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Research Article

Coupled Invariant Point Theorems in Bicomplex Parametric Partial Metric Space Amongst Dominate Function

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Abstract. In this paper, bicomplex parametric partial metric space is discussed. The new contractive condition is discussed for coupled invariant point concerning self-mapping on the said spaces with particular conditions. We support our tracking down through models. Our outcomes sum up the common coupled invariant point in bicomplex valued parametric metric spaces by Saluja (Coupled fixed point results based on control function in complex partial metric spaces, *Functional Analysis, Approximation and Computation* **15**(2) (2023), 1 – 15). Established results generalize previously known results for specified conditions.

Keywords. Bicomplex Parametric Partial Metric Space (*BCPPMS*), Dominate function, Coupled invariant point, Generalized contraction conditions

Mathematics Subject Classification (2020). 47H10

1. Introduction

There is part of progress and speculation work has been finished by a few researchers on invariant point results and common fixed point brings about metric space. Choi *et al.* [6] proposed the idea of bicomplex numbers and their rudimentary capabilities, which was speculation of complex numbers and made some common invariant statement hypotheses regarding two feebly viable planning. Later, in 2019, Jebril *et al.* [8, 9] demonstrated a variety of invariant point outcomes in bicomplex valued metric spaces by means of mixed type contractions. Mani *et al.* [11] exhibited some normal FPTs in bicomplex valued metric spaces in 2022. In 2023, Bharathi *et al.* [3] found coupled invariant focuses results in bicomplex partial metric space and its applications and Saluja [18] exhibited some coupled invariant point results complex partial metric space in view of control capability. Poornavel *et al.* [16] recently discovered some significant invariant point results on tricomplex valued parametric metric spaces in the year 2024. After that few specialists distributed a significant amount of the review, e.g., Beg *et al.* [2], Bharathi*et al.* [3],Bhaskar and Lakshmikantam [4], Bhatt *et al.* [5], Choi*et al.* [6], Jebrial*et al.* [8, 9], Mukheimer [12], Ramaswamy*et al.* [17], Saluja [18], and Venkatesh and Raju [20]. In this paper, we present bicomplex parametric partial metric space and established some fixed-point results.

2. Preliminaries

Now we recall some basic definition which will be utilized in our paper:

Definition 2.1 ([19]). Let C be the set of complex numbers and $\varrho, v \in C$. Define a partial order $\leq$ on C as $\varrho \leq v$ if $\mathrm{Re}(\varrho) \leq \mathrm{Re}(v)$, $\mathrm{Img}(z_1) \leq \mathrm{Img}(v)$. It follows that $\varrho \leq v$. If one of the following conditions are holds:

(i) $\mathrm{Re}(\varrho) = Re(v)$ and $\mathrm{Img}(\varrho) = \mathrm{Img}(v)$,

(ii) $\mathrm{Re}(\varrho) < Re(v)$ and $\mathrm{Img}(\varrho) = \mathrm{Img}(v)$,

(iii) $\mathrm{Re}(\varrho) = \mathrm{Re}(v)$ and $\mathrm{Img}(\varrho) < \mathrm{Img}(v)$,

(iv) $\mathrm{Re}(\varrho) < \mathrm{Re}(v)$ and $\mathrm{Img}(\varrho) < \mathrm{Img}(v)$.

We write $\varrho \lesssim v_2$ if $\varrho \neq v$ and one of (ii) and (iii) are satisfied and we write $\varrho < v$ if only (iv) is satisfied.

We denote the symbol $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$ as a bunch of genuine, complex and bicomplex numbers separately, a bunch of genuine, complex and bicomplex numbers separately. The arrangement of bicomplex numbers characterized as:

$$\begin{aligned}\mathcal{E}_2 &= \{w : w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathcal{E}_0\}\\ &= \{w : w = \varrho + v i_2, \varrho, v \in \mathcal{E}_1\},\end{aligned}$$

where $\varrho = a_1 + a_2 i_1$, $v = a_3 + a_4 i_1$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product of all units are commutative and satisfy:

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

The inverse of $u = u_1 + i_2 u_2$ exits if $u_1^2 + u_2^2 \neq 0$, i.e., $|u_1^2 + u_2^2| \neq 0$ and it is defined as $u^{-1} = \frac{1}{u} = \frac{u_1 - i_2 u_2}{u_1^2 + u_2^2}$, and then u is called invertible.

For a bicomplex number $w = \varrho + \upsilon i_2$, the norm is denoted by $\|\varrho + \upsilon i_2\|$ and defined by

$$(|\varrho|^2 + |\upsilon|^2)^{\frac{1}{2}} = (|\varrho - \upsilon i_1|^2 + |\varrho + \upsilon i_1|^2)^{\frac{1}{2}}.$$

If we take $w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$ then the norm of w is defined by

$$\|w\| = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{\frac{1}{2}}.$$

The partial order relation $\lesssim_{i_2}$ on $\mathcal{E}_2$ was defined by Choi *et al*. [6] as $\tau \lesssim_{i_2} \sigma$ if and only if $\varrho \le w_1$ and $\upsilon \le w_2$, for $\tau = \varrho + \upsilon i_2$and $\sigma = w_1 + w_2 i_2$ be two bicomplex numbers. It follows that $\tau \lesssim_{i_2} \upsilon$ if one of the following conditions is satisfied:

(i) $\varrho = w_1$, $\upsilon = w_2$,

(ii) $\varrho \prec w_1$, $\upsilon = w_2$,

(iii) $\varrho = w_1$, $\upsilon \prec w_2$,

(iv) $\varrho \prec w_1$, $\upsilon \prec w_2$.

We write $\tau \lnsim_{i_2} \sigma$ if $\tau \lesssim_{i_2} \sigma$ and $\tau \neq \sigma$ and one of (ii), (iii) and (iv) are satisfied and we will write $\tau \prec \sigma$ if only (iv) is satisfied. The following statements hold for any two bicomplex numbers:

(1) $\tau \lesssim_{i_2} \sigma \Longrightarrow \|\tau\| \le \|\sigma\|$.

(2) $\|\tau + \sigma\| \le \|\tau\| + \|\sigma\|$.

(3) $\|a\tau\| = a\|\tau\|$, where a is nonnegative real number.

(4) $\|\tau\sigma\| \le \sqrt{2}\|\tau\|\|\sigma\|$, and the equally holds only when at least one of τ and σ is degenerated.

(5) $\|\tau^{-1}\| = \|\tau\|^{-1}$, if τ is a degenerated bicomplex number with $0 \prec \tau$.

(6) $\left\|\frac{\tau}{\upsilon}\right\| = \frac{\|\tau\|}{\|\upsilon\|}$, if υ is a degenerated bicomplex number.

Now we recall some notations basic concept that will be utilized in the sequel.

Definition 2.2 ([6])**.** Let Ω be a nonempty set the function $\mathfrak{F} : \Omega \times \Omega \to \mathcal{E}_2$ satisfies the following conditions: If for all $\varrho, \upsilon, \eta \in \Omega$,

(i) $0 \lesssim_{i_2} \mathfrak{F}(\varrho, \upsilon)$ and $\mathfrak{F}(\varrho, \upsilon) = 0 \Longleftrightarrow \varrho = \upsilon$.

