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Research Article

On the Spectrum of Generalized Zero-Divisor Graph of the Ring $\mathbb{Z}_{p^\alpha q^\beta}$

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Abstract. The generalized zero-divisor graph of a commutative ring R, denoted by $\Gamma'(R)$, is a simple (undirected) graph with vertex set $Z^*(R)$, the set of all nonzero zero-divisors of R and two distinct vertices x and y are adjacent if $x^n y = 0$ or $y^n x = 0$, for some positive integer n. In this paper, we determine the adjacency spectrum of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$, where p,q are distinct primes and α,β are positive integers. Also, we obtain the clique number, stability number, diameter, and the girth of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

Keywords. Zero-divisor graph, Adjacency matrix, Eigenvalues

Mathematics Subject Classification (2020). Primary: 05C25; Secondary: 13A70, 05E40

1. Introduction

Let $G = (V,E)$ be a simple graph with n vertices. The clique number $\omega(G)$ of G is the number of vertices in a maximum clique in G. The stability number $\alpha(G)$ of G is the largest number of pairwise non-adjacent vertices in G. In a graph G, the distance between two distinct vertices x and y, denoted by $d(x,y)$, is the length of a shortest path from x to y if it exists, otherwise $d(x,y) = \infty$. The diameter of a graph G is $\operatorname{diam}(G) = \sup\{d(x,y) \mid x \text{ and } y \text{ are vertices of } G\}$. The girth of G, denoted by $\operatorname{gr}(G)$, is the length of the shortest cycle in G. The girth of G is ∞ if G contains no cycle. The adjacency matrix of a graph G with n vertices, denoted by $A(G) = [a_{ij}]_{n\times n}$, is the matrix with $a_{ij} = 1$, if (i,j) is an edge and $a_{ij} = 0$, otherwise. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the adjacency matrix of G. The multiset of eigenvalues $\sigma_A(G) = \{\lambda_1^{(s_1)}, \dots, \lambda_n^{(s_n)}\}$ of $A(G)$ is called the adjacency spectrum of G. We refer to Anderson *et al.* [2], Atiyah and

MacDonald [3], and Godsil and Royle [7] for concepts in graphs from rings, ring theory and graph theory, respectively.

The concept of a zero-divisor graph for a commutative ring was first introduced by Beck [5] in 1988. Beck defined the zero-divisor graph of a ring R as a graph with vertex set R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Being motivated by Beck, in 1999, Anderson and Livingston [1] defined the zero-divisor graph for a commutative ring R, denoted by $\Gamma(R)$, as a simple (undirected) graph, with vertex set $Z^*(R)$, the set of nonzero zero-divisors of R and two vertices x and y are adjacent if and only if $xy = 0$. Redmond [15], defined the zero-divisor graph for a non-commutative ring R, denoted by $\overline{\Gamma(R)}$, as a simple graph with vertex set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $xy = 0$ or $yx = 0$. Patil and Waphare [12] introduced the zero-divisor graph for a ring R with involution $*$, denoted by $\Gamma^*(R)$, as a simple (undirected) graph with vertex set being all nonzero left zero-divisors in R and x and y are adjacent if and only if $xy^* = 0$. They studied the properties of $\Gamma^*(R)$ for Rickart $*$-ring R and obtained sufficient conditions for the zero-divisor graph $\Gamma^*(R)$ to be connected. Kumbhar *et al.* [9] introduced the strong zero-divisor graph for a ring with involution. They associated a simple undirected graph to a $*$-ring R, denoted by $\Gamma_s^*(R)$, whose vertex set is $V(\Gamma_s^*(R)) = \{0 \neq a \in R \mid r_R(aR) \neq \{0\}\}$ and two distinct vertices a and b are adjacent if and only if $aRb^* = 0$. Beaugris *et al.* [4] introduced the weak zero-divisor graph of finite commutative rings denoted by $\Omega(R)$. It is a graph with a vertex set consisting of nonzero elements u and v of a ring R and such that the vertices u and v are adjacent if and only if $(uv)^n = 0$ for some positive integer n, and studied diameter, girth, center, and their domination number for $\Omega(R)$. In [10], Lande and Khairnar introduced the generalized zero-divisor graph for a $*$-ring R, denoted by $\Gamma'(R)$, as a simple (undirected) graph with vertex set $Z^*(R)$, and two distinct vertices x and y are adjacent in $\Gamma'(R)$ if and only if $x^n y^* = 0$ or $y^n x^* = 0$ for some positive integer n. The study of zero-divisor graphs and the investigation of the spectra of zero-divisor graphs can be seen in Cardoso *et al.* [6], Khairnar and Waphare [8], Magi *et al.* [11], Pirzada *et al.* [13, 14]. For a positive integer n, $\mathbb{Z}_n$ denotes the ring of integers modulo n. Magi *et al.* [11] obtained the adjacency spectra of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$ for distinct primes p, q. They found the girth, diameter, clique number, and the stability number of $\Gamma(\mathbb{Z}_{p^2q^2})$. Pirzada *et al.* [14] determined the spectrum of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^M q^N})$, where p and q are distinct primes and M, N are positive integers.

