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Research Article

Decomposition of Bipolar Pythagorean Fuzzy Matrices

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Abstract. This paper presents novel findings on modal operators through the use of max-min composition, analyzing properties such as reflexivity, symmetry, transitivity, and idempotency related to necessity and possibility. It explores the necessary and sufficient conditions for transitive and c-transitive closure matrices using modal operators. Additionally, a new composition operator, labeled as '$\wedge_m$' is introduced and its algebraic properties are thoroughly discussed. The study also achieves a decomposition of a BPyFM utilizing the new composition operator and modal operators.

Keywords. Intuitionistic fuzzy matrix, Bipolar Pythagorean fuzzy matrix, Modal operator

Mathematics Subject Classification (2020). 03E72, 15B15, 15B99

1. Introduction

Over the years, theories have been developed to address a wide range of uncertainty. These updated theories are implemented and improved as inadequacies are identified, allowing for the formation of fresh frameworks to solve complicated uncertainty. Probability theory stands as a pivotal theory in examining stochastic phenomena. In 1965, the concept of the Fuzzy set was introduced by Zadeh [16], representing an extension of the classical notion of set. Fuzzy sets have proven to be a valuable instrument in addressing ambiguity. The *Intuitionistic Fuzzy Matrix* (IFM) extends the fuzzy matrix proposed by Thomason [14] and has proven beneficial in various domains including decision-making, relational equations, and clustering analysis. Consequently, a new concept known as the intuitionistic fuzzy set was developed, developed by Atanassov [1]. Hashimoto [6] has examined the problem of decomposing fuzzy rectangular matrices and has demonstrated certain properties associated with this decomposition. Fuzzy matrix theory was developed by Kim and Roush [8] as an extension of Boolean matrix theory.

The structures of intuitionistic fuzzy relations were studied by Bustince and Burillo [3]. The intuitionistic fuzzy matrix was defined by Pal *et al*. [13]. The discussion on the period of power of square intuitionistic fuzzy matrices is extensively covered, including various results concerning equivalence IFMs, as examined by Jeong and Park [7]. Generalized intuitionistic fuzzy matrices were studied by Bhowmik and Pal [2]. An attempt was made by Mondal and Pal [10] to study the similarity relations, invertibility, and eigenvalues of intuitionistic fuzzy matrices. Lee and Jeong [9] investigated how a transitive intuitionistic fuzzy matrix can be decomposed into a sum of nilpotent and symmetric IFM. The decomposition process of an intuitionistic fuzzy matrix into a product consisting of idempotent IFM and rectangular IFM was explored by Murugadas and Lalitha [11]. A promising research direction was opened when Atanassov [1] introduced modal operators, previously considered meaningless in fuzzy set theory.

Bipolar Intuitionistic Fuzzy Sets (BIFSs) were introduced by Ezhilmaran and Sankar [5], who elucidated their operations and defined bipolar intuitionistic fuzzy relations. The *Bipolar Pythagorean Fuzzy Matrix* (BPyFM) and some of its novel operations were defined by Chinnadurai *et al*. [4] using bipolar pythagorean fuzzy set theory. A decomposition of an intuitionistic fuzzy matrix was obtained by Muthuraji *et al*. [12] using the new decomposition operator and modal operator. The necessary and sufficient conditions for a transitive and c-transitive closure matrix are examined in terms of modal operators. Moreover, further results are investigated utilizing modal operators for BPyFM under max-min composition, accompanied by discussions on similarity relations and idempotency. Ultimately, a decomposition of a BPyFM is accomplished using modal operators through the introduction of a new composition operator, with some properties of this new operator substantiated.

2. Preliminaries

Some fundamental definitions and results are recalled for subsequent utilization.

Definition 2.1 ([13])**.** An *Intuitionistic Fuzzy Matrix* (IFM) is a matrix of ordered pair $X = (\langle x^{\mu}_{ij}, x^{\gamma}_{ij}\rangle)$ of non-negative real number satisfying $0 \le x^{\mu}_{ij} + x^{\gamma}_{ij} \le 1$, for all i, j.

Definition 2.2 ([13])**.** An IFM $J = (\langle 1,0\rangle)$ for all entries is known as the universal matrix and an IFM $O = (\langle 0,1\rangle)$ for all entries is known as zero matrix. Denote the set of all IFMs of order $m \times n$ by F_{mn} and square matrix of order n by F_n. The identity IFM $I = (\langle \delta^{\mu}_{ij}, \delta^{\gamma}_{ij}\rangle)$ is defined by $\langle \delta^{\mu}_{ij}, \delta^{\gamma}_{ij}\rangle = \langle 1,0\rangle$ if $i = j$ and $\langle \delta^{\mu}_{ij}, \delta^{\gamma}_{ij}\rangle = \langle 0,1\rangle$ if $i \ne j$.

Definition 2.3 ([1])**.** Let $\langle x^{\mu}, x^{\gamma}\rangle, \langle y^{\mu}, y^{\gamma}\rangle \in$ IFS. Then

$$\langle x^{\mu}, x^{\gamma}\rangle \leftarrow \langle y^{\mu}, y^{\gamma}\rangle = \begin{cases} \langle 1,0\rangle, & \text{if } \langle x^{\mu}, x^{\gamma}\rangle \ge \langle y^{\mu}, y^{\gamma}\rangle, \\ \langle x^{\mu}, x^{\gamma}\rangle, & \text{if } \langle x^{\mu}, x^{\gamma}\rangle \le \langle y^{\mu}, y^{\gamma}\rangle. \end{cases}$$

Definition 2.4 ([12])**.** Let $X = (\langle x^{\mu}_{ij}, x^{\gamma}_{ij}\rangle)_{m\times n}$ and $Y = (\langle y^{\mu}_{ij}, y^{\gamma}_{ij}\rangle)_{n\times p}$ are IFMs. Then

$$XY = \left(\left\langle \sum_{k=1}^{n} (x^{\mu}_{ik} y^{\mu}_{kj}), \prod_{k=1}^{n} (x^{\gamma}_{ik} + y^{\gamma}_{kj}) \right\rangle\right).$$

Also, $X^2 = XX$, $X^k = X^{k-1}X$ for max-min composition and $X^{[2]} = XX$, $X^k = X^{k-1}X$ for min-max composition.