(ii) $\mathfrak{F}(\varrho, \upsilon) = \mathfrak{F}(\upsilon, \varrho)$.

(iii) $\mathfrak{F}(\varrho, \upsilon) \lesssim_{i_2} \mathfrak{F}(\varrho, \eta) + \mathfrak{F}(\eta, \upsilon)$.

Then the pair $(\Omega, \mathfrak{F})$ is called bicomplex valued metric spaces.

Definition 2.3 ([11])**.** Let Ω be a nonempty set the function $\mathfrak{F} : \Omega \times \Omega \to \mathcal{E}_2$ satisfies the following conditions: If for all $\varrho, \upsilon, \eta \in \Omega$,

(i) $0 \lesssim_{i_2} \mathfrak{F}(\varrho, \varrho)$ and $\mathfrak{F}(\varrho, \upsilon)$.

(ii) $\mathfrak{F}(\varrho, \upsilon) = \mathfrak{F}(\upsilon, \varrho)$.

(iii) $\mathfrak{F}(\varrho, \varrho) = \mathfrak{F}(\varrho, \upsilon) = \mathfrak{F}(\upsilon, \upsilon) \Longleftrightarrow \varrho = \upsilon$.

(iv) $\mathfrak{F}(\varrho, \upsilon) \lesssim_{i_2} \mathfrak{F}(\varrho, \eta) + \mathfrak{F}(\eta, \upsilon) - \mathfrak{F}(\eta, \eta)$.

Then the pair $(\Omega, \mathfrak{F})$ is called bicomplex partial metric spaces.

Definition 2.4 ([16]). Let Ω be a nonempty set the function $\mathfrak{F}:\Omega\times\Omega\times(0,\infty)\to\mathcal{E}_2$ satisfies the following conditions: If for all $\varrho,\upsilon,\eta\in\Omega$ *and* $t>0$.

(i) $0\precsim_{i_2}\mathfrak{F}(\varrho,\upsilon,t)$ and $\mathfrak{F}(\varrho,\upsilon,t)=0\Longleftrightarrow\varrho=\upsilon$.

(ii) $\mathfrak{F}(\varrho,\upsilon,t)=\mathfrak{F}(\upsilon,\varrho,t)$.

(iii) $\mathfrak{F}(\varrho,\upsilon,t)\precsim_{i_2}\mathfrak{F}(\varrho,\eta,t)+\mathfrak{F}(\eta,\upsilon,t)$.

Then the pair $(\Omega,\mathfrak{F})$ is called bicomplex parametric metric spaces.

Definition 2.5. Let Ω be a nonempty set the function $\mathfrak{F}:\Omega\times\Omega\to\mathcal{E}_2$ satisfies the following conditions: If for all $\varrho,\upsilon,\eta\in\Omega$ and $t>0$.

(i) $0\precsim_{i_2}\mathfrak{F}(\varrho,\varrho,t)$ and $\mathfrak{F}(\varrho,\upsilon,t)$.

(ii) $\mathfrak{F}(\varrho,\upsilon,t)=\mathfrak{F}(\upsilon,\varrho,t)$.

(iii) $\mathfrak{F}(\varrho,\varrho,t)=\mathfrak{F}(\varrho,\upsilon,t)=\mathfrak{F}(\upsilon,\upsilon,t)\Longleftrightarrow\varrho=\upsilon$.

(iv) $\mathfrak{F}(\varrho,\upsilon,t)\precsim_{i_2}\mathfrak{F}(\varrho,\eta,t)+\mathfrak{F}(\eta,\upsilon,t)-\mathfrak{F}(\eta,\eta,t)$.

Then the pair $(\Omega,\mathfrak{F})$ is called *Bicomplex Parametric Partial Metric Spaces* (*BCPPMS*).

Example 2.6. Suppose $\Omega=[0,1]$ and $\mathfrak{F}:\Omega\times\Omega\times(0,\infty)\to\mathcal{E}_2$ defined by $\mathfrak{F}(\varrho,\upsilon,t)=t|\varrho-\upsilon|e^{\frac{i_2\pi}{3}}$, for all $t>0$. Then $(\Omega,\mathfrak{F})$ is a *BCPPMS*.

Example 2.7. Let $(\Omega,\mathfrak{F})$ be a *BCPPMS* and $\mathfrak{D}\subseteq\Omega$,

(p_1) $\mathfrak{z}\in\mathfrak{D}$ is called an interior point of $\mathfrak{D}$ whenever for each $0\prec_{i_2}c\in\mathcal{E}_2$, such that $\mathcal{N}(\mathfrak{z},c,p)\subseteq\mathfrak{D}$, where $\mathcal{N}(\mathfrak{z},c,p)=\{\mathcal{E}\in\Omega:\mathfrak{F}(\mathfrak{z},\mathcal{E},p)\precsim_{i_2}c\}$, $p>0$.

(p_2) $\mathfrak{y}\in\Omega$ is called limit point of $\mathfrak{D}$ whenever there is $0\prec_{i_2}c\in\mathcal{E}_2$,

$$\mathcal{N}(\mathfrak{z},c,p)\cup(\mathfrak{D}-\Omega)\neq\emptyset,\quad\text{for all } p>0.$$

(p_3) $\mathcal{A}\in\Omega$ is called open whenever every element of $\mathcal{A}$ is an interior point of $\mathcal{A}$ and $\mathfrak{D}\subseteq\Omega$ is called closed whenever each limit point of $\mathfrak{D}$ belongs to $\mathfrak{D}$. The family

$$F=\{\mathcal{N}(\mathfrak{z},c,p):\mathfrak{z}\in\Omega,\,0\prec_{i_2}c\in\mathcal{E}_2,\,p>0\}$$

is a sub-basis for a topology on Ω.

Definition 2.8 ([11]). Let $(\Omega,\mathfrak{F})$ be a *BCPPMS* and $\{\upsilon_n\}$ be a sequence in Ω, then

(i) A sequence $\{\upsilon_n\}$ in X is said to be convergent, $\{\upsilon_n\}$ convergence to υ. If for any $0\prec_{i_2}c\in\mathcal{E}_2$ then there exist $n_0\in\mathfrak{N}$ such that $\mathfrak{F}(\upsilon_n,\ \upsilon)\prec_{i_2}c$, for all $n>n_0$ and we can denote this $\lim_{n\to\infty}\upsilon_n=\upsilon$ or $\upsilon_n\to\upsilon$ as $n\to\infty$.

(ii) A sequence $\{\upsilon_n\}$ in X is said to be Cauchy sequence if and only if $d(\upsilon_n,\upsilon_{n+\ell})\prec_{i_2}c$, where $\ell,n\in\mathfrak{N}$.

(iii) If every Cauchy sequence is convergent in $(\Omega,\mathfrak{F})$ then $(\Omega,\mathfrak{F})$ is said to be a Complete *BCPPMS*.