Definition 1.1. The generalized zero-divisor graph of a commutative ring R, denoted by $\Gamma'(R)$, is a simple (undirected) graph with the vertex set $Z^*(R)$ of nonzero zero-divisors in R and two distinct vertices x and y are adjacent in $\Gamma'(R)$ if $x^n y = 0$ or $xy^n = 0$, for some positive integer n.

For a commutative ring R, we observe that $\Gamma(R)$ is a subgraph of $\Gamma'(R)$. If $F_i, i = 1, 2 \ldots, n$ are finite fields and $R = \bigoplus_{i=1}^{n} F_i$, then $\Gamma'(R) = \Gamma(R)$.

Example 1.1. Let $R = \mathbb{Z}_{125}$. The zero-divisor graph $\Gamma(R)$ is depicted in Figure 1. The generalized zero-divisor graph $\Gamma'(R)$ is isomorphic to the complete graph K_{24}.

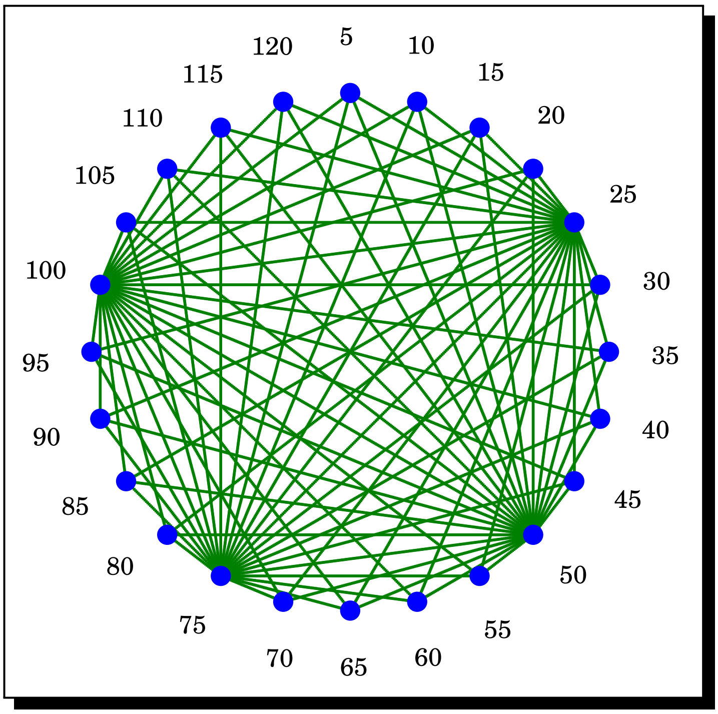


Figure 1. $\Gamma(R)$

The following result gives a characterization for the completeness of the generalized zero-divisor graph $\Gamma'(\mathbb{Z}_n)$.

Theorem 1.1. *The graph* $\Gamma'(\mathbb{Z}_n)$ *is complete if and only if* $n = p^r$*, where* p *is a prime and* r *is a positive integer. Moreover,* $\Gamma'(\mathbb{Z}_{p^r}) = K_{p^{r-1}-1}$.

Proof. Assume that $\Gamma'(\mathbb{Z}_n)$ is a complete graph and n is not a power of a prime. Let p and q be distinct primes that divide n. Let $p < q$. Then the vertices p and $2p$ are not adjacent in $\Gamma'(\mathbb{Z}_n)$, a contradiction to the fact that $\Gamma'(\mathbb{Z}_n)$ is a complete graph. Thus, $n = p^r$. Conversely, let $n = p^r$, for some prime p and positive integer r. Let x be any element in $V(\Gamma'(\mathbb{Z}_n))$. Then $x = kp^i$, for some integers k and i. Therefore, $x^r y = 0$, for any $y \in V(\Gamma'(\mathbb{Z}_n))$. Thus, $\Gamma'(\mathbb{Z}_n)$ is a complete graph. Since all the non-units are zero-divisors in a finite ring, we have $\Gamma'(\mathbb{Z}_{p^r}) = K_{p^{r-1}-1}$. □

Proposition 1.1. *Let* p *and* q *be distinct primes. Then* $\Gamma'(\mathbb{Z}_{pq}) = K_{p-1,q-1}$.

