunivalent functions, Coefficient bounds, Fekete-Szegö inequality, (p,q)-Gegenbauer polynomials.

Mathematics Subject Classification (2020). 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z|\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{U}. \tag{1.1}$$

Consider a class $\mathcal{S}$ of univalent functions in $\mathcal{U}$ such that $\mathcal{S}$ is the subclass of class $\mathcal{A}$.

The Köebe one-quarter theorem guarantees that every univalent function f has a disc of radius $\frac{1}{4}$ in it's image. As a result, every univalent function f has an inverse f^{-1} satisfying

$f^{-1}(f(z)) = z$, $z \in \mathcal{U}$ and

$$f^{-1}(f(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

If both f and f^{-1} are univalent in $\mathcal{U}$, f is said to be biunivalent in $\mathcal{U}$. The class of biunivalent functions defined in the unit disc is denoted by σ. The Köebe function univalently maps the unit disc $\mathcal{U}$ onto the entire complex plane minus an opening along the line from $-\frac{1}{4}$ to $-\infty$. Therefore, the Köebe function is not an element of σ. Hence the picture domain doesnot contain the unit disc $\mathcal{U}$. In 1985, Bieberbach conjecture was proved by Louis de Branges [5], which asserts that for each $\mathcal{S}$ generated by the series (1.1), the following coefficient inequality is true

$$|a_n| \leq n \quad (n \in \mathbb{N} - 1).$$

The set of positive integers is denoted by $\mathbb{N}$. Lewin [9] was the first to introduce and study the class of analytic bi-univalent functions, proving that $|a_2| \leq 1.51$. In [9] Lewin's result was refined to $|a_2| \leq \sqrt{2}$. Bi-univalent function subclasses generated by strongly starlike, starlike and convex functions were studied by Brannan and Clunie [2], and Pommerenke [16]. They defined non-sharp estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, and presented bi-starlike and bi-convex functions.

For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f(z) \prec g(z)$, if there exists a schwarz function ω, which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.

Let λ be a real non-zero constant, the function which generates Gegenbauer polynomials is given by $G_\lambda(x,z) = \frac{1}{(1-2xz+z^2)^\lambda}$, where $x \in [-1,1]$ and $z \in \mathcal{U}$.

Since G_λ is analytic in unit disc $\mathcal{U}$ for a fixed value of x, we can expand G_λ by Taylor's series expansion which gives

$$G_\lambda = \sum_{n=0}^{\infty} v_n^\lambda(x) z^n, \tag{1.3}$$

where v_n^λ is Gegenbauer polynomials of degree n.
Gegenbauer polynomials can also be expanded by the relation,

$$v_n^\lambda(x) = \frac{1}{n}[2x(n+\lambda-1)v_{n-1}^\lambda(x) - (n+2\lambda-2)v_{n-2}^\lambda(x). \tag{1.4}$$

For the values of n, initial Gegenbauer polynomials can be written as:

$$v_0^\lambda(x) = 1,$$
$$v_1^\lambda(x) = 2\lambda x,$$
$$v_2^\lambda(x) = 2\lambda(1+\lambda)x^2 - \lambda.$$

Generalized equation of (1.4) for $0 < q \leq p \leq 1$ is given by

$$v_0^\lambda(x,s,p,q) = 1,$$
$$v_1^\lambda(x,s,p,q) = 2(p+q)x,$$

$$v_2^{\lambda}(x,s,p,q) = \frac{1}{2}[\lambda(1+\lambda)(p^2+q^2)(p+q)x^2 + 2\lambda pqs],$$

which are known as (p,q)-Gegenbauer polynomials which become Chebyshev polynomials for $p = q = \lambda = 1$ and Legendre polynomials for $p = q = 1$ and $\lambda = \frac{1}{2}$.

In present, work we define two new classes of bi-univalent functions with the help of (p,q) Gegenbauer polynomials. To prove our results we use the following lemma.

Lemma 1.1 ([1]). *If $w(z) = c_1z + c_2z^2 + c_3z^3 + \dots$ which is analytic on the unit disc $\mathfrak{U}$ with w(0)=0 and $|w(z)| \le 1$ then $|c_i| \le 1$, for all $j \in \mathbb{N}$.*

First, we define a class of convex bi-univalent functions associated with (p,q) Gegenbauer polynomials as below.