Definition 2.5 ([12])**.** For any IFM $X \in F_n$,

(i) X is reflexive if and only if $X \geq I_n$,

(ii) X is symmetric if and only if $X = X^T$,

(iii) X is transitive if and only if $X \geq X^2$,

(iv) X is idempotent if and only if $X = X^2$,

(v) X is irreflexive if $\langle x^{\mu}_{ii}, x^{\gamma}_{ii}\rangle = \langle 0,1\rangle$, for all $i = j$,

(vi) X is c-transitive if $X \leq X^{[2]}$.

Definition 2.6 ([15])**.** An IFM X is said to be an intuitionistic fuzzy equivalence matrix if it satisfy reflexive, symmetry and transitivity.

Definition 2.7 ([12])**.** For an IFM X, $\Box X = \langle x^{\mu}_{ij}, 1 - x^{\mu}_{ij}\rangle$ and $\Diamond X = \langle 1 - x^{\gamma}_{ij}, x^{\gamma}_{ij}\rangle$.

Definition 2.8 ([12])**.** Let $X \in$ IFM, the transitive closure and c-transitive closure of X is defined by $X^{\infty} = X \vee X^2 \vee X^3 \vee \ldots \vee X^n$ and $X_{\infty} = X^c \wedge (X^c)^{[2]} \wedge (X^c)^{[3]} \wedge \ldots \wedge (X^c)^{[n]}$.

Definition 2.9 ([12])**.** For any two element $\langle x^{\mu}, x^{\gamma}\rangle, \langle y^{\mu}, y^{\gamma}\rangle \in$ IFS we introduce the operation '$\wedge_m$' as $\langle x^{\mu}, x^{\gamma}\rangle \wedge_m \langle y^{\mu}, y^{\gamma}\rangle = \langle \min(x^{\mu}, y^{\mu}), \min(x^{\gamma}, y^{\gamma})\rangle$.

Lemma 2.10 ([12])**.** $1 - \prod_{k=1}^{n}(x^{\mu}_{ik} + y^{\mu}_{kj}) = \sum_{k=1}^{n}(1 - x^{\mu}_{ik})(1 - y^{\mu}_{kj})$, *for all* $i, j, x^{\mu}_{ik}, y^{\mu}_{kj} \in [0,1]$.

Lemma 2.11 ([12])**.** $1 - \sum_{k=1}^{n} x^{\mu}_{ik} y^{\mu}_{kj} = \prod_{k=1}^{n}(1 - x^{\mu}_{ik}) + (1 - y^{\mu}_{kj})$, *for all* $i, j, x^{\mu}_{ik}, y^{\mu}_{kj} \in [0,1]$.

Lemma 2.12 ([12])**.** *If X^{∞} is the transitive closure of X, then the transitive closure of $\Box X$ is $\Box X^{\infty}$.*

Proof. For $X = (\langle x^{\mu}_{ij}, x^{\gamma}_{ij}\rangle), Y = (\langle y^{\mu}_{ij}, y^{\gamma}_{ij}\rangle)$,

$$XY = \left(\left\langle \sum_{k=1}^{n}(x^{\mu}_{ik} y^{\mu}_{kj}), \prod_{k=1}^{n}(x^{\gamma}_{ik} + y^{\gamma}_{kj})\right\rangle\right),$$

$$\left(\left\langle \sum_{k=1}^{n}(x^{\mu}_{ik} y^{\mu}_{kj}), \prod_{k=1}^{n}(x^{\gamma}_{ik} + y^{\gamma}_{kj})\right\rangle\right) = \begin{cases} \langle 1,0\rangle, & \text{if } i = j, \\ \langle y^{\mu}, y^{\gamma}\rangle, & \text{if } i \neq j. \end{cases}$$

Thus $XY = Y \Rightarrow XX \leq XY = Y$, that is, $X^2 \leq Y$. Continuing in this way, we have $X^3 \leq Y$, $X^4 \leq Y, \ldots$ and also $X \vee X^2 \vee X^3 \vee \ldots \vee X^n \leq Y$ and hence $X^{\infty} \leq Y$. □

Lemma 2.13 ([12])**.** *For an* IFM X, *the following inequalities are true*

(i) $\Box X_{\infty} = (\Box X)_{\infty}$,

(ii) $\Diamond X^{\infty} = (\Diamond X)^{\infty}$,

(iii) $\Diamond X_{\infty} = (\Diamond X)_{\infty}$.

Definition 2.14 ([4])**.** A *Bipolar Pythagorean Fuzzy Matrix* (BPyFM) of size $r \times s$ is $A = [(x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp})]$, where $x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp} \in [-1,1]$ are positive and negative membership values of the element $x^{\mu}_{ij}, x^{\gamma}_{ij}$,

$$0 \le (x^{\mu}_{ijp})^2 + (x^{\gamma}_{ijp})^2 \le 1, \quad -1 \le -[(x^{\mu}_{ijn})^2 + (x^{\gamma}_{ijn})^2] \le 0.$$

Definition 2.15 ([4])**.** Let A and B are two BPyFMs of same size, then we write $A \ge B$, if $(x^{\mu}_{ijp}) \ge (y^{\mu}_{ijp})$, $(x^{\gamma}_{ijp}) \le (y^{\gamma}_{ijp})$, $(x^{\mu}_{ijn}) \ge (y^{\mu}_{ijn})$ and $(x^{\gamma}_{ijn}) \le (y^{\gamma}_{ijn})$, for all i,j.

3. More Properties of Modal Operators in Bipolar Pythagorean Fuzzy Matrix

New results on modal operators under max-min composition are explored in this section. Additionally, properties such as reflexivity, symmetry, transitivity, and idempotency of necessity and possibility are discussed.