Proof. Observe that $V(\Gamma'(\mathbb{Z}_{pq})) = \{q, 2q, \dots, (p-1)q, p, 2p, \dots, (q-1)p\}$. Let $V_1 = \{q, 2q, \dots, (p-1)q\}$ and $V_2 = \{p, 2p, \dots, (q-1)p\}$ be a partition of $V(\Gamma'(\mathbb{Z}_{pq}))$. There is no edge between any two vertices in V_1 or any two vertices in V_2. For any $x \in V_1$ and $y \in V_2$, we have $xy = 0$. Therefore, every vertex from V_1 is adjacent to every vertex in V_2. Thus, $\Gamma'(\mathbb{Z}_{pq})$ is a complete bipartite graph $K_{p-1,q-1}$. □

The rest of the paper is organized as follows. In Section 2, we discuss the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ for distinct primes p, q, and integers α, β. We determine the multiplicities of the eigenvalues 0 and -1 and give the matrix for the remaining eigenvalues. In Section 3, we obtain the clique number, the stability number, diameter, and girth of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

2. Adjacency Spectrum of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$

Let p and q be distinct primes. In this section, we determine the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$, where α and β are positive integers. Further, we determine the multiplicities of eigenvalues 0 and -1 for the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

Let $\phi(n)$ denote the Euler's totient function, that is, the number of positive integers less than n and relatively prime to n. Note that the number of nonzero zero-divisors of $\mathbb{Z}_n$ is $n-\phi(n)-1$, that is, $|Z^*(\mathbb{Z}_n)| = n-\phi(n)-1$. Let $T_d = \{a \in \mathbb{Z}_n : (a,n) = d\}$, where (a,n) denotes the greatest common divisor of a and n. Then the cardinality of T_d is $\phi\left(\frac{n}{d}\right)$ (see, Young [16]). The canonical decomposition of an integer $n>1$ is given by $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are distinct primes and $k_1, k_2, \ldots, k_r$ are positive integers. We know that $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$.

We recall the following definition by Cardoso *et al.* [6].

Let $G_1, G_2, \ldots, G_n$ be graphs and H be a graph of order n and vertex set $\{1,2,\ldots,n\}$. The H-generalized join of the graphs $G_1, G_2, \ldots, G_n$ is denoted by $\bigvee_H \{G_1, G_2, \ldots, G_n\}$. It is a graph obtained by replacing each vertex i of H with the graph G_i and joining any two vertices of G_i and G_j if and only if the vertices i and j are adjacent in H.

Let N be the set of all nonzero nilpotent elements in a commutative ring R. Let Γ_1 be the induced subgraph of $\Gamma'(R)$ on the set of all non-nilpotent elements and $K_{|N|}$ be the complete graph on $|N|$ number of vertices.

In the following theorem, we determine the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ for integers $\alpha > 1$, $\beta > 1$ and also we find its spectrum.

Theorem 2.1. *Let p, q be distinct primes and $\alpha > 1, \beta > 1$ be integers:*

(a): *The adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is*

$$A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = \begin{bmatrix} 0_{n_1,n_1} & 0_{n_1,n_2} & 0_{n_1,n_3} & 1_{n_1,n_4} & 1_{n_1,n_5} \\ 0_{n_2,n_1} & 0_{n_2,n_2} & 1_{n_2,n_3} & 0_{n_2,n_4} & 1_{n_2,n_5} \\ 0_{n_3,n_1} & 1_{n_3,n_2} & 0_{n_3,n_3} & 1_{n_3,n_4} & 1_{n_3,n_5} \\ 1_{n_4,n_1} & 0_{n_4,n_2} & 1_{n_4,n_3} & 0_{n_4,n_4} & 1_{n_4,n_5} \\ 1_{n_5,n_1} & 1_{n_5,n_2} & 1_{n_5,n_3} & 1_{n_5,n_4} & (1-I)_{n_5,n_5} \end{bmatrix}, \tag{2.1}$$

where 0 is a matrix of all zeros, 1 is a matrix of all ones and I is an identity matrix.

(b): *0 and -1 are eigenvalues of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ with multiplicities $p^{\alpha-1}q^{\beta-1}(p+q-2)-4$ and $p^{\alpha-1}q^{\beta-1}-2$, respectively.*

(c): *The remaining 5 eigenvalues are the eigenvalues of the matrix*

$$M = \begin{bmatrix} 0 & 0 & 0 & \sqrt{n_1 n_4} & \sqrt{n_1 n_5} \\ 0 & 0 & \sqrt{n_2 n_3} & 0 & \sqrt{n_2 n_5} \\ 0 & \sqrt{n_2 n_3} & 0 & \sqrt{n_3 n_4} & \sqrt{n_3 n_5} \\ \sqrt{n_1 n_4} & 0 & \sqrt{n_3 n_4} & 0 & \sqrt{n_4 n_5} \\ \sqrt{n_1 n_5} & \sqrt{n_2 n_5} & \sqrt{n_3 n_5} & \sqrt{n_4 n_5} & n_5 - 1 \end{bmatrix}. \tag{2.2}$$