Lemma 2.9 ([11]). *Let $(\Omega,\mathfrak{F})$ be a BCPPMS and let $\{\upsilon_n\}$ be a sequence in Ω then $\{\upsilon_n\}$ is convergent and converges to υ if and only if $\lim_{n\to\infty}\|\mathfrak{F}(\upsilon_n,\upsilon,t)\|=0$.*

Lemma 2.10 ([11]). *Let $(\Omega,\mathfrak{F})$ be a* BCPPMS *and let $\{\upsilon_n\}$ be a sequence in Ω then $\{\upsilon_n\}$ is a Cauchy sequence if and only if $\lim_{n\to\infty}\|\mathfrak{F}(\upsilon_n,\upsilon_{n+\ell},t)\|=0$, $\ell,n\in\mathfrak{N}$ and $t>0$.*

Definition 2.11. Let $(\Omega,\mathfrak{F})$ be a *BCPPMS*. Then an element $\mathfrak{F}(\varrho,\upsilon,t)\in\Omega\times\Omega$ is said to be a common coupled invariant point of $\mathfrak{F}:\Omega\times\Omega\to\Omega$ if $\varrho=\mathfrak{F}(\varrho,\upsilon,t)$, $\upsilon=\mathfrak{F}(\varrho,\upsilon,t)$.

Example 2.12. Consider $\Omega=[0,+\infty)$ and $\mathfrak{F}:\Omega\times\Omega\to\Omega$ be defined by $\mathfrak{F}(\varrho,\upsilon,t)=\frac{\varrho+\upsilon}{3}$, for all $\varrho,\upsilon\in\Omega$. Then $\mathfrak{F}$ has unique coupled invariant point.

3. Main Results

Before starting the main results, we introduced '*Dominate function*' used in main theorem.
Let dominate function $\mathbf{\Psi}$ denotes the set of all function $\varpi:[0,+\infty)\to[0,+\infty)$ satisfying the following conditions:

$\mathbf{\Psi}_{(\boldsymbol{a})}$: ϖ is continuous on$[0,+\infty)$,

$\mathbf{\Psi}_{(\boldsymbol{b})}$: $\varpi(\alpha)<\alpha$ for all $\alpha>0$.

In fact $\varpi(0)=0$ and $\varpi(\alpha)\le\alpha$, for all $\alpha\ge 0$.

In this paper, we draw upon rational type contractive conditions in the framework of *Bicomplex Valued Parametric Metric Space* (*BCPPMS*) using dominate function and prove unique coupled invariant point theorem.

Theorem 3.1. *Suppose a mapping $\mathfrak{F}:\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ in bicomplex parametric partial metric space $(\mathfrak{G},\partial)$, such that*

$$\partial\{\mathfrak{F}(\varrho,\upsilon),\mathfrak{F}(\eta,\kappa),t\}\precsim_{i_2}\mathcal{N}\mho_1(\varrho,\upsilon,\eta,\kappa)+\mathcal{K}\mho_2(\varrho,\upsilon,\eta,\kappa),$$

where $\mho_1(\varrho,\upsilon,\eta,\kappa)=\varpi\left(\dfrac{(\mho+\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))\partial(\eta,\mathfrak{F}(\eta,\kappa),t)}{\mho+\partial(\varrho,\eta,t)}\right)$, *and*

$$\begin{aligned}\mho_2(\varrho,\upsilon,\eta,\kappa)=\max\Bigg[&\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)),\varpi(\partial(\eta,\mathfrak{F}(\varrho,\upsilon),t)),\varpi(\partial(\eta,\mathfrak{F}(\varrho,\upsilon),t)),\\&\varpi(\partial(\mathfrak{F}(\upsilon,\varrho),\kappa,t)),\varpi(\partial(\varrho,\eta,t)),\varpi(\partial(\upsilon,\kappa,t)),\\&\min\Bigg\{\varpi(\partial(\kappa,\mathfrak{F}(\upsilon,\varrho),t)),\varpi(\partial(\mathfrak{F}(\eta,\kappa),\eta,t)),\\&\frac{\varpi(\partial(\eta,\mathfrak{F}(\varrho,\upsilon),t))\varpi(\partial(\mathfrak{F}(\eta,\kappa),\varrho,t))}{1+\varpi(\partial(\varrho,\eta,t))+\varpi(\partial(\upsilon,\kappa,t))}\Bigg\}\Bigg],\end{aligned}$$

for all $\varrho,\upsilon,\eta,\kappa\in\mathfrak{G}$ with $(\mathcal{N}+\mathcal{K})<1$. Then $\mathfrak{F}$ allows unique coupled invariant point.

Proof. Choose $(\varrho_0,\upsilon_0)\in\mathfrak{G}\times\mathfrak{G}$ be an arbitrary. Construct sequences $\{\varrho_n\}$ and $\{\upsilon_n\}$ in $\mathfrak{G}$ as follows: $\mathfrak{F}(\varrho_n,\upsilon_n)=\varrho_{n+1}$, $\mathfrak{F}(\upsilon_n,\varrho_n)=\upsilon_{n+1}$, $\mathfrak{F}(\varrho_{n+1},\upsilon_{n+1})=\varrho_{n+2}$, $\mathfrak{F}(\upsilon_{n+1},\varrho_{n+1})=\upsilon_{n+2}$,and also let $\partial(\varrho_n\varrho_{n+1},t)=\mathcal{Y}_n$ and $\partial(\upsilon_n\upsilon_{n+1},t)=\mathcal{S}_n$, for ≥ 0,

$$\partial(\varrho_n,\varrho_{n+1},t)=\partial\{(\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1},t),\mathfrak{F}(\varrho_n\upsilon_n,t))\}$$

$$\precsim_{i_2} \mathcal{N}\mho_1(\varrho_{n-1},\upsilon_{n-1},\varrho_n,\upsilon_n)+\mathcal{K}\mho_2(\varrho_{n-1},\upsilon_{n-1},\varrho_n,\upsilon_n).$$

Now

$$\begin{aligned}
\mho_1(\varrho_{n-1},\upsilon_{n-1},\varrho_n,\upsilon_n) &= \varpi\left(\frac{\mathcal{O}+\partial(\varrho_{n-1},\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1}),t)\partial(\varrho_n,\mathfrak{F}(\varrho_n,\upsilon_n),t)}{\mathcal{O}+\partial(\varrho_{n-1},\varrho_n\, t)}\right)\\
&= \varpi\left(\frac{(\mathcal{O}+\partial(\varrho_{n-1},\varrho_n,t))\partial(\varrho_n,\varrho_{n+1},t)}{\mathcal{O}+\partial(\varrho_{n-1},\varrho_n\, t)}\right)\\
&= \varpi\partial(\varrho_n,\varrho_{n+1},t)\\
&= \varpi(\mathcal{Y}_n)
\end{aligned}$$