Definition 3.1. Let X be a BPyFM. Then,

(i) the *necessity operation* of X is defined as

$$\Box X = [\langle x^{\mu}_{ijn}, \sqrt{1-(x^{\mu}_{ijn})^2}\rangle, \langle x^{\mu}_{ijp}, \sqrt{1-(x^{\mu}_{ijp})^2}\rangle],$$

(ii) the *possibility operation* of X is defined as

$$\Diamond X = [\langle \sqrt{1-(x^{\gamma}_{ijn})^2}, x^{\gamma}_{ijn}\rangle, \langle \sqrt{1-(x^{\gamma}_{ijp})^2}, x^{\gamma}_{ijp}\rangle].$$

Definition 3.2. Let $X = \langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle$, $Y = \langle y^{\mu}_{ijn}, y^{\mu}_{ijp}, y^{\gamma}_{ijn}, y^{\gamma}_{ijp}\rangle \in$ BPyFM. Then,

$$X \leftarrow Y = \begin{cases} \langle -1,1,0,0\rangle, & \text{if } X \ge Y, \\ \langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle, & \text{if } X \le Y. \end{cases}$$

Here, $X \ge Y$, $x^{\mu}_{ijn} \ge y^{\mu}_{ijn}$, $x^{\mu}_{ijp} \ge y^{\mu}_{ijp}$, $x^{\gamma}_{ijn} \le y^{\gamma}_{ijn}$, $x^{\gamma}_{ijp} \le y^{\gamma}_{ijp}$.

Theorem 3.3. *Let X and Y be two BPyFMs, then*

$$\Box(X \leftarrow Y) = \Box X \leftarrow \Box Y. \tag{3.1}$$

Proof. Case (i): If $X \ge Y$, then

$$\Box(\langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle \leftarrow \langle y^{\mu}_{ijn}, y^{\mu}_{ijp}, y^{\gamma}_{ijn}, y^{\gamma}_{ijp}\rangle) = \Box\langle -1,1,0,0\rangle = \langle -1,1,0,0\rangle. \tag{3.2}$$

Since, $X \ge Y$, $x^{\mu}_{ijn} \ge y^{\mu}_{ijn}$, $x^{\mu}_{ijp} \ge y^{\mu}_{ijp}$, $x^{\gamma}_{ijn} \le y^{\gamma}_{ijn}$ and $x^{\gamma}_{ijp} \le y^{\gamma}_{ijp}$.
Therefore,

$$\sqrt{1-(x^{\mu}_{ijn})^2} \le \sqrt{1-(y^{\mu}_{ijn})^2},$$

$$\sqrt{1-(x^{\mu}_{ijp})^2} \le \sqrt{1-(y^{\mu}_{ijp})^2},$$

$$(x^{\mu}_{ijn}, x^{\mu}_{ijp}, \sqrt{1-(x^{\mu}_{ijn})^2}, \sqrt{1-(x^{\mu}_{ijp})^2}) \ge (y^{\mu}_{ijn}, y^{\mu}_{ijp}, \sqrt{1-(y^{\mu}_{ijn})^2}, \sqrt{1-(y^{\mu}_{ijp})^2}).$$

Implies

$$\Box(x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}) \ge \Box(y^{\mu}_{ijn}, y^{\mu}_{ijp}, y^{\gamma}_{ijn}, y^{\gamma}_{ijp}).$$

Thus

$$\Box(\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle)\leftarrow\Box(\langle y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}\rangle)=\langle -1,1,0,0\rangle. \tag{3.3}$$

From equation (3.2) and equation (3.3), equation (3.1) holds.

Case (ii): If $X\le Y$, then

$$\begin{aligned}\Box(\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle &\leftarrow \langle y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}\rangle)\\ &=\Box\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle\\ &=(x^{\mu}_{ijn},x^{\mu}_{ijp},\sqrt{1-(x^{\mu}_{ijn})^2},\sqrt{1-(x^{\mu}_{ijp})^2}),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Box(x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp})&\leftarrow\Box(y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp})\\ &=(x^{\mu}_{ijn},x^{\mu}_{ijp},\sqrt{1-(x^{\mu}_{ijn})^2},\sqrt{1-(x^{\mu}_{ijp})^2})\leftarrow(y^{\mu}_{ijn},y^{\mu}_{ijp},\sqrt{1-(y^{\mu}_{ijn})^2},\sqrt{1-(y^{\mu}_{ijp})^2})\\ &=(x^{\mu}_{ijn},x^{\mu}_{ijp},\sqrt{1-(x^{\mu}_{ijn})^2},\sqrt{1-(x^{\mu}_{ijp})^2}).\end{aligned} \tag{3.5}$$

From equation (3.4) and equation (3.5), equation (3.1) holds. □

Theorem 3.4. *Let X and Y be two BPyFMs, then*

$$\Diamond(X\leftarrow Y)=\Diamond X\leftarrow\Diamond Y. \tag{3.6}$$

Proof. Case (i): If $X\ge Y$, then

$$\Diamond(\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle\leftarrow\langle y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}\rangle)=\Diamond\langle -1,1,0,0\rangle=\langle -1,1,0,0\rangle. \tag{3.7}$$

Since, $X\ge Y$, $x^{\mu}_{ijn}\ge y^{\mu}_{ijn}$, $x^{\mu}_{ijp}\ge y^{\mu}_{ijp}$, $x^{\gamma}_{ijn}\le y^{\gamma}_{ijn}$ and $x^{\gamma}_{ijp}\le y^{\gamma}_{ijp}$.
Therefore,

$$\sqrt{1-(x^{\gamma}_{ijn})^2}\ge\sqrt{1-(y^{\gamma}_{ijn})^2},\quad \sqrt{1-(x^{\gamma}_{ijp})^2}\ge\sqrt{1-(y^{\gamma}_{ijp})^2},$$

$$(\sqrt{1-(x^{\gamma}_{ijn})^2},\sqrt{1-(x^{\gamma}_{ijp})^2},x^{\gamma}_{ijn},x^{\gamma}_{ijp})\ge(\sqrt{1-(y^{\gamma}_{ijn})^2},\sqrt{1-(y^{\gamma}_{ijp})^2},y^{\gamma}_{ijn},y^{\gamma}_{ijp}).$$

So,

$$\Diamond(x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp})\ge\Diamond(y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}).$$

Thus

$$\Diamond(\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle)\leftarrow\Diamond(\langle y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}\rangle)=\langle -1,1,0,0\rangle. \tag{3.8}$$

From equation (3.7) and equation (3.8), equation (3.6) holds.