Definition 1.2. *A function $f \in \sigma$ is said to be in the class $B_{\lambda}^{c}(p,q)$ if it satisfies the following subordination for all $z, w \in \mathfrak{U}$,*

$$1 + \frac{zf''(z)}{f'(z)} \prec G_{(p,q)}^{\lambda}(x,z) \tag{1.5}$$

and

$$1 + \frac{wg''(w)}{g'(w)} \prec G_{(p,q)}^{\lambda}(x,w), \tag{1.6}$$

where $x \in \left(\frac{1}{2},1\right]$, $G_{(p,q)}^{\lambda}$ is the generating function of the (p,q)-Gegenbauer polynomials and $g(w) = f^{-1}(w)$.

Next, we define the following class which consists of starlike bi-univalent functions associated with (p,q)-Gegenbauer polynomials.

Definition 1.3. *A function $f \in \sigma$ is said to be in the class $B_{\lambda}^{*}(p,q)$ if the following subordinations hold for all $z, w \in \mathfrak{U}$,*

$$\frac{zf'(z)}{f(z)} \prec G_{(p,q)}^{\lambda}(x,z) \tag{1.7}$$

and

$$\frac{wg'(w)}{g(w)} \prec G_{(p,q)}^{\lambda}(x,w). \tag{1.8}$$

Now we estimate the bounds of initial coefficients for the above two classes.

2. Main Results

Theorem 2.1. *Let the function $f \in \sigma$ in the class $B_{\lambda}^{c}(x,s,p,q)$ then*

$$|a_2| \le \frac{1}{\sqrt{2}} \frac{|\lambda||p+q|^{3/2}|x|^{3/2}}{\sqrt{(p+q)((\lambda+1)p^2 - \lambda p + ((\lambda+1)p - \lambda)q)x^2 + 2pqs}}$$

and

$$|a_3| \le \frac{2}{13} \left| \frac{x(p+q)[(p+q)((\lambda+1)p^2 - 4\lambda p + q((\lambda+1)q - 4\lambda))x^2 + 2pqs]\lambda}{(p+q)((\lambda+1)p^2 - \lambda p + ((\lambda+1)p - \lambda)q)x^2 + 2pqs} \right|.$$

Proof. Let $f \in B^{c}_{\lambda}(p,q)$ then by definition, we have

$$1+\frac{zf''(z)}{f'(z)} = G^{\lambda}_{(p,q)}(x,c(z)) \tag{2.1}$$

and

$$1+\frac{wg''(w)}{g'(w)} = G^{\lambda}_{(p,q)}(x,d(w)) \tag{2.2}$$

for some functions $c(z)=c_1z+c_2z^2+c_3z^3+\dots$ and $d(w)=d_1w+d_2w^2+\dots$ which are anayltic on the unit disk $\mathcal{U}$ with $c(0)=d(0)=0$, $|c(z)|<1$, $|d(w)|<1$ $(z,w\in\mathcal{U})$. By virtue of the generating function of the (p,q)-Gegenbauer polynomials $G^{\lambda}_{(p,q)}$ defined already, the equations (2.1) and (2.2) become,

$$1+\frac{zf''(z)}{f'(z)} = v^{\lambda}_0(x,s,p,q)+v^{\lambda}_1(x,s,p,q)c(z)+v^{\lambda}_2(x,s,p,q)c^2(z)+\dots \tag{2.3}$$

and

$$1+\frac{wg''(w)}{g'(w)} = v^{\lambda}_0(x,s,p,q)+v^{\lambda}_1(x,s,p,q)d(w)+v^{\lambda}_2(x,s,p,q)d^2(w)+\dots\ . \tag{2.4}$$

After some substitution and simplification, we have

$$\begin{aligned}&1+2a_2z+(-4a_2^2+a_3)z^2+(-6a_2a_3+12a_4+2(4a_2^2-6a_3)a_2)z^3+\dots\\ &\quad=1+v^{\lambda}_1(x,s,p,q)c_1z+[v^{\lambda}_1(x,s,p,q)c_2+v^{\lambda}_2(x,s,p,q)c_1^2]z^2+\dots\end{aligned}$$

and

$$\begin{aligned}&1-2a_2w+(8a_2^2-6a_3)w^2+(-60a_2^3+60a_2a_3-12a_4+2a_2(6a_2^2-3a_3)-2(-8a_2^2+6a_3)a_2)w^3+\dots\\ &\quad=1+v^{\lambda}_1(x,s,p,q)d_1w+[v^{\lambda}_1(x,s,p,q)d_2+v^{\lambda}_2(x,s,p,q)d_1^2]w^2+\dots.\end{aligned}$$