and

$$\begin{aligned}
\mho_2(\varrho_{n-1},\upsilon_{n-1},\varrho_n,\upsilon_n) &= \max\Big[\varpi(\partial(\varrho_{n-1},\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1}),t)),\varpi(\partial(\varrho_n,\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1}),t)),\\
&\qquad \varpi(\partial(\varrho_n,\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1}),t)),\varpi(\partial(\mathfrak{F}(\upsilon_{n-1},\varrho_{n-1}),\upsilon_n,t)),\\
&\qquad \varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t)),\\
&\qquad \min\Big\{\varpi(\partial(\upsilon_n,\mathfrak{F}(\upsilon_{n-1},\varrho_{n-1}),t)),\varpi(\partial(\mathfrak{F}(\varrho_n,\upsilon_n),\varrho_n,t)),\\
&\qquad \frac{\varpi(\partial(\varrho_n,\mathfrak{F}(\varrho_{n-1},\upsilon_{n-1}),t))\varpi(\partial(\mathfrak{F}(\varrho_n,\upsilon_n),\varrho_{n-1},t))}{1+\varpi(\partial(\varrho_{n-1},\varrho_n,t))+\varpi(\partial(\upsilon_{n-1},\upsilon_n,t))}\Big\}\Big]\\
&= \max\Big[\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\varrho_n,\varrho_n,t)),\varpi(\partial(\varrho_n,\varrho_n,t)),\\
&\qquad \varpi(\partial(\upsilon_n,\upsilon_n,t)),\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t)),\\
&\qquad \min\Big\{\varpi(\partial(\upsilon_n,\upsilon_n,t)),\varpi(\partial(\varrho_{n+1},\varrho_n,t)),\\
&\qquad \frac{\varpi(\partial(\varrho_n,\varrho_n,t)).\varpi(\partial(\varrho_{n+1},\varrho_{n-1},t))}{1+\varpi(\partial(\varrho_{n-1},\varrho_n,t))+\varpi(\partial(\upsilon_{n-1},\upsilon_n,t))}\Big\}\Big]\\
&= \max\Big[\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(0),\varpi(0),\\
&\qquad \varpi(0),\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t)),\\
&\qquad \min\Big\{\varpi(\partial(\upsilon_n,\upsilon_n,t)),\varpi(\partial(\varrho_{n+1},\varrho_n,t)),\\
&\qquad \frac{\varpi(0)\varpi(\partial(\varrho_{n+1},\varrho_{n-1},t))}{1+\varpi(\partial(\varrho_{n-1},\varrho_n,t))+\varpi(\partial(\upsilon_{n-1},\upsilon_n,t))}\Big\}\\
&= \max\Big[\varpi(\partial(\varrho_{n-1},\varrho_n,t)),0,0,0,\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t)),\\
&\qquad \min\Big\{\varpi(\partial(\upsilon_n,\upsilon_n,t)),\varpi(\partial(\varrho_{n+1},\varrho_n,t)),\\
&\qquad \frac{0.\varpi(\partial(\varrho_{n+1},\varrho_{n-1},t))}{1+\varpi(\partial(\varrho_{n-1},\varrho_n,t))+\varpi(\partial(\upsilon_{n-1},\upsilon_n,t))}\Big\}\Big]\\
&= \max[\varpi(\partial(\varrho_{n-1},\varrho_n,t)),0,0,0,\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t)),0]
\end{aligned}$$

$$= \max[\varpi(\partial(\varrho_{n-1},\varrho_n,t)),\varpi(\partial(\upsilon_{n-1},\upsilon_n,t))]$$
$$= \max[\varpi(\mathcal{Y}_{n-1}),\varpi(\mathcal{S}_{n-1})].$$

Therefore,

$$\|\partial(\varrho_n,\varrho_{n+1},t)\| \precsim_{i_2} \mathcal{N}\|\varpi(\mathcal{Y}_n)\| + \mathcal{K}\|\max[\varpi(\mathcal{Y}_{n-1}),\varpi(\mathcal{S}_{n-1})]\|,$$
$$\|\mathcal{Y}_n\| \precsim_{i_2} \mathcal{N}\|\varpi(\mathcal{Y}_n)\| + \mathcal{K}\psi_1. \tag{3.1}$$

In the same manner we obtain,

$$\partial(\upsilon_n,\upsilon_{n+1},t) = \partial\{(\mathfrak{F}(\upsilon_{n-1},\varrho_{n-1},t),\mathfrak{F}(\upsilon_n,\varrho_n,t)\}$$
$$\precsim_{i_2} \mathcal{N}\mho_1(\upsilon_{n-1},\varrho_{n-1},\upsilon_n,\varrho_n) + \mathcal{K}\mho_2(\upsilon_{n-1},\varrho_{n-1},\upsilon_n,\varrho_n),$$

where

$$\mho_1(\upsilon_{n-1},\varrho_{n-1},\upsilon_n,\varrho_n) = \varpi(\mathcal{S}_n)$$

and

$$\mho_2(\upsilon_{n-1},\varrho_{n-1},\upsilon_n,\varrho_n) = \max[\varpi(\mathcal{S}_{n-1}),\varpi(\mathcal{Y}_{n-1})].$$

Therefore,

$$\|\partial(\upsilon_n,\upsilon_{n+1},t)\| \precsim_{i_2} \mathcal{N}\|\varpi(\mathcal{S}_n)\| + \mathcal{K}\|\max[\varpi(\mathcal{S}_{n-1}),\varpi(\mathcal{Y}_{n-1})]\|,$$
$$\|\mathcal{S}_n\| \precsim_{i_2} \mathcal{N}\|\varpi(\mathcal{S}_n)\| + \mathcal{K}\psi_2.. \tag{3.2}$$

Put

$$\Phi_n = \|\partial(\varrho_n,\varrho_{n+1},t)\| + \|\partial(\upsilon_n,\upsilon_{n+1},t)\| = \|\mathcal{Y}_n\| + \|\mathcal{S}_n\|,$$
$$\psi_1 = \|\max[\varpi(\mathcal{Y}_{n-1}),\varpi(\mathcal{S}_{n-1})]\|,$$
$$\psi_2 = \|\max[\varpi(\mathcal{S}_{n-1}),\varpi(\mathcal{Y}_{n-1})]\|.$$

Only following cases may possible:

Case (i): If $\psi_1 = \|\varpi(\mathcal{Y}_{n-1})\|$ and $\psi_2 = \|\varpi(\mathcal{Y}_{n-1})\|$.
Then from equation (3.1) and (3.2),

$$\Phi_n = \|\mathcal{Y}_n\| + \|\mathcal{S}_n\| \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{Y}_{n-1})\| + \|\varpi(\mathcal{Y}_{n-1})\|),$$
$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{Y}_{n-1})\| + \|\varpi(\mathcal{Y}_{n-1})\|).$$

By using definition of dominate function $\varpi(\mathfrak{z}) < \mathfrak{z}$,for all $\mathfrak{z} > 0$, we get

$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\mathcal{Y}_n\| + \|\mathcal{S}_n\|) + \mathcal{K}(\|\mathcal{Y}_{n-1}\| + \|\mathcal{Y}_{n-1}\|);$$
$$\Phi_n \precsim_{i_2} \mathcal{N}\Phi_n + 2\mathcal{K}\mathcal{Y}_{n-1};$$
$$(1-\mathcal{N})\Phi_n \precsim_{i_2} 2\mathcal{K}\mathcal{Y}_{n-1};$$
$$\Phi_n \precsim_{i_2} \frac{2\mathcal{K}}{(1-\mathcal{N})}\mathcal{Y}_{n-1}. \tag{3.3}$$

Case (ii): If $\psi_1 = \|\varpi(\mathcal{S}_{n-1})\|$and $\psi_2 = \|\varpi(\mathcal{S}_{n-1})\|$.
Then from equation (3.1) and (3.2),

$$\Phi_n = \|\mathcal{Y}_n\| + \|\mathcal{S}_n\| \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{S}_{n-1})\| + \|\varpi(\mathcal{S}_{n-1})\|);$$