Case (ii): If $X\le Y$, then

$$\begin{aligned}\Diamond(\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle\leftarrow\langle y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp}\rangle)&=\Diamond\langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle\\ &=(\sqrt{1-(x^{\gamma}_{ijn})^2},\sqrt{1-(x^{\gamma}_{ijp})^2},x^{\gamma}_{ijn},x^{\gamma}_{ijp}),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Diamond(x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp})&\leftarrow\Diamond(y^{\mu}_{ijn},y^{\mu}_{ijp},y^{\gamma}_{ijn},y^{\gamma}_{ijp})\\ &=(\sqrt{1-(x^{\gamma}_{ijn})^2},\sqrt{1-(x^{\gamma}_{ijp})^2},x^{\gamma}_{ijn},x^{\gamma}_{ijp})\leftarrow(\sqrt{1-(y^{\gamma}_{ijn})^2},\sqrt{1-(y^{\gamma}_{ijp})^2},y^{\gamma}_{ijn},y^{\gamma}_{ijp})\\ &=(\sqrt{1-(x^{\gamma}_{ijn})^2},\sqrt{1-(x^{\gamma}_{ijp})^2},x^{\gamma}_{ijn},x^{\gamma}_{ijp}).\end{aligned} \tag{3.10}$$

From equation (3.9) and equation (3.10), equation (3.6) holds. □

Theorem 3.5. *X is reflexive matrix if and only if $\Box X$ is reflexive matrix.*

Proof. X is reflexive matrix

$$\Leftrightarrow X \geq I$$
$$\Leftrightarrow \langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle \geq \langle \delta^{\mu}_{ijn}, \delta^{\mu}_{ijp}, \delta^{\gamma}_{ijn}, \delta^{\gamma}_{ijp}\rangle, \text{ for all } i,j$$
$$\Leftrightarrow (x^{\mu}_{ijn}, x^{\mu}_{ijp}, \sqrt{1-(x^{\mu}_{ijn})^2}, \sqrt{1-(x^{\mu}_{ijp})^2}) \geq (\delta^{\mu}_{ijn}, \delta^{\mu}_{ijp}, \sqrt{1-(\delta^{\mu}_{ijn})^2}, \sqrt{1-(\delta^{\mu}_{ijp})^2})$$
$$\Leftrightarrow \Box X \geq \Box I \Leftrightarrow \Box X \text{ is reflexive.}$$ □

The proof of the following theorem is obvious from Theorem 3.5.

Theorem 3.6. *X is reflexive matrix if and only if $\Diamond X$ is reflexive matrix.*

Theorem 3.7. *X is reflexive if and only if $\Box X^c$ is irreflexive.*

Proof. It is obvious that if X is reflexive if and only if X^c is irreflexive and so $\Box X^c$ is irreflexive. Similarly, $\Diamond X^c$ is irreflexive if and only if X is reflexive. □

Theorem 3.8. *X is symmetric matrix if and only if $\Box X$ is symmetric matrix.*

Proof. X is symmetric

$$\Leftrightarrow X = X^T$$
$$\Leftrightarrow \langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle = \langle x^{\mu}_{jin}, x^{\mu}_{jip}, x^{\gamma}_{jin}, x^{\gamma}_{jip}\rangle$$
$$\Leftrightarrow (x^{\mu}_{ijn}, x^{\mu}_{ijp}, \sqrt{1-(x^{\mu}_{ijn})^2}\sqrt{1-(x^{\mu}_{ijp})^2}) = (x^{\mu}_{jin}, x^{\mu}_{jip}, \sqrt{1-(x^{\mu}_{jin})^2}\sqrt{1-(x^{\mu}_{jip})^2})$$
$$\Leftrightarrow \Box X = (\Box X)^T.$$

Thus X is symmetric matrix if and only if $\Box X$ is symmetric matrix. □

The proof of the following theorem is obvious from Theorem 3.8.

Theorem 3.9. *X is symmetric matrix if and only if $\Diamond X$ is symmetric matrix.*

Theorem 3.10. *X is transitive matrix if and only if $\Box X$ is transitive matrix.*

Proof. X is transitive

$$\Leftrightarrow X \geq X^2$$
$$\Leftrightarrow \langle x^{\mu}_{ijn}, x^{\mu}_{ijp}, x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle = \Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}), \sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}),$$
$$\prod_{k=1}^{n}(\sqrt{(x^{\gamma}_{ikn})^2+(y^{\gamma}_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^{\gamma}_{ikp})^2+(y^{\gamma}_{kjp})^2})\Bigg), \text{ for all } i,j$$
$$\langle x^{\mu}_{ijn}, x^{\mu}_{ijp}\rangle \geq \Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}), \sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp})\Bigg),$$
$$\langle x^{\gamma}_{ijn}, x^{\gamma}_{ijp}\rangle \leq \Bigg(\prod_{k=1}^{n}(\sqrt{(x^{\gamma}_{ikn})^2+(y^{\gamma}_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^{\gamma}_{ikp})^2+(y^{\gamma}_{kjp})^2})\Bigg)$$
$$\Leftrightarrow (x^{\mu}_{ijn}, x^{\mu}_{ijn}, \sqrt{1-(x^{\mu}_{ijn})^2}, \sqrt{1-(x^{\mu}_{ijp})^2}) \geq \Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}), \sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}), \sqrt{1-\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn})^2},$$

$$\sqrt{1-\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp})^2}\Bigg)$$

$$=\Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}),\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}),$$

$$\prod_{k=1}^{n}(\sqrt{1-(x^{\mu}_{ikn})^2}+\sqrt{1-(x^{\mu}_{kjn})^2}),$$

$$\prod_{k=1}^{n}(\sqrt{1-(x^{\mu}_{ikp})^2}+\sqrt{1-(x^{\mu}_{kjp})^2})\Bigg)\text{ (by Lemma 2.11)}$$

$\Leftrightarrow \Box X \geq \Box X^2$.