Equating coefficients, we get

$$2a_2 = v^{\lambda}_1(x,s,p,q)c_1, \tag{2.5}$$

$$6a_3-4a_2^2 = v^{\lambda}_1(x,s,p,q)c_2+v^{\lambda}_2(x,s,p,q)c_1^2, \tag{2.6}$$

$$-2a_2 = v^{\lambda}_1(x,s,p,q)d_1, \tag{2.7}$$

$$8a_2^2-6a_3 = v^{\lambda}_1(x,s,p,q)d_2+v^{\lambda}_2(x,s,p,q)d_1^2. \tag{2.8}$$

From (2.5) and (2.7), we have

$$c_1=-d_1 \ \text{ and } \ 8a_2^2=[v^{\lambda}_1(x,s,p,q)]^2(c_1^2+d_1^2), \tag{2.9}$$

by adding (2.6) and (2.8) we get

$$4a_2^2 = v^{\lambda}_1(x,s,p,q)(c_2+d_2)+v^{\lambda}_2(x,s,p,q)(c_1^2+d_1^2). \tag{2.10}$$

By substituting for $c_1^2+d_2^2$ in (2.10), we get

$$\left[4-\frac{8v^{\lambda}_2(x,s,p,q)}{v^{\lambda}_1}\right]a_2^2 = v^{\lambda}_1(x,s,p,q)(c_2+d_2).$$

By using Lemma 1.1 and after further simplification, we have

$$|a_2| \le \frac{1}{\sqrt{2}}\frac{|\lambda|\,|p+q|^{3/2}|x|^{3/2}}{\sqrt{(p+q)((\lambda+1)p^2-\lambda p+((\lambda+1)p-\lambda)q)x^2+2pqs}}.$$

From (2.6) and (2.8), we have

$$12a_3 - 12a_2^2 = v_1^\lambda(x,s,p,q)(c_2 - d_2) + v_2^\lambda(x,s,p,q)(c_1^2 - d_1^2). \tag{2.11}$$

After substituting a_2^2 and simple calculations it follows that

$$|a_3| \le \frac{2}{13}\left|\frac{x(p+q)[(p+q)((\lambda+1)p^2 - 4\lambda p + q((\lambda+1)q - 4\lambda))x^2 + 2pqs]\lambda}{(p+q)((\lambda+1)p^2 - \lambda p + ((\lambda+1)p - \lambda)q)x^2 + 2pqs}\right|. \qquad \square$$

By taking $\lambda = 1$, $p = q = 1$ in above theorem, we get the following corollary.

Corollary 2.2. *Let the function $f \in \sigma$ given by* (1.1) *be in the class $B_c(1)$. Then*

$$|a_2| \le 2x\sqrt{x},$$
$$|a_3| \le x^2 + \frac{x}{3}.$$

2.1 Fekete-Szegö Inequality for the Class $B_\lambda^c(x,s,p,q)$

In study of coefficients of univalent analytic functions, Fekete-Szegö inequality is a very interesting problem. In this section we find sharp bound of Fekete-Szegö inequality for the class $B_\lambda^c(x,s,p,q)$ of bi-univalent functions.

Theorem 2.3. *Let the function $f(z) \in \sigma$ be in the class $B_\lambda^c(x,s,p,q)$ then for some $\eta \in \mathbb{R}$,*

$$|a_3 - \eta a_2^2| = |v_1^\lambda(x,s,p,q)|\left|\left(h - \frac{1}{12}\right)d_2 + \left(h + \frac{1}{12}\right)c_2\right|$$
$$\le \begin{cases} \frac{|\lambda|(p+q)x}{6}, & |h(\eta)| \le \frac{1}{12}, \\ 2|\lambda|(p+q)xh(\eta), & |h(\eta)| \ge \frac{1}{12}. \end{cases}$$