$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{S}_{n-1})\| + \|\varpi(\mathcal{S}_{n-1})\|).$$

By using definition of dominate function $\varpi(\mathfrak{z}) < z$, for all $\mathfrak{z} > 0$, we get

$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\mathcal{Y}_n\| + \|\mathcal{S}_n\|) + \mathcal{K}(\|\mathcal{S}_{n-1}\| + \|\mathcal{S}_{n-1}\|);$$

$$\Phi_n \precsim_{i_2} \mathcal{N}\Phi_n + 2\mathcal{K}\mathcal{S}_{n-1};$$

$$(1-\mathcal{N})\Phi_n \precsim_{i_2} 2\mathcal{K}\mathcal{S}_{n-1};$$

$$\Phi_n \precsim_{i_2} \frac{2\mathcal{K}}{(1-\mathcal{N})}\mathcal{S}_{n-1}. \tag{3.4}$$

Now adding (3.3) and (3.4),

$$2\Phi_n \precsim_{i_2} \frac{2\mathcal{K}}{(1-\mathcal{N})}(\mathcal{Y}_{n-1} + \mathcal{S}_{n-1});$$

$$\Phi_n \precsim_{i_2} \frac{\mathcal{K}}{(1-\mathcal{N})}\Phi_{n-1};$$

$$\Phi_n \precsim_{i_2} \vartheta\Phi_{n-1}, \quad \text{say } \vartheta = \frac{\mathcal{K}}{(1-\mathcal{N})} < 1.$$

Case (iii): If $\psi_1 = \|\varpi(\mathcal{Y}_{n-1})\|$ and $\psi_2 = \|\varpi(\mathcal{S}_{n-1})\|$.
Then from equations (3.1) and (3.2),

$$\Phi_n = \|\mathcal{Y}_n\| + \|\mathcal{S}_n\| \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{Y}_{n-1})\| + \|\varpi(\mathcal{S}_{n-1})\|),$$

$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\varpi(\mathcal{Y}_n)\| + \|\varpi(\mathcal{S}_n)\|) + \mathcal{K}(\|\varpi(\mathcal{Y}_{n-1})\| + \|\varpi(\mathcal{S}_{n-1})\|).$$

By using definition of dominate function $\varpi(\mathfrak{z}) < \mathfrak{z}$, for all $\mathfrak{z} > 0$, we get

$$\Phi_n \precsim_{i_2} \mathcal{N}(\|\mathcal{Y}_n\| + \|\mathcal{S}_n\|) + \mathcal{K}(\|\mathcal{Y}_{n-1}\| + \|\mathcal{S}_{n-1}\|);$$

$$\Phi_n \precsim_{i_2} \mathcal{N}\Phi_n + \mathcal{K}\Phi_{n-1};$$

$$(1-\mathcal{N})\Phi_n \precsim_{i_2} \mathcal{K}\Phi_{n-1};$$

$$\Phi_n \precsim_{i_2} \frac{\mathcal{K}}{(1-\mathcal{N})}\Phi_{n-1},$$

$$\Phi_n \precsim_{i_2} \vartheta\Phi_{n-1}. \tag{3.5}$$

Case (iv): If $\psi_1 = \|\varpi(\mathcal{S}_{n-1})\|$ and$\psi_2 = \|\varpi(\mathcal{Y}_{n-1})\|$.
Then, we get same result of *Case* (iii).
From the observation of all cases, for each $n \in \mathcal{N}$, we have

$$\Phi_n \precsim_{i_2} \vartheta\Phi_{n-1} \precsim_{i_2} \vartheta^2\Phi_{n-2} \precsim_{i_2} \cdots \precsim_{i_2} \vartheta^n\Phi_0,$$

where $\Phi_0 = 0$, then $\|\partial(\varrho_0,\varrho_1,t)\| + \|\partial(\upsilon_0,\upsilon_1,t)\| = 0$. Hence, we get $\varrho_0 = \varrho_1 = \mathfrak{F}(\varrho_0,\upsilon_0)$ and $\upsilon_0 = \upsilon_1 = \mathfrak{F}(\upsilon_0,\varrho_0)$, which shows that (ϱ_0,υ_0) is coupled invariant point of $\mathfrak{F}$. Now we assume that $\Phi_0 > 0$, for each $n \geq p$, where $n,p \in \mathcal{N}$,

$$\begin{aligned}\|\partial(\varrho_n,\varrho_p,t)\| \precsim_{i_2} & \|\partial(\varrho_n,\varrho_{n-1},t)\| + \|\partial(\varrho_{n-1},\varrho_{n-2},t)\| + \cdots + \|\partial(\varrho_{p+1},\varrho_p,t)\| - \|\partial(\varrho_{n-1},\varrho_{n-1},t)\| \\ & - \|\partial(\varrho_{n-2},\varrho_{n-2},t)\| - \cdots - \|\partial(\varrho_{p+1},\varrho_{p+1},t)\| \\ \precsim_{i_2} & \|\partial(\varrho_n,\varrho_{n-1},t)\| + \|\partial(\varrho_{n-1},\varrho_{n-2},t)\| + \cdots + \|\partial(\varrho_{p+1},\varrho_p,t)\|.\end{aligned}$$

Similarly,

$$\|\partial(\upsilon_n,\upsilon_p,t)\| \precsim_{i_2} \|\partial(\upsilon_n,\upsilon_{n-1},t)\| + \|\partial(\upsilon_{n-1},\upsilon_{n-2},t)\| + \cdots + \|\partial(\upsilon_{p+1},\upsilon_p,t)\|.$$

Thus,

$$\begin{aligned}\Phi_n &= \|\partial(\varrho_n,\varrho_p,t)\| + \|\partial(\upsilon_n,\upsilon_p,t)\| \\ &\precsim_{i_2} \Phi_{n-1}+\Phi_{n-2}+\Phi_{n-3}+\cdots+\Phi_p \\ &\precsim_{i_2} (\vartheta^{n-1}+\vartheta^{n-2}+\vartheta^{n-3}+\cdots+\vartheta^p)\Phi_0 \\ &\precsim_{i_2} \left(\frac{\vartheta^p}{1-\vartheta}\right)\Phi_0 \\ &\precsim_{i_2} \left(\frac{\vartheta^n}{1-\vartheta}\right)\Phi_0 \to 0 \ \text{ as } n\to+\infty.\end{aligned}$$

Since $0 \le \vartheta < 1$ so that $\{\varrho_n\}$ and $\{\upsilon_n\}$ are Cauchy sequences in. Since the *BCPPMS* $(\mathfrak{G},\partial)$ is complete, so there exist $\varrho,\upsilon \in \mathfrak{G}$ such that $\{\varrho_n\}\to\varrho$ and $\{\upsilon_n\}\to\upsilon$ as $n\to\infty$ and

$$\partial(\varrho,\varrho,t) = \lim_{n\to\infty}\partial(\varrho,\varrho_n,t) = \lim_{m,n\to\infty}\partial(\varrho_n,\varrho_m,t) = 0$$

and

$$\partial(\upsilon,\upsilon,t) = \lim_{n\to\infty}\partial(\upsilon,\upsilon_n,t) = \lim_{m,n\to\infty}\partial(\upsilon_n,\upsilon_m,t) = 0.$$