Proof. (a): Let p, q be distinct primes, $\alpha > 1, \beta > 1$ be integers and $n = p^\alpha q^\beta$. We partition the vertex set of $\Gamma'(\mathbb{Z}_n)$ into five sets in terms of zero divisors of $\mathbb{Z}_n$. Let

$X_1 = \{x \in \mathbb{Z}_n : \gcd(x,n) = p^i,\ i = 1,2,\ldots,\alpha-1\}$,

$$X_2 = \{x \in \mathbb{Z}_n : \gcd(x,n) = q^i,\ i = 1,2,\dots,\beta-1\},$$
$$X_3 = \{x \in \mathbb{Z}_n : \gcd(x,n) = p^\alpha\},$$
$$X_4 = \{x \in \mathbb{Z}_n : \gcd(x,n) = q^\beta\},$$
$$X_5 = \{x \in \mathbb{Z}_n \setminus \{0\} : x = kp^i q^j,\ i = 1,2,\dots,\alpha,\ j = 1,2,\dots,\beta\}.$$

Note that X_5 is the set of all nonzero nilpotent elements in $\mathbb{Z}_{p^\alpha q^\beta}$. Observe that all the sets X_1, X_2, X_3, X_4, X_5 are mutually disjoint. Thus,

$$P = \{X_1, X_2, X_3, X_4, X_5\} \tag{2.3}$$

forms a partition of the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

Now, we find the cardinality of the sets X_1, X_2, X_3, X_4, X_5. Assume that $x \in X_1$. Then $\gcd(x,n) = p^i$, for some $i \in \{1,2,\dots,\alpha-1\}$. The number of elements in X_1 with $\gcd(x,n) = p^i$ is

$$\phi\left(\frac{p^\alpha q^\beta}{p^i}\right) = \phi(p^{\alpha-i}q^\beta) = p^{\alpha-i-1}q^{\beta-1}(p-1)(q-1)), \quad \text{for } i = 1,2,\dots,\alpha-1.$$

Therefore,

$$\begin{aligned}
n_1 = |X_1| &= \phi\left(\frac{p^\alpha q^\beta}{p}\right) + \phi\left(\frac{p^\alpha q^\beta}{p^2}\right) + \dots + \phi\left(\frac{p^\alpha q^\beta}{p^{\alpha-1}}\right) \\
&= p^{\alpha-2}q^{\beta-1}(p-1)(q-1) + p^{\alpha-3}q^{\beta-1}(p-1)(q-1) + \dots + q^{\beta-1}(p-1)(q-1) \\
&= q^{\beta-1}(p-1)(q-1)(p^{\alpha-2} + p^{\alpha-3} + \dots + p + 1) \\
&= q^{\beta-1}(p-1)(q-1)\left(\frac{p^{\alpha-1}-1}{p-1}\right) \\
&= q^{\beta-1}(q-1)(p^{\alpha-1}-1).
\end{aligned}$$

Similarly, if $x \in X_2$, then $\gcd(x,n) = q^i$, for some $i \in \{1,2,\dots,\beta-1\}$. So,

$$\begin{aligned}
n_2 = |X_2| &= \phi\left(\frac{p^\alpha q^\beta}{q}\right) + \phi\left(\frac{p^\alpha q^\beta}{q^2}\right) + \dots + \phi\left(\frac{p^\alpha q^\beta}{q^{\beta-1}}\right) \\
&= p^{\alpha-1}(p-1)(q^{\beta-1}-1).
\end{aligned}$$

If $x \in X_3$, then $\gcd(x,n) = p^\alpha$ and so $n_3 = |X_3| = \phi\left(\frac{p^\alpha q^\beta}{p^\alpha}\right) = q^{\beta-1}(q-1)$.

Similarly, $n_4 = |X_4| = \phi\left(\frac{p^\alpha q^\beta}{q^\beta}\right) = p^{\alpha-1}(p-1)$.