Thus X is transitive matrix if and only if $\Box X$ is transitive matrix. □

The proof of the following theorem is obvious from Theorem 3.10.

Theorem 3.11. *X is transitive matrix if and only if $\Diamond X$ is transitive matrix.*

Theorem 3.12. *X is idempotent matrix if and only if $\Box X$ is idempotent matrix.*

Proof. X is idempotent

$\Leftrightarrow X = X^2$

$$\Leftrightarrow \langle x^{\mu}_{ijn},x^{\mu}_{ijp},x^{\gamma}_{ijn},x^{\gamma}_{ijp}\rangle=\Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}),\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}),$$

$$\prod_{k=1}^{n}(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}),\prod_{k=1}^{n}(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2})\Bigg)\quad\text{for all } i,j$$

$$\Leftrightarrow (x^{\mu}_{ijn},x^{\mu}_{ijp},\sqrt{1-(x^{\mu}_{ijn})^2},\sqrt{1-(x^{\mu}_{ijp})^2})=\Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}),\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}),$$

$$\sqrt{1-\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn})^2},\sqrt{1-\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp})^2}\Bigg)$$

$$=\Bigg(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}),\sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}),$$

$$\prod_{k=1}^{n}(\sqrt{1-(x^{\mu}_{ikn})^2}+\sqrt{1-(x^{\mu}_{kjn})^2}),$$

$$\prod_{k=1}^{n}(\sqrt{1-(x^{\mu}_{ikp})^2}+\sqrt{1-(x^{\mu}_{kjp})^2})\Bigg)\text{ (by Lemma 2.11)}$$

$\Leftrightarrow \Box X = \Box X^2$.

Thus X is idempotent matrix if and only if $\Box X$ is idempotent matrix. □

The proof of the following theorem is obvious from Theorem 3.12.

Theorem 3.13. *X is idempotent matrix if and only if $\Diamond X$ is idempotent matrix.*

Remark 3.14. If X is a bipolar Pythagorean fuzzy equivalence matrix then $\Box X$ and $\Diamond X$ are also bipolar Pythagorean fuzzy equivalence matrix.

4. Transitive Closure and c-Transitive Closures on Bipolar Pythagorean Fuzzy Matrix

In this section, an investigation is underway to explore the necessary and sufficient conditions for transitive closure and c-transitive closure matrices, utilizing modal operators.

Theorem 4.1. *Let X be a BPyFM, $X_\infty = (X^\infty)^c$.*

Proof. By Definition 2.9,

$$\begin{aligned}(X^\infty)^c &= (X \vee X^2 \vee X^3 \vee \ldots \vee X^n)^c \\ &= X^c \wedge (X^2)^c \wedge (X^3)^c \wedge \ldots \wedge (X^n)^c.\end{aligned}$$

Let us prove,

$$(X^2)^c = (X^c)^{[2]},$$

$$X^2 = \left(\sum_{k=1}^{n}(x^\mu_{ikn}y^\mu_{kjn}), \sum_{k=1}^{n}(x^\mu_{ikp}y^\mu_{kjp}), \prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikn})^2+(y^\gamma_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikp})^2+(y^\gamma_{kjp})^2})\right),$$

$$(X^2)^c = \left(\prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikn})^2+(y^\gamma_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikp})^2+(y^\gamma_{kjp})^2}), \sum_{k=1}^{n}(x^\mu_{ikn}y^\mu_{kjn}), \sum_{k=1}^{n}(x^\mu_{ikp}y^\mu_{kjp})\right). \quad (4.1)$$

Also, $X^c = \langle x^\gamma_{ikn}, x^\gamma_{ikp}, x^\mu_{ikn}, x^\mu_{ikp}\rangle$ gives by the definition of $X^{[2]}$,

$$(X^c)^{[2]} = \left(\prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikn})^2+(y^\gamma_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^\gamma_{ikp})^2+(y^\gamma_{kjp})^2}), \sum_{k=1}^{n}(x^\mu_{ikn}y^\mu_{kjn}), \sum_{k=1}^{n}(x^\mu_{ikp}y^\mu_{kjp})\right). \quad (4.2)$$

Thus by equation (4.1) and equation (4.2), $(X^2)^c = (X^c)^{[2]}$, so in general $(X^n)^c = (X^c)^{[n]}$.
By the definition,

$$\begin{aligned}(X^\infty)^c &= (X \vee X^2 \vee X^3 \vee \ldots \vee X^n)^c \\ &= X^c \wedge (X^2)^c \wedge (X^3)^c \wedge \ldots (X^n)^c \\ &= X^c \wedge (X^c)^{[2]} \wedge (X^c)^{[3]} \wedge \ldots \wedge (X^c)^{[n]} \\ &= X_\infty.\end{aligned}$$

□

The proof of the following lemma is obvious from the definition of transitive and c-transitive.

Theorem 4.2. *X is transitive if and only if X^c is c-transitive and so $\Box X^c$ is c-transitive.*

Theorem 4.3. *If X is reflexive, then*

(i) *X^T is reflexive,*

(ii) *$X \vee Y$ is reflexive,*

(iii) *$X \wedge Y$ is reflexive if and only if Y is reflexive.*

Proof. (i) and (ii) are obvious from the definition of reflexive.