Now to show that $\varrho = \mathfrak{F}(\varrho,\upsilon)$ and $\upsilon = \mathfrak{F}(\upsilon,\varrho)$.
Suppose $\varrho \ne \mathfrak{F}(\varrho,\upsilon)$ and $\upsilon \ne \mathfrak{F}(\upsilon,\varrho)$ so that

$$0 < (\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)) = j_1 \ \text{ and } \ 0 < (\partial(\upsilon,\mathfrak{F}(\upsilon,\varrho),t)) = j_2$$

then

$$\begin{aligned}j_1 &= (\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)) \\ &\precsim_{i_2} (\partial(\varrho,\varrho_{n+1},t)) + (\partial(\varrho_{n+1},\mathfrak{F}(\varrho,\upsilon),t)) - (\partial(\varrho_{n+1},\varrho_{n+1},t)) \\ &= (\partial(\varrho,\varrho_{n+1},t)) + (\partial(\varrho_{n+1},\mathfrak{F}(\varrho,\upsilon),t)) \\ &= (\partial(\varrho_{n+1},\mathfrak{F}(\varrho,\upsilon),t)) + (\partial(\varrho,\varrho_{n+1},t)) \\ &= (\partial(\mathfrak{F}(\varrho_n\upsilon_n),\mathfrak{F}(\varrho,\upsilon),t)) + (\partial(\varrho,\varrho_{n+1},t)) \\ &\precsim_{i_2} \mathcal{N}\mho_1(\varrho_n,\upsilon_n,\varrho,\upsilon) + \mathcal{K}\mho_2(\varrho_n,\upsilon_n,\varrho,\upsilon) + (\partial(\varrho,\varrho_{n+1},t)),\end{aligned} \tag{3.6}$$

where $\mho_1(\varrho_n,\upsilon_n,\varrho,\upsilon) = \varpi\left(\dfrac{\mho + \partial(\varrho_n\mathfrak{F}(\varrho_n,\upsilon_n),t)\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)}{\mho + \partial(\varrho_n,\varrho,t)}\right)$.

Letting limit as $n\to\infty$ and use dominate property, then, we get

$$\mho_1(\varrho_n,\upsilon_n,\varrho,\upsilon) \to \varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))$$

and

$$\begin{aligned}\mho_2(\varrho_n,\upsilon_n,\varrho,\upsilon) = \max\Big[&\varpi(\partial(\varrho_n,\mathfrak{F}(\varrho_n,\upsilon_n),t)), \varpi(\partial(\varrho,\mathfrak{F}(\varrho_n,\upsilon_n),t)), \varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)), \\ &\varpi(\partial(\mathfrak{F}(\upsilon_n,\varrho_n),\upsilon,t)), \varpi(\partial(\varrho_n,\varrho,t)), \varpi(\partial(\upsilon_n,\upsilon,t)), \\ &\min\Big\{\varpi(\partial(\upsilon,\mathfrak{F}(\upsilon_n,\varrho_n),t)), \varpi(\partial(\mathfrak{F}(\varrho,\upsilon),\varrho,t)),\end{aligned}$$

$$\left.\left.\frac{\varpi(\partial(\varrho,\mathfrak{F}(\varrho_n,\upsilon_n),t))\varpi(\partial(\mathfrak{F}(\varrho,\upsilon),\varrho_n,t))}{1+\varpi(\partial(\varrho_n,\varrho,t))+\varpi(\partial(\upsilon_n,\upsilon,t))}\right\}\right]$$

$$\begin{aligned}=\max\Big[&\varpi(\partial(\varrho_n,\varrho_{n+1},t)),\varpi(\partial(\varrho,\varrho_{n+1},t)),\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)),\\&\varpi(\partial(\upsilon_{n+1},\upsilon,t)),\varpi(\partial(\varrho_n,\varrho,t)),\varpi(\partial(\upsilon_n,\upsilon,t)),\\&\min\Big\{\varpi(\partial(\upsilon,\upsilon_{n+1},t)),\varpi(\partial(\mathfrak{F}(\varrho,\upsilon),\varrho,t)),\\&\frac{\varpi(\partial(\varrho,\varrho_{n+1},t))\varpi(\partial(\mathfrak{F}(\varrho,\upsilon),\varrho_n,t))}{1+\varpi(\partial(\varrho_n,\varrho,t))+\varpi(\partial(\upsilon_n,\upsilon,t))}\Big\}\Big].\end{aligned}$$

Letting limit as $n\to\infty$. Then, we get

$$\mho_2(\varrho_n,\upsilon_n,\varrho,\upsilon)\to\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)).$$

From equation (3.6),

$$j_1\precsim_{i_2}\mathcal{N}\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))+\mathcal{K}\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))+(\partial(\varrho,\varrho_{n+1},t)).$$

Letting limit as $n\to\infty$ and use dominate property,

$$\begin{aligned}\|j_1\|&\precsim_{i_2}(\mathcal{N}+\mathcal{K})\|\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))\|\\&\precsim_{i_2}(\mathcal{N}+\mathcal{K})\|\varpi(j_1)\|\\&\precsim_{i_2}(\mathcal{N}+\mathcal{K})\|j_1\|.\end{aligned}\tag{3.7}$$

Similarly, one can find

$$\begin{aligned}\|j_2\|&\precsim_{i_2}(\mathcal{N}+\mathcal{K})\|\varpi(j_2)\|\\&\precsim_{i_2}(\mathcal{N}+\mathcal{K})\|j_2\|.\end{aligned}\tag{3.8}$$

By adding equation (3.7) and (3.8), then we obtain

$$\|j_1\|+\|j_2\|\precsim_{i_2}(\mathcal{N}+\mathcal{K})(\|j_1\|+\|j_2\|),$$

which is contradiction since $(\mathcal{N}+\mathcal{K})<1$. Thus

$$\begin{aligned}\|j_1\|+\|j_2\|=0&\Longrightarrow\|(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))\|+\|(\partial(\upsilon,\mathfrak{F}(\upsilon,\varrho),t))\|=0\\&\Longrightarrow\|(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t))\|=0\text{ and }\|(\partial(\upsilon,\mathfrak{F}(\upsilon,\varrho),t))\|=0.\end{aligned}$$

Therefore, $\varrho=\mathfrak{F}(\varrho,\upsilon)$ and $\upsilon=\mathfrak{F}(\upsilon,\varrho)$.
Hence $\mathfrak{F}$ has coupled invariant point.

Uniqueness: Suppose (ϱ_1,υ_2) be another coupled invariant fixed point of $\mathfrak{F}$ such that$(\varrho,\upsilon)\neq(\varrho_1,\upsilon_2)$.
Then

$$\begin{aligned}\partial(\varrho,\varrho_1,t)&=(\partial(\mathfrak{F}(\varrho,\upsilon),\mathfrak{F}(\varrho_1,\upsilon_2),t))\\&\precsim_{i_2}\mathcal{N}\mho_1(\varrho,\upsilon,\varrho_1,\upsilon_1)+\mathcal{K}\mho_2(\varrho,\upsilon,\varrho_1,\upsilon_1),\end{aligned}$$

where $\mho_1(\varrho,\upsilon,\varrho_1,\upsilon_1)=\varpi\left(\dfrac{\mathcal{O}+\partial(\varrho,\varrho_1,t)\partial(\varrho_1,\varrho_1,t)}{\mathcal{O}+\partial(\varrho,\varrho_1 t)}\right)$.