Clearly, the number of multiples of pq in $\mathbb{Z}_{p^\alpha q^\beta}$ is $p^{\alpha-1}q^{\beta-1}$. Therefore,

$$n_5 = |X_5| = p^{\alpha-1}q^{\beta-1} - 1.$$

Since $\phi(p^\alpha q^\beta) = p^{\alpha-1}q^{\beta-1}(p-1)(q-1)$, therefore, the number of nonzero zero-divisors in

$$\begin{aligned}
\mathbb{Z}_{p^\alpha q^\beta} &= n - \phi(n) - 1 \\
&= p^\alpha q^\beta - p^{\alpha-1}q^{\beta-1}(p-1)(q-1) - 1 \\
&= p^{\alpha-1}q^{\beta-1}(pq - (p-1)(q-1)) - 1 \\
&= p^{\alpha-1}q^{\beta-1}(p+q-1) - 1.
\end{aligned} \tag{2.4}$$

Let $P = \{X_1, X_2, X_3, X_4, X_5\}$ be a partition of $\mathbb{Z}_{p^\alpha q^\beta}$. Let $X, Y \in P$. If every element of X is adjacent to every element of Y, we denote it as $X \sim Y$. If no element of X is adjacent to any element in Y, we denote it as $X \nsim Y$. We have the following observations.

(1): Since X_5 is the set of all nonzero nilpotent elements, so each element in X_5 is adjacent to all the remaining vertices. Thus, $X_5 \sim X_1, X_2, X_3, X_4, X_5$.

(2): Let $a \in X_1, b \in X_4$. Then there exist a positive integer m such that $a^m b = 0$. Also, for any $b \in X_1 \cup X_2 \cup X_3$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, every element in X_1 is adjacent to every element in X_4 and no element of X_1 is adjacent to any element of X_1, X_2 and X_3. Thus, $X_1 \sim X_4$, $X_1 \nsim X_1$, $X_1 \nsim X_2$, $X_1 \nsim X_3$.

(3): Let $a \in X_2$, $b \in X_3$. Then there exist a positive integer m such that $a^m b = 0$. Also, for any $b \in X_1 \cup X_2 \cup X_4$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, every element in X_2 is adjacent to every element in X_3 and no element of X_2 is adjacent to any element of X_1, X_2 and X_4. So, $X_2 \sim X_3$, $X_2 \nsim X_1$, $X_2 \nsim X_2$, $X_2 \nsim X_4$.

(4): Let $a \in X_3$, $b \in X_2$. Then there exist a positive integer m such that $ab^m = 0$. If $b \in X_4$, then $ab = 0$. Also, for any $b \in X_1 \cup X_3$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, every element in X_3 is adjacent to every element in X_2, X_4 and no element of X_3 is adjacent to any element of X_1 and X_3. So, $X_3 \sim X_2$, $X_3 \sim X_4$, $X_3 \nsim X_1$, $X_3 \nsim X_3$.

(5): Let $a \in X_4$, $b \in X_1$. Then there exist a positive integer m such that $a^m b = 0$. If $b \in X_3$, then $ab = 0$. Also, for any $b \in X_2 \cup X_4$ and for any positive integer k, $a^k b \neq 0$ and $b^k a \neq 0$. Therefore, every element in X_4 is adjacent to every element in X_1, X_3 and no element of X_4 is adjacent to any element of X_2 and X_4. Thus, $X_4 \sim X_1$, $X_4 \sim X_3$, $X_4 \nsim X_2$, $X_4 \nsim X_4$.

Since all the vertices of X_1 are adjacent to all the vertices in X_4, we get a block of ones corresponding to the row X_1 and the column X_4. Also, no vertex of X_1 is adjacent to any vertex of X_2, we get a block of zeros corresponding to the row X_1 and the column X_2. Similarly, we get blocks of zeros and ones for the remaining vertices. For nilpotent elements, we have to consider the diagonal entries as zero. Thus, corresponding to the row X_5 and the column X_5 we get a block of $1-I$, where 1 is a matrix of all ones and I is an identity matrix. Therefore, the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ with row and column headings X_1, X_2, X_3, X_4, X_5 is

$$A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{array} \begin{array}{c} \begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \end{array} \\ \begin{bmatrix} 0_{n_1,n_1} & 0_{n_1,n_2} & 0_{n_1,n_3} & 1_{n_1,n_4} & 1_{n_1,n_5} \\ 0_{n_2,n_1} & 0_{n_2,n_2} & 1_{n_2,n_3} & 0_{n_2,n_4} & 1_{n_2,n_5} \\ 0_{n_3,n_1} & 1_{n_3,n_2} & 0_{n_3,n_3} & 1_{n_3,n_4} & 1_{n_3,n_5} \\ 1_{n_4,n_1} & 0_{n_4,n_2} & 1_{n_4,n_3} & 0_{n_4,n_4} & 1_{n_4,n_5} \\ 1_{n_5,n_1} & 1_{n_5,n_2} & 1_{n_5,n_3} & 1_{n_5,n_4} & (1-I)_{n_5,n_5} \end{bmatrix} \end{array}. \tag{2.5}$$