(iii): If Y is not reflexive, then

$$\langle y^\mu_{kjn}, y^\mu_{kjp}, y^\gamma_{kjn}, y^\gamma_{kjp}\rangle \neq \langle -1,1,0,0\rangle,$$

for at least one i, that is

$$\langle y^\mu_{kjn}, y^\mu_{kjp}, y^\gamma_{kjn}, y^\gamma_{kjp}\rangle < \langle -1,1,0,0\rangle.$$

Thus

$$\langle x^{\mu}_{kjn}, x^{\mu}_{kjp}, x^{\gamma}_{kjn}, x^{\gamma}_{kjp}\rangle \wedge \langle y^{\mu}_{kjn}, y^{\mu}_{kjp}, y^{\gamma}_{kjn}, y^{\gamma}_{kjp}\rangle < \langle -1,1,0,0\rangle.$$

Therefore, Y is reflexive is necessary, the sufficient part is trivial. □

Theorem 4.4. *If X and Y be two BPyFMs, where X is reflexive and symmetric, Y is reflexive, symmetric and transitive and $X \le Y$, then $X^{\infty} \le Y$.*

Proof. For $X = \langle x^{\mu}_{kjn}, x^{\mu}_{kjp}, x^{\gamma}_{kjn}, x^{\gamma}_{kjp}\rangle$, $Y = \langle y^{\mu}_{kjn}, y^{\mu}_{kjp}, y^{\gamma}_{kjn}, y^{\gamma}_{kjp}\rangle$,

$$XY = \left(\sum_{k=1}^{n}(x^{\mu}_{ikn}y^{\mu}_{kjn}), \sum_{k=1}^{n}(x^{\mu}_{ikp}y^{\mu}_{kjp}), \prod_{k=1}^{n}\left(\sqrt{(x^{\gamma}_{ikn})^2+(y^{\gamma}_{kjn})^2}\right), \prod_{k=1}^{n}\left(\sqrt{(x^{\gamma}_{ikp})^2+(y^{\gamma}_{kjp})^2}\right)\right)$$
$$= \begin{cases} \langle -1,1,0,0\rangle, & \text{if } i=j,\\ \langle y^{\mu}_{kjn}, y^{\mu}_{kjp}, y^{\gamma}_{kjn}, y^{\gamma}_{kjp}\rangle, & \text{if } i\neq j \end{cases}$$

Thus $XY = Y \Rightarrow XX \le XY = Y$ that is $X^2 \le Y$. Continuing in this way, we have $X^3 \le Y$, $X^4 \le Y \ldots$ and also $X \vee X^2 \vee X^3 \vee \ldots \vee X^n \le Y$ and hence $X^{\infty} \le Y$. □

The proof of the following theorem is obvious from Lemma 2.11.

Theorem 4.5. *If X^{∞} is the transitive closure of X, then the transitive closure of $\Box X$ is $\Box X^{\infty}$.*

Theorem 4.6. *Let X be a BPyFM, $[(\Box X)^c]^{\infty} = [(\Box X)_{\infty}]^c$.*

Proof. As we know $(\Box X)^c = \Diamond X^c$,

$$[(\Box X)^c]^{\infty} = [\Diamond X^c]^{\infty} = \Diamond X^c \vee (\Diamond X^c)^2 \vee (\Diamond X^c)^3 \vee \ldots \vee (\Diamond X^c)^n$$

$$(\Diamond X^c)^2 = \left(\sum_{k=1}^{n}\left(\sqrt{1-(x^{\mu}_{ikn})^2}\right)\left(\sqrt{1-(x^{\mu}_{kjn})^2}\right), \sum_{k=1}^{n}\left(\sqrt{1-(x^{\mu}_{ikp})^2}\right)\left(\sqrt{1-(x^{\mu}_{kjp})^2}\right),\right.$$
$$\left.\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), \prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right)\right) \quad \text{(by Lemma 2.10)}$$
$$= \left(1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), 1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right),\right.$$
$$\left.\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), \prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right)\right). \tag{4.3}$$

By definition

$$X^{[2]} = \left(\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), \prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right), \sum_{k=1}^{n}(x^{\mu}_{ikn}x^{\mu}_{kjn}), \sum_{k=1}^{n}(x^{\mu}_{ikp}x^{\mu}_{kjp})\right)$$

and so

$$\Box X^{[2]} = \left(\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), \prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right),\right.$$
$$\left.1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), 1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right)\right)$$

which yields

$$(\Box X^{[2]})^c = \left(1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}\right), 1-\prod_{k=1}^{n}\left(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2}\right),\right.$$

$$\prod_{k=1}^{n}(\sqrt{(x^{\mu}_{ikn})^2+(x^{\mu}_{kjn})^2}), \prod_{k=1}^{n}(\sqrt{(x^{\mu}_{ikp})^2+(x^{\mu}_{kjp})^2})\Big). \tag{4.4}$$

From equation (4.3) and equation (4.4), we get, therefore $(\lozenge X^c)^2 = (\Box X^{[2]})^c$, so in general $(\lozenge X^c)^n = (\Box X^{[n]})^c$,

$$\begin{aligned}[(\Box X)^c]^{\infty} &= [\lozenge X^c]^{\infty}\\ &= \lozenge X^c \vee (\lozenge X^c)^2 \vee (\lozenge X^c)^3 \vee \ldots \vee (\lozenge X^c)^n\\ &= (\Box X)^c \vee (\Box X^{[2]})^c \vee (\Box X^{[3]})^c \vee \ldots \vee (\Box X^{[n]})^c\\ &= (\Box X \wedge \Box X^{[2]} \wedge \Box X^{[3]} \wedge \ldots \wedge \Box X^{[n]})^c\\ &= (\Box X_{\infty})^c.\end{aligned}$$ □

The proof of the following theorem is obvious from Theorem 4.7.

Theorem 4.7. *Let X be a BPyFM,* $[(\lozenge X)_{\infty}]^c = [(\lozenge X)^c]^{\infty}$.

5. Decomposition of BPyFM Using a New Composition Operator in Terms of Modal Operators

In this section, a new composition operator denoted by '$\wedge_m$' is introduced, with a discussion of its algebraic properties. Finally, a decomposition of a BPyFM is achieved using the new composition operator and modal operators.