Letting limit as $n \to \infty$ and use dominate property, then $\mho_1(\varrho,\upsilon,\varrho_1,\upsilon_1) \to 0$, and

$$\begin{aligned}\mho_2(\varrho,\upsilon,\varrho_1,\upsilon_1) = \max\Big[&\varpi(\partial(\varrho,\mathfrak{F}(\varrho,\upsilon),t)),\varpi(\partial(\varrho_1,\mathfrak{F}(\varrho,\upsilon),t)),\varpi(\partial(\varrho_1,\mathfrak{F}(\varrho,\upsilon),t)),\\ &\varpi(\partial(\mathfrak{F}(\upsilon,\varrho),\upsilon_1,t)),\varpi(\partial(\varrho,\varrho_1,t)),\varpi(\partial(\upsilon,\upsilon_1,t)),\\ &\min\Big\{\varpi(\partial(\upsilon_1,\mathfrak{F}(\upsilon,\varrho),t)),\varpi(\partial(\mathfrak{F}(\varrho_1,\upsilon_1),\varrho_1,t)),\\ &\frac{\varpi(\partial(\varrho_1,\mathfrak{F}(\varrho,\upsilon),t))\varpi(\partial(\mathfrak{F}(\varrho_1,\upsilon_1),\varrho,t))}{1+\varpi(\partial(\varrho,\varrho_1,t))+\varpi(\partial(\upsilon,\upsilon_1,t))}\Big\}\Big]\\ = \max\Big[&\varpi(\partial(\varrho,\varrho,t)),\varpi(\partial(\varrho_1,\varrho,t)),\varpi(\partial(\varrho_1,\varrho,t)),\\ &\varpi(\partial(\upsilon,\upsilon_1,t)),\varpi(\partial(\varrho,\varrho_1,t)),\varpi(\partial(\upsilon,\upsilon_1,t)),\\ &\min\Big\{\varpi(\partial(\upsilon_1,\upsilon,t)),\varpi(\partial(\varrho_1,\varrho_1,t)),\\ &\frac{\varpi(\partial(\varrho_1,\varrho,t)).\varpi(\partial(\varrho,\varrho,t))}{1+\varpi(\partial(\varrho,\varrho_1,t))+\varpi(\partial(\upsilon,\upsilon_1,t))}\Big\}\Big].\end{aligned}$$

Letting limit as $n \to \infty$ and use dominate property, then

$$\mho_2(\varrho,\upsilon,\varrho_1,\upsilon_1) \to \max\{\partial(\varrho,\varrho_1,t),\partial(\upsilon,\upsilon_1,t),0\}.$$

Therefore,

$$\|\partial(\varrho,\varrho_1,t)\| \lesssim_{i_2} \mathcal{K}\|\max\{\partial(\varrho,\varrho_1,t),\partial(\upsilon,\upsilon_1,t),0\}\| \Longrightarrow \|A\| \lesssim_{i_2} \mathcal{K}\theta_1.$$

Similarly,

$$\|\partial(\upsilon,\upsilon_1,t)\| \lesssim_{i_2} \mathcal{K}\|\max\{\partial(\upsilon,\upsilon_1,t),\partial(\varrho,\varrho_1,t),0\}\| \Longrightarrow \|B\| \lesssim_{i_2} \mathcal{K}\theta_2.$$

Put

$$C = \|\partial(\varrho,\varrho_1,t)\| + \|\partial(\upsilon,\upsilon_1,t)\| = \|A\| + \|B\|,$$

$$\theta_1 = \|\max\{A,B,0\}\|, \quad \theta_2 = \|\max\{B,A,0\}\|.$$

Following cases may possible:

Case (a): If $\theta_1 = \|\max\{A,B,0\}\| = A$ and $\theta_2 = \|\max\{B,A,0\}\| = A$.
Then $C = \|A\| + \|B\| \lesssim_{i_2} \mathcal{K}(\theta_1+\theta_2)$,

$$C \lesssim_{i_2} 2A\mathcal{K}.$$

Case (b): If $\theta_1 = \|\max\{A,B,0\}\| = B$ and $\theta_2 = \|\max\{B,A,0\}\| = B$.
Then $C = \|A\| + \|B\| \lesssim_{i_2} \mathcal{K}(\theta_1+\theta_2)$,

$$C \lesssim_{i_2} 2B\mathcal{K}.$$

Adding *Case* (a) and *Case* (b), then

$$2C \lesssim_{i_2} 2\mathcal{K}(A+B) \Longrightarrow C \lesssim_{i_2} \mathcal{K}(A+B) \Longrightarrow C \lesssim_{i_2} C\mathcal{K},$$

which is contradiction since $(A+B) < 1$.
Hence, $C = 0$, i.e.,

$$\|\partial(\varrho,\varrho_1,t)\| + \|\partial(\upsilon,\upsilon_1,t)\| = 0$$

$$\implies \|\partial(\varrho,\varrho_1,t)\| = 0 \text{ and } \|\partial(\upsilon,\upsilon_1,t)\| = 0$$
$$\implies \varrho = \varrho_1 \text{ and } \upsilon = \upsilon_1.$$

Case (c): If $\theta_1 = \|\max\{A,B,0\}\| = A$ and $\theta_2 = \|\max\{B,A,0\}\| = B$.
Then $C = \|A\| + \|B\| \precsim_{i_2} \mathcal{K}(\theta_1+\theta_2)$, $C \precsim_{i_2} \mathcal{K}(A+B) \Longrightarrow C \precsim_{i_2} C\mathcal{K}$.
Then, we get same result of *Case* (b).

Case (d): If $\theta_1 = \|\max\{A,B,0\}\| = B$ and $\theta_2 = \|\max\{B,A,0\}\| = A$.
Then we get same result of Case (b).

Thus, from all the above cases, we obtain $\varrho = \varrho_1$ and $\upsilon = \upsilon_1$. Therefore, (ϱ,υ) is unique common coupled invariant point. □

Example 3.2. Define $\mathfrak{F} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ by $\partial(\varrho,\upsilon,t) = (1+i_2)\max\{\varrho,\upsilon,t\}$, for $\varrho,\upsilon \in \mathfrak{G}$ and $\mathfrak{G} = [0,1]$ with partial order $\precsim_{i_2}$. Clearly, $(\mathfrak{G},\varrho)$ is a complete bicomplex parametric partial metric space. Consider a mapping $\mathfrak{F} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ define by $\partial(\varrho,\eta,t) = \begin{cases} \frac{\varrho+\upsilon}{4}, & \varrho \geq \upsilon, \\ 0, & \varrho < \upsilon, \end{cases}$ $\varrho,\upsilon,\eta,\kappa \in \mathfrak{G}$ such that $\varrho \leq \eta$, $\upsilon \geq \kappa$.

Following cases may arise for verify our theorem.