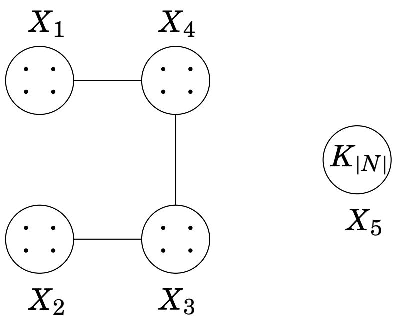
(b): The adjacency matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is given in equation (2.5). Since $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is a real and symmetric matrix, the algebraic multiplicities and the geometric multiplicities of all the eigenvalues are the same. By performing elementary row operations, the rank of the matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is less than its size $p^{\alpha-1}q^{\beta-1}(p+q-1)-1$. Therefore, $\det A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = 0$. Hence 0 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$. The geometric multiplicity of an eigenvalue 0 is the nullity of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$. By performing elementary row transformations on $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$, the number of zero rows in the transformed matrix

$$\begin{aligned} &= |X_1| + |X_2| + |X_3| + |X_4| - 4 \\ &= q^{\beta-1}(q-1)(p^{\alpha-1}-1) + p^{\alpha-1}(p-1)(q^{\beta-1}-1) + q^{\beta-1}(q-1) + p^{\alpha-1}(p-1) \\ &= p^{\alpha-1}q^{\beta-1}(p+q-2) - 4. \end{aligned}$$

Therefore, the nullity of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is $p^{\alpha-1}q^{\beta-1}(p+q-2)-4$. Thus, the multiplicity of an eigenvalue 0 is $p^{\alpha-1}q^{\beta-1}(p+q-2)-4$.

The geometric multiplicity of an eigenvalue -1 is the nullity of the matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))+I$. By performing elementary row operations on $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))+I$, the number of zero rows in the transformed matrix is $|X_5|-1 = p^{\alpha-1}q^{\beta-1}-1-1$. Thus, -1 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ with the multiplicity $p^{\alpha-1}q^{\beta-1}-2$.

(c): From equation (2.5), we express $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ as a generalized join of graphs Γ_1 and $K_{|N|}$ as follows. $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}) = \Gamma_1 \bigvee_{K_2} K_{|N|}$. This is because every nilpotent element is adjacent to every other vertex. We express $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ as a generalized join of two graphs as depicted in Figure 2 and 3.



(a) $\Gamma_1(\mathbb{Z}_{p^\alpha q^\beta})$ **(b)** $K_{|N|}$

Figure 2

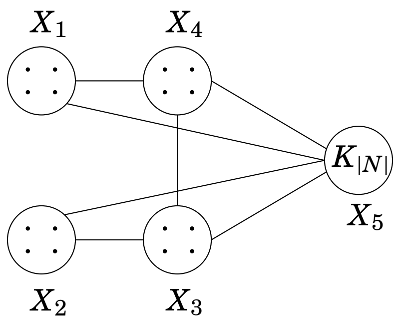


Figure 3. $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$

Also, we make use of the following result by Cardoso *et al.* [6].

Let G be a graph with vertices $\{1,2,\dots,n\}$ and G_i be n pairwise disjoint r_i-regular graphs of order n_i respectively. Then the adjacency spectrum of $G = \bigvee\{G_1, G_2, \dots, G_n\}$ is given by

$$\sigma_A(G) = \left(\bigcup_{i=1}^{n}(\sigma_A(G_i)\setminus\{r_i\})\right)\bigcup \sigma(C_A(G)),$$

where

$$C_A(G) = (c_{ij})_{n\times n} = \begin{cases} r_i, & i=j, \\ \sqrt{n_i n_j}, & ij \in E(G), \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

As $|A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))| = p^{\alpha-1}q^{\beta-1}(p+q-1)-1$ and the sum of the eigenvalues 0 and -1 is $p^{\alpha-1}q^{\beta-1}(p+q-1)-6$, therefore, there are 5 more eigenvalues of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$. Since the sum of all the eigenvalues is the trace of the matrix, so the sum of the remaining 5 eigenvalues is

$$p^{\alpha-1}q^{\beta-1}-2. \tag{2.7}$$

Using the above observations, equation (2.6) and the adjacency matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ given in equation (2.5), the remaining 5 eigenvalues of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ are the eigenvalues of the matrix M, where

$$M = \begin{bmatrix} 0 & 0 & 0 & \sqrt{n_1n_4} & \sqrt{n_1n_5} \\ 0 & 0 & \sqrt{n_2n_3} & 0 & \sqrt{n_2n_5} \\ 0 & \sqrt{n_2n_3} & 0 & \sqrt{n_3n_4} & \sqrt{n_3n_5} \\ \sqrt{n_1n_4} & 0 & \sqrt{n_3n_4} & 0 & \sqrt{n_4n_5} \\ \sqrt{n_1n_5} & \sqrt{n_2n_5} & \sqrt{n_3n_5} & \sqrt{n_4n_5} & n_5-1 \end{bmatrix}. \tag{2.8}$$