Definition 5.1. For any two element $\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle, \langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle \in BPyFSs$ we introduce the operation '$\wedge_m$' as

$$\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle = \langle \min(x^{\mu}_n, y^{\mu}_n), \min(x^{\mu}_p, y^{\mu}_p), \min(x^{\gamma}_n, y^{\gamma}_n), \min(x^{\gamma}_p, y^{\gamma}_p)\rangle.$$

The proof of the following theorem is obvious from the definition.

Theorem 5.2. *The operation* $\wedge_m$ *is commutative on BPyFSs.*

Theorem 5.3. *The operation* $\wedge_m$ *is associative on BPyFSs.*

Proof. Consider any three elements on BPyFSs as $\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle$, $\langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle$, $\langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle$. To prove $\wedge_m$ is associative it is enough to prove:

$$\begin{aligned}&\{\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle\} \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle\\ &\quad = \langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \{\langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle\}.\end{aligned}$$

Case (1): If $\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \geq \langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle$ and $\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \geq \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle$,

$$\{\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle\} \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle = \langle y^{\mu}_n, y^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle.$$

Subcase (1.1): If $\langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle \leq \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle$ then

$$\langle y^{\mu}_n, y^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle = \langle y^{\mu}_n, y^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle.$$

Also,

$$\begin{aligned}\langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \{\langle y^{\mu}_n, y^{\mu}_p, y^{\gamma}_n, y^{\gamma}_p\rangle \wedge_m \langle z^{\mu}_n, z^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle\} &= \langle x^{\mu}_n, x^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle \wedge_m \langle y^{\mu}_n, y^{\mu}_p, z^{\gamma}_n, z^{\gamma}_p\rangle\\ &= \langle y^{\mu}_n, y^{\mu}_p, x^{\gamma}_n, x^{\gamma}_p\rangle.\end{aligned}$$

Subcase (1.2): If $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$ then

$$\langle y_n^\mu, y_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle.$$

Also,

$$\begin{aligned}\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\} &= \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle \\ &= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle.\end{aligned}$$

In this case $\wedge_m$ is associative.

Case (2): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$,

$$\{\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\} \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle x_n^\mu, x_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle.$$

Subcase (2.1): If $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$ then

$$\begin{aligned}&\{\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\} \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\} = \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle.\end{aligned}$$

Subcase (2.2): If $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$ then

$$\begin{aligned}&\{\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\} \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\} = \langle x_n^\mu, x_p^\mu, y_n^\gamma, y_p^\gamma\rangle.\end{aligned}$$

In this case also $\wedge_m$ is associative.

Case (3): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, we have

$$\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle.$$

Now

$$\begin{aligned}&\{\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\} \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\} = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle.\end{aligned}$$

Case (4) If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, we have

$$\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$$

and

$$\begin{aligned}&\{\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\} \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\} = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle.\end{aligned}$$

Hence $\wedge_m$ is associative. □

Theorem 5.4. (i) *The operation $\wedge_m$ is right distributive over addition in BPyFSs.*

(ii) *The operation $\wedge_m$ is left distributive over addition in BPyFSs.*

Proof. (i): For any $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle, \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle, \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \in$ BPyFS,

$$\begin{aligned}&\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle \max(x_n^\mu, y_n^\mu), \max(x_p^\mu, y_p^\mu), \min(x_n^\gamma, y_n^\gamma), \min(x_p^\gamma, y_p^\gamma)\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \\ &\quad = \langle \min\{\max(x_n^\mu, y_n^\mu), z_n^\mu\}, \min\{\max(x_p^\mu, y_p^\mu), z_p^\mu\}, \min\{\min(x_n^\gamma, y_n^\gamma), z_n^\gamma\}, \min\{\min(x_p^\gamma, y_p^\gamma), z_p^\gamma\}\rangle.\end{aligned} \tag{5.1}$$

Case (1): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.
Then, the right-hand side of equation (5.1) is $\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.
Now consider

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \begin{cases}\langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle, & \text{if } \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle,\\ \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle, & \text{if } \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle,\end{cases}\\ &\quad= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle \end{aligned} \tag{5.2}$$

In this case, it is distributive.

Case (2): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.
Then, the left-hand side of equation (5.1) reduce to $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.

Subcase (2.1): If $\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$, then

$$\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \wedge_m \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle.$$

Now

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad= \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle.\end{aligned}$$

Thus distributive holds.

Subcase (2.2): If $\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$.
Then the left-hand side of equation (5.2) becomes $\langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle$ and right-hand side of equation (5.2) becomes

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad= \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle.\end{aligned}$$

Thus, it is distributive in this case also.

Case (3): If $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, the left-hand side becomes,

$$\begin{aligned}(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle &= \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\\ &= \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle.\end{aligned}$$

Also,

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad= \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle.\end{aligned}$$

Thus, it is distributive in this case too.

Case (4): If $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \geq \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, the left-hand side reduces to

$$\begin{aligned}(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle &= \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle\\ &= \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle\end{aligned}$$

and

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle.\end{aligned}$$

Thus distributive holds for all cases.

(ii): The proof of (ii) is similar to (i). □

Theorem 5.5. (i) *The operation $\wedge_m$ is right distributive over multiplication in BPyFSs.*
(ii) *The operation $\wedge_m$ is left distributive over multiplication in BPyFSs.*

Proof. (i) Here it is enough to prove

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$$
$$= (\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle).$$

Case (1): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle.$$

Subcase (1.1): Suppose $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \geq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$,

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle$$

and

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)$$
$$= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle.$$

Subcase (1.2): Suppose $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$,

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle$$

and

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)$$
$$= \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle.$$

In this case it is is distributive.

Case (2): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ and $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then the left-hand side of distributive property reduces to $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.

Subcase (2.1): If $\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \leq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$, then the right-hand side becomes $\langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.

Subcase (2.2): If $\langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$, then the right-hand side becomes $\langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle$.
Hence distributivity holds.