Proof. Let $f \in B_c^\lambda(x,s,p,q)$ by using equation (2.10) and (2.11) for some $\eta \in \mathbb{R}$, we arrive at

$$a_3 - \eta a_2^2 = (1-\eta)\left[\frac{[v_1^\lambda(x,s,p,q)^3(d_2 + c_2)]}{4[v_1^\lambda(x,s,p,q)]^2 - 8v_2^\lambda(x,s,p,q)}\right] + \frac{v_1^\lambda(x,s,p,q)}{12}(c_2 - d_2).$$

After some simple calculation, we get

$$a_3 - \eta a_2^2 = v_1^\lambda(x,s,p,q)\left[h(\eta)d_2 - \frac{d_2}{12} + h(\eta)c_2 + \frac{c_2}{12}\right],$$

where $h(\eta) = \frac{(1-\eta)(v_1^\lambda(x,s,p,q))^2}{4(v_1^\lambda(x,s,p,q))^2 - 8v_2^\lambda(x,s,p,q)}$.

We have

$$|a_3 - \eta a_2^2| = |v_1^\lambda(x,s,p,q)|\left|\left(h - \frac{1}{12}\right)d_2 + \left(h + \frac{1}{12}\right)c_2\right|$$
$$\le \begin{cases} \frac{|\lambda|(p+q)x}{6}, & |h(\eta)| \le \frac{1}{12}, \\ 2|\lambda|(p+q)xh(\eta), & |h(\eta)| \ge \frac{1}{12}. \end{cases} \qquad \square$$

If we replace $\eta = 1$, $p = q = 1$ in Theorem 2.3, we get the following corollary.

Corollary 2.4. *Let f be function such that $f \in \sigma$ given by* (1.1) *which belongs to the class $B_c(\alpha)$. Then $|a_3 - a_2^2| \le \frac{|\alpha|x}{2}$.*

The next theorem is about bounds of initial coefficients of the class $B_\lambda^*(x,s,p,q)$ of starlike bi-univalent functions.

Theorem 2.5. *Let the function $f \in \sigma$ be in the class $B^*_\lambda(x,s,p,q)$ then*
$$|a_2| \le \frac{\sqrt{2}|\lambda|(p+q)x\sqrt{(p+q)x}}{\sqrt{|[(\lambda+1)p^2-2\lambda p+((\lambda+1)q-2\lambda)q](p+q)x^2+2pqs|}}$$
and
$$|a_3| \le \frac{1}{2}|x\lambda(1+2\lambda(p+q)x)(p+q)|.$$

Proof. Let $f \in B^*_\lambda(x,s,p,q)$, we have
$$\frac{zf'(z)}{f(z)} = G^\lambda_{(p,q)}(x,s,c(z))$$
and
$$\frac{wg'(w)}{g(w)} = G^\lambda_{(p,q)}(x,s,d(w)),$$
where $c(z)$ and $d(w)$ are Schwartz functions such that $c(0)=d(0)=1$, $|c(z)|<1$, $(z \in \mathcal{U})$ and $|c(z)|<1$, $|d(w)|<1$, $(w \in \mathcal{U})$.

We can write the above equations as,
$$\frac{zf'(z)}{f(z)} = 1+v^\lambda_1(x,s,p,q)c_1z+[v^\lambda_1(x,s,p,q)c_2+v^\lambda_2(x,s,p,q)c_1^2]z^2+\dots$$
and
$$\frac{wg'(w)}{g(w)} = 1+v^\lambda_1(x,s,p,q)+[v^\lambda_1(x,s,p,q)d_2+v^\lambda_2(x,s,p,q)d_1^2]w^2+\dots.$$
By comparing coefficients we can write,
$$a_2 = v^\lambda_1(x,s,p,q)c_1, \tag{2.12}$$
$$2a_3-a_2^2 = v^\lambda_1(x,s,p,q)c_2+v^\lambda_2(x,s,p,q)c_1^2, \tag{2.13}$$
$$-2a_2 = v^\lambda_1(x,s,p,q)d_1, \tag{2.14}$$
$$3a_2^2-2a_3 = v^\lambda_1(x,s,p,q)d_2+v^\lambda_2(x,s,p,q)d_1^2. \tag{2.15}$$
From (2.11) and (2.13), we have
$$c_1=-d_1 \ \text{ and } \ 2a_2^2=[v^\lambda_1(x,s,p,q)]^2(c_1^2+d_1^2). \tag{2.16}$$
By adding (2.12) and (2.14), we get
$$2a_2^2 = v^\lambda_1(x,s,p,q)(c_2+d_2)+v^\lambda_2(x,s,p,q)(c_1^2+d_1^2). \tag{2.17}$$
By substituting (2.15) in (2.16), we get
$$\left[2-\frac{2v^\lambda_2(x,s,p,q)}{v^\lambda_1(x,s,p,q)^2}\right]a_2^2 = v^\lambda_1(x,s,p,q)(c_2+d_2).$$
After applying lemma and some calculations we arrive at
$$|a_2| \le \frac{\sqrt{2}|\lambda|(p+q)x\sqrt{(p+q)x}}{\sqrt{|[(\lambda+1)p^2-2\lambda p+((\lambda+1)q-2\lambda)q](p+q)x^2+2pqs|}}.$$
Next by subtracting (2.14) by (2.12), we get
$$4a_3-4a_2^2 = v^\lambda_1(x,s,p,q)(c_2-d_2)+v^\lambda_2(x,s,p,q)(c_1^2-d_1^2).$$