□

The order of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is the sum of the number of rows of all five blocks. Thus, the order of $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$ is m, where

$$\begin{aligned} m &= n_1+n_2+n_3+n_4+n_5 \\ &= q^{\beta-1}(q-1)(p^{\alpha-1}-1)+p^{\alpha-1}(p-1)(q^{\beta-1}-1)+q^{\beta-1}(q-1)+p^{\alpha-1}(p-1)+p^{\alpha-1}q^{\beta-1}-1 \\ &= p^{\alpha-1}q^{\beta-1}(p+q-1)-1. \end{aligned}$$

Observe that this is the same as the number of nonzero zero-divisors given in equation (2.4).
As an illustration, we find the eigenvalues of the adjacency matrix of $\Gamma'(\mathbb{Z}_{72})$.

Example 2.1. Consider the ring $\mathbb{Z}_{72}$. Let $p=2$, $q=3$, $\alpha=3$, $\beta=2$. Then $n_1=18$, $n_2=8$, $n_3=6$, $n_4=4$, $n_5=11$. The adjacency matrix of $\Gamma'(\mathbb{Z}_{72})$ is

$$A(\Gamma'(\mathbb{Z}_{72})) = \begin{bmatrix} 0_{18,18} & 0_{18,8} & 0_{18,6} & 1_{18,4} & 1_{18,11} \\ 0_{8,18} & 0_{8,8} & 1_{8,6} & 0_{8,4} & 1_{8,11} \\ 0_{6,18} & 1_{6,8} & 0_{6,6} & 1_{6,4} & 1_{6,11} \\ 1_{4,18} & 0_{4,8} & 1_{4,6} & 0_{4,4} & 1_{4,11} \\ 1_{11,18} & 1_{11,8} & 1_{11,6} & 1_{11,4} & (1-I)_{11,11} \end{bmatrix}.$$

Here 0 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{72}))$ with multiplicity $p^{\alpha-1}q^{\beta-1}(p+q-2)-4 = 32$ and -1 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{72}))$ with multiplicity $p^{\alpha-1}q^{\beta-1}-2 = 10$. By equation (2.8), the remaining 5 eigenvalues of $A(\Gamma'(\mathbb{Z}_{72}))$ are the eigenvalues of the matrix M with sum $p^{\alpha-1}q^{\beta-1}-2 = 2^2.3^1-2 = 10$ (by equation (2.7)), where

$$M = \begin{bmatrix} 0 & 0 & 0 & \sqrt{72} & \sqrt{191} \\ 0 & 0 & \sqrt{48} & 0 & \sqrt{88} \\ 0 & \sqrt{48} & 0 & \sqrt{24} & \sqrt{66} \\ \sqrt{72} & 0 & \sqrt{24} & 0 & \sqrt{44} \\ \sqrt{191} & \sqrt{88} & \sqrt{66} & \sqrt{44} & 10 \end{bmatrix}.$$

Using MAPLE software, we find the eigenvalues of M. The characteristic polynomial of M is

$$\left.\begin{aligned} &x^5-10x^4-533x^3-(-1440+24\sqrt{191}\sqrt{2}\sqrt{11}+16\sqrt{22}\sqrt{3}\sqrt{66}+8\sqrt{6}\sqrt{11}\sqrt{66})x^2 \\ &\quad -(-32520+24\sqrt{2}\sqrt{6}\sqrt{66}\sqrt{191}+64\sqrt{3}\sqrt{6}\sqrt{11}\sqrt{22})x \\ &\quad -34560-192\sqrt{2}\sqrt{3}\sqrt{6}\sqrt{22}\sqrt{191}+1152\sqrt{191}\sqrt{2}\sqrt{11}+1152\sqrt{22}\sqrt{3}\sqrt{66}, \end{aligned}\right\} \tag{2.9}$$

and the eigenvalues of M are

$-13.66143422,\ -9.163998769,\ -2.041694066,\ 5.530632118,\ 29.33649494.$

Note that the sum of the five eigenvalues

$-13.66143422,\ -9.163998769,\ -2.041694066,\ 5.530632118,\ 29.33649494$ is 10.

The adjacency spectrum of $A(\Gamma'(\mathbb{Z}_{72}))$ is the multiset
$\{0^{(32)}, -1^{(10)}, -13.66143422^{(1)}, -9.163998769^{(1)}, -2.041694066^{(1)}, 5.530632118^{(1)}, 29.33649494^{(1)}\}$.