Case (3): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \geq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ but $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, we have $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \leq \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$. Now

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle.$$

Also,

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)$$
$$= \langle x_n^\mu, x_p^\mu, z_n^\gamma, z_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle y_n^\mu, y_p^\mu, z_n^\gamma, z_p^\gamma\rangle.$$

Case (4): If $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \le \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ but $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \ge \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$, then, we have $\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \ge \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle$. Now

$$(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle) \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle.$$

Also,

$$\begin{aligned}&(\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle) + (\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \wedge_m \langle z_n^\mu, z_p^\mu, z_n^\gamma, z_p^\gamma\rangle)\\ &\quad = \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle + \langle z_n^\mu, z_p^\mu, y_n^\gamma, y_p^\gamma\rangle = \langle z_n^\mu, z_p^\mu, x_n^\gamma, x_p^\gamma\rangle.\end{aligned}$$

Hence in all the above cases $\wedge_m$ is right distributive.

(ii): The proof of (ii) is similar to (i). □

Definition 5.6. For any two elements $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle, \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \in$ BPyFS, we define the inequality '$\preceq$' as $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \preceq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle$ means $x_n^\mu \le y_n^\mu, x_p^\mu \le y_p^\mu, x_n^\gamma \le y_n^\gamma, x_p^\gamma \le y_p^\gamma$.

Remark 5.7. The elements in the set $\{\langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle \in \text{BPyFS} \mid \langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle \preceq \langle y_n^\mu, y_p^\mu, y_n^\gamma, y_p^\gamma\rangle\}$ are identity element of $\langle x_n^\mu, x_p^\mu, x_n^\gamma, x_p^\gamma\rangle$ with respect to $\wedge_m$, that is, we have multiplied identity element.

Remark 5.8. Any IFM X can be decomposed into two bipolar Pythagorean fuzzy matrices $\Box X$ and $\Diamond X$ by means of $\wedge_m$, that is, $X = (\Box X) \wedge_m (\Diamond X)$.

Remark 5.9. For any two BPyFMs X and Y, $(X \vee Y) \wedge_m (X \wedge Y) = (X \wedge_m Y)$.

6. Conclusion

New results on modal operators under max-min composition were explored. Additionally, properties such as reflexivity, symmetry, transitivity, and idempotency of necessity and possibility were discussed. An investigation was conducted to explore the necessary and sufficient conditions for transitive closure and c-transitive closure matrices, utilizing modal operators. Furthermore, a new composition operator denoted by '$\wedge_m$' was introduced, with a detailed discussion of its algebraic properties. Finally, a decomposition of a BPyFM was achieved using the new composition operator and modal operators.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20**(1) (1986), 87 – 96, DOI: 10.1016/S0165-0114(86)80034-3.

[2] M. Bhowmik and M. Pal, Generalized intuitionistic fuzzy matrices, *Far East Journal of Mathematical Sciences* **29**(3) (2008), 533 – 554.

[3] H. Bustince and P. Burillo, Structure on intuitionistic fuzzy relations, *Fuzzy Sets and Systems* **78**(3) (1996), 293 – 303, DOI: 10.1016/0165-0114(96)84610-0.

[4] V. Chinnadurai, G. Thirumurugan and M. Umamaheswari, Novel operations in bipolar pythagorean fuzzy matrices, *International Journal of Advanced Science and Technology* **29**(4) (2020), 5387 – 5401, URL: http://sersc.org/journals/index.php/IJAST/article/view/25979.

[5] D. Ezhilmaran and K. Sankar, Morphism of bipolar intuitionistic fuzzy graphs, *Journal of Discrete Mathematical Science and Cryptography* **18**(5) (2015), 605 – 621, DOI: 10.1080/09720529.2015.1013673.

[6] H. Hashimoto, Decomposition of fuzzy matrices, *SIAM Journal of Algebraic and Discrete Methods* **6**(1) (1985), 32 – 38, DOI: 10.1137/0606003.

[7] N. G. Jeong and S. W. Park, The equivalence intuitionistic fuzzy matrix and idempotent, *Far East Journal of Mathematical Sciences* **11**(3) (2003), 355 – 365.

[8] K. H. Kim and R. W. Roush, Generalized fuzzy matrices, *Fuzzy Sets and System* **4**(3) (1980), 293 – 315, DOI: 10.1016/0165-0114(80)90016-0.

[9] H.-Y. Lee and N.-G. Jeong, Canonical form of an transitive intuitionistic fuzzy matrices, *Honam Mathematical Journal* **27**(4) (2005), 543 – 550, URL: https://koreascience.kr/article/JAKO200510102453756.page.

[10] S. Mondal and M. Pal, Similarity relations, invertibility and eigenvalues of intuitionistic fuzzy matrix, *Fuzzy Information and Engineering* **5**(4) (2013), 431 – 443, DOI: 10.1007/s12543-013-0156-y.

[11] P. Murugadas and K. Lalitha, Decomposition of an intuitionistic fuzzy matrices using implication operators, *Annals of Fuzzy Mathematics and Informatics* **11**(1) (2016), 11 – 18, URL: http://www.afmi.or.kr/papers/2016/Vol-11_No-01/PDF/AFMI-11-1(11-18)-H-150309R2.pdf.

[12] T. Muthuraji, S. Sriram and P. Murugadas, Decomposition of intuitionistic fuzzy matrices, *Fuzzy Information and Engineering* **8**(3) (2016), 345 – 354, DOI: 10.1016/j.fiae.2016.09.003.

[13] M. Pal, S. Khan and A. K. Shyamal, Intuitionistic fuzzy matrices, *Notes on Intuitionistic Fuzzy Sets* **8**(2) (2002), 51 – 62, URL: https://ifigenia.org/images/2/24/NIFS-08-2-51-62.pdf.

[14] M. G. Thomason, Convergence of powers of fuzzy matrix, *Journal of Mathematical Analysis and Applications* **57**(2) (1977), 476 – 480, DOI: 10.1016/0022-247X(77)90274-8.

[15] Z. Xu, *Intuitionistic Fuzzy Aggregation and Clustering*, Studies in Fuzziness and Soft Computing series, Springer, Berlin — Heidelberg, ix + 278 pages (2012), DOI: 10.1007/978-3-642-28406-9.

[16] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.