Since $c_1 = -d_1$, the above equation becomes

$$a_3 = a_2^2 + \frac{v_1^\lambda(x,s,p,q)}{4}(c_2 - d_2). \tag{2.18}$$

After using Lemma 1.1 with further simplification, we get

$$|a_3| \le \frac{1}{2}|x\lambda\big(1+2\lambda(p+q)x\big)(p+q)|. \qquad \square$$

By taking $\lambda = 1$, $p = q = 1$ in above theorem, we get the following corollary.

Corollary 2.6. *Let the function $f \in \sigma$ given by* (1.1) *be in the class $B_\lambda(1)$. Then*

$$|a_2| \le 2x\sqrt{2x},$$
$$|a_3| \le 4x^2 + x.$$

Now we can find the sharp bounds of Fekete-Szegö functional $a_3 - \eta a_2^2$ which is defined for the class $B_\lambda^*(x,s,p,q)$.

2.2 Fekete-Szegö Inequality for the Class $B_\lambda^*(x,s,p,q)$

Theorem 2.7. *Let the function $f \in \sigma$ be in the class $B_\lambda^*(x,s,p,q)$. Then for $\eta \in \mathbb{R}$, we have*

$$|a_3 - \eta a_2^2| \le \begin{cases} |\lambda|(p+q)x, & |h(\eta)| \le \frac{1}{4}, \\ 2|\lambda|(p+q)x|h(\eta)|, & |h(\eta)| \ge \frac{1}{4}. \end{cases}$$

Proof. Let $f \in B_\lambda^*(x,s,p,q)$ then by using the equations (2.16) and (2.18) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)a_2^2 + \frac{v_1^\lambda(x,s,p,q)}{4}(c_2 - d_2) \\ &= v_1^\lambda(x,s,p,q)\left[\left(h(\eta)+\frac{1}{4}\right)c_2 + \left(h(\eta)-\frac{1}{4}\right)d_2\right], \end{aligned}$$

where $h(\eta) = \frac{(v_1^\lambda(x,s,p,q))^2(1-\eta)}{2(v_1^\lambda(x,s,p,q))^2 - 2v_1^\lambda(x,s,p,q)}$, we have

$$\begin{aligned} |a_3 - \eta a_2^2| &\le |v_1^\lambda(x,s,p,q)|\left|h(\eta)+\frac{1}{4}+h(\eta)-\frac{1}{4}\right| \\ &\le \begin{cases} |\lambda|(p+q)x, & |h(\eta)| \le \frac{1}{4}, \\ 2|\lambda|(p+q)x|h(\eta)|, & |h(\eta)| \ge \frac{1}{4}. \end{cases} \qquad \square \end{aligned}$$

If we replace $\eta = 1$, $p = q = 1$ in Theorem 2.7, we get the following corollary.

Corollary 2.8. *Let f be function such that $f \in \sigma$ given by* (1.1) *which belongs to the class $B_c(\alpha)$. Then $|a_3 - a_2^2| \le |\alpha|x$.*

3. Conclusion

We have obtained initial bounds for two new subclasses of biunivalent functions and have obtained bounds for Fekete-Szegö functional.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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