Case (i): Let $\varrho \geq \upsilon$ but $\varrho \leq \eta$, $\upsilon \geq \kappa$ then combine $\eta \geq \varrho \geq \upsilon \geq \kappa$. Now

$$\begin{aligned}
\partial\{\mathfrak{F}(\varrho,\upsilon),\mathfrak{F}(\eta,\kappa),t\} &= (1+i_2)\max\left\{\frac{\varrho+\upsilon}{4},\frac{\eta+\kappa}{4}\right\} \\
&= (1+i_2)\frac{\varrho+\eta}{4} \\
&= \frac{1}{4}[(1+i_2)\max(\varrho,\upsilon)+(1+i_2)\max(\eta,\kappa)] \\
&= \frac{1}{4}[\partial(\varrho,\upsilon,t)+\partial(\eta,\kappa,t)] \\
&\precsim_{i_2} \frac{1}{4}\,\mho_1(\varrho,\upsilon,\eta,\kappa) \\
&\precsim_{i_2} \mathcal{N}\mho_1(\varrho,\upsilon,\eta,\kappa)+\mathcal{K}\mho_2(\varrho,\upsilon,\eta,\kappa).
\end{aligned}$$

Case (ii): When $\varrho < \upsilon$ but $\upsilon \geq \kappa$ then $\varrho > \kappa$,

$$\begin{aligned}
\partial\{\mathfrak{F}(\varrho,\upsilon),\mathfrak{F}(\eta,\kappa),t\} &= (1+i_2)\max\left\{0,\frac{\eta+\kappa}{4}\right\} \\
&= (1+i_2)\frac{\eta+\kappa}{4} \\
&= \frac{1}{4}[(1+i_2)\max(\varrho,\upsilon)+(1+i_2)\max(\eta,\kappa)] \\
&= \frac{1}{4}[\partial(\varrho,\upsilon,t)+\partial(\eta,\kappa,t)] \\
&\precsim_{i_2} \frac{1}{4}\,\mho_1(\varrho,\upsilon,\eta,\kappa) \\
&\precsim_{i_2} \mathcal{N}\mho_1(\varrho,\upsilon,\eta,\kappa)+\mathcal{K}\mho_2(\varrho,\upsilon,\eta,\kappa).
\end{aligned}$$

Case (iii): When $\varrho \geq \upsilon$ and $\eta < \kappa$ the n does not arise, because $\varrho \leq \eta$, $\upsilon \geq \kappa$.

Case (iv): When $\varrho < \upsilon$ and $\eta < \kappa$ then

$$\partial\{\mathfrak{F}(\varrho,\upsilon),\mathfrak{F}(\eta,\kappa),t\} = 0 \Longrightarrow \mathfrak{F}(\varrho,\upsilon) = 0,\ \mathfrak{F}(\eta,\kappa) = 0.$$

From above cases validated out theorem with $\mathcal{N} = \frac{1}{4}$, for any values of $\mathcal{K} \geq 0$.
Therefore (ϱ,υ) is unique common coupled invariant point.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, *Numerical Functional Analysis and Optimization* **32**(3) (2011), 243 – 253, DOI: 10.1080/01630563.2011.533046.

[2] I. Beg, S.K. Datta and P. Dipankar, Fixed point in bicomplex valued metric spaces, *International Journal of Nonlinear Analysis and Applications* **12**(2) (2021), 717 – 727, DOI: 10.22075/ijnaa.2019.19003.2049.

[3] A. Bharathi, M.V.R. Kameswari and B. Padmavathi, Coupled fixed points in bicomplex partial metric space and an application, *Eur. Shem. Bull.* **12**(10) (2023), 5510 – 5520.

[4] T.G. Bhaskar and V. Lakshmikantam, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods & Applications* **65**(7) (2006), 1379 – 1393, DOI: 10.1016/j.na.2005.10.017.

[5] S. Bhatt, S. Chaukiyal and R.C. Dimiri, Common fixed point of satisfying rational inequality in complex valued metric spaces, *International Journal of Pure and Applied Mathematics* **73**(2) (2011), 159 – 164.

[6] J. Choi, S.K. Datta, T. Biswas and N. Islam, Some fixed point theorems in connection with two weakly compatible mapping in bicomplex valued metric spaces, *Honam Mathematical Journal* **30**(1) (2017), 115 – 126, DOI: 10.5831/HMJ.2017.39.1.115.

[7] H.A. Hammad and M. De la Sen, Analytical solution of Urysohn integral equations by fixed point technique in complex valued metric spaces, *Mathematics* **7**(9) (2019), 852, DOI: 10.3390/math7090852.

[8] I.H. Jebrial, R. Datta, S.K. Sarkar and N. Biswas, Common fixed point theorems under rational contractions using two mappings and six mappings and coupled fixed point theorem in bicomplex valued b-metric space, *TWMS Journal of Applied and Engineering Mathematics* **11**(3) (2011), 804 – 813.

[9] I.H. Jebrial, S.K. Datta, R. Sarkar and N. Biswas, Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, *Journal of Interdisciplinary Mathematics* **22**(7) (2019), 1071 – 1082, DOI: 10.1080/09720502.2019.1709318.

[10] F. Kamalov and H.H. Leung, *Fixed point theory: A review*, arXiv:2309.03226, DOI: 10.48550/arXiv.2309.03226.

[11] G. Mani, A.J. Gnanaprakasam, K. Javed, M. Arshad and F. Jarad, Solving a Fredholm integral equation via coupled fixed point on bicomplex partial metric space, *AIMS Mathematics* **7**(8) (2022), 15402 – 15416, DOI: 10.3934/math.2022843.

[12] A.A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, *The Scientific Word Journal* **2014** (2014), Article ID 587825, 6 pages, DOI: 10.1155/2014/587825.

[13] H.K. Nashine, Coupled common fixed point results in ordered G-metric spaces, *Journal of Nonlinear Science and Applications* **1** (2012), 1 – 13.

[14] H.K. Nashine, J.K. Kim, A.K. Sharma and G.S. Saluja, Some coupled fixed point without mixed monotone mapping, *Nonlinear Functional Analysis and Applications* **21**(2) (2016), 235 – 247.

[15] H.K. Pandey and S.K. Tiwari, A common fixed point theorem in complex valued metric spaces, *Turkish Journal of Computer and Mathematics Education* **12**(14) (2021), 3150 – 3155.

[16] S. Poornavel, N. Balamurguan and H.A. Hammad, Some common fixed point theorems on tricomplex valued parametric metric space, *Annals of Mathematics and Computer Science* **20** (2024), 120 – 134, DOI: 10.56947/amcs.v20.248.

[17] R. Ramaswamy, M. Gunaseelan M., A.R. Gnanaprakasam, O.A.A. Abdelnaby and S. Radenović, An application to fixed-point results in tricomplex-valued metric spaces using control functions, *Mathematics* **10**(18) (2022), 3344, DOI: 10.3390/math10183344.

[18] G.S. Saluja, Coupled fixed point results based on control function in complex partial metric spaces, *Functional Analysis, Approximation and Computation* **15**(2) (2023), 1 – 15.

[19] B. Samet and H. Yazidi, Coupled fixed point theorems for contraction involving rational expressions in partially ordered metric spaces, *arXiv:1005.3142* [math GN], DOI: 10.48550/arXiv.1005.3142.

[20] D. Venkatesh and V.N. Raju, Some fixed point results in bicomplex valued metric spaces, *Mathematics and Statistics* **10**(6) (2022), 1334 – 1339, DOI: 10.13189/ms.2022.100620.