Now $\Gamma'(\mathbb{Z}_{pq})$ is a complete bipartite graph $K_{p-1,q-1}$ with the vertex set $V = X_1 \cup X_2$, where $X_1 = \{kp : k = 1,2,\dots,q-1\}$ and $X_2 = \{kq : k = 1,2,\dots,p-1\}$. Therefore, its adjacency spectrum is a multiset

$$\{0^{(p+q-4)}, \sqrt{(p-1)(q-1)}^{(1)}, -\sqrt{(p-1)(q-1)}^{(1)}\}.$$

Now, we find the adjacency matrix for $\Gamma'(\mathbb{Z}_{p^\alpha q})$, for $\alpha > 1$ and we also obtain its spectrum.

Theorem 2.2. *Let p,q be distinct primes and $\alpha > 1$ be an integer:*

(a): *The adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ is*

$$A(\Gamma'(\mathbb{Z}_{p^\alpha q})) = \begin{bmatrix} 0_{n_1,n_1} & 1_{n_1,n_2} & 1_{n_1,n_3} \\ 1_{n_2,n_1} & 0_{n_2,n_2} & 1_{n_2,n_3} \\ 1_{n_3,n_1} & 1_{n_3,n_2} & (1-I)_{n_3,n_3} \end{bmatrix}, \tag{2.10}$$

where 0 is a matrix of all zeros, 1 is a matrix of all ones and I is an identity matrix.

(b): *0 and -1 are eigenvalues of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ with multiplicities $p^{\alpha-1}(p+q-2)-2$ and $p^{\alpha-1}-2$, respectively.*

(c): *The remaining 3 eigenvalues are the eigenvalues of the matrix*

$$M = \begin{bmatrix} 0 & \sqrt{n_1 n_2} & \sqrt{n_1 n_3} \\ \sqrt{n_1 n_2} & 0 & \sqrt{n_2 n_3} \\ \sqrt{n_1 n_3} & \sqrt{n_2 n_3} & n_3 - 1 \end{bmatrix}. \tag{2.11}$$

Proof. (a): Let $m = p^\alpha q$. We partition the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ into three sets in terms of zero divisors of $\mathbb{Z}_m$. Let

$X_1 = \{x \in \mathbb{Z}_m : \gcd(x,m) = p^i,\ i = 1,2,\dots,\alpha\}$,

$X_2 = \{x \in \mathbb{Z}_m : \gcd(x,m) = q\}$,

$X_3 = \{x \in \mathbb{Z}_m : x = kp^i q,\ i = 1,2,\dots,\alpha-1,\ k = 1,2,\dots,q-1\}$.

Observe that X_3 is the set of all nonzero nilpotent elements in $\mathbb{Z}_{p^\alpha q}$ and all the sets X_1, X_2, X_3 are mutually disjoint. Therefore,

$$P_1 = \{X_1, X_2, X_3\} \tag{2.12}$$

forms a partition of the vertex set of $\Gamma'(\mathbb{Z}_{p^\alpha q})$.

Assume that $x \in X_1$. Then $\gcd(x,m) = p^i$, for some $i = 1,2,\dots,\alpha-1$. The number of elements in X_1 with $\gcd(x,m) = p^i$ is $\phi(\frac{m}{p^i}) = \phi(\frac{p^\alpha q}{p^i}) = p^{\alpha-i-1}(p-1)(q-1)$. Therefore,

$$n_1 = |X_1| = \phi\left(\frac{p^\alpha q}{p}\right) + \phi\left(\frac{p^\alpha q}{p^2}\right) + \cdots + \phi\left(\frac{p^\alpha q}{p^\alpha}\right)$$

$$= p^{\alpha-2}(p-1)(q-1) + p^{\alpha-3}(p-1)(q-1) + \cdots + (p-1)(q-1) + (q-1)$$
$$= p^{\alpha-1}(q-1).$$

Let $x \in X_2$. Then $\gcd(x,m) = q$. Therefore, the number of elements in X_2 is

$$n_2 = |X_2| = \phi\left(\frac{p^\alpha q}{q}\right) = p^{\alpha-1}(p-1).$$

The number of multiples of pq in $p^\alpha q$ is $p^{\alpha-1}$. Therefore, $n_3 = |X_3| = p^{\alpha-1} - 1$. The number of nonzero zero-divisors in $\mathbb{Z}_{p^\alpha q}$ is $|X_1| + |X_2| + |X_3| = p^{\alpha-1}(p+q-1) - 1$. Similar to the proof of Theorem 2.1(a), we have the following observations:

$$X_3 \sim X_1, X_3 \sim X_2, X_3 \sim X_3,$$
$$X_1 \sim X_2, X_1 \sim X_3, X_1 \nsim X_1,$$
$$X_2 \sim X_1, X_2 \sim X_3, X_2 \nsim X_2.$$

Therefore, the adjacency matrix of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ with the row and the column headings X_1, X_2, X_3 is

$$A(\Gamma'(\mathbb{Z}_{p^\alpha q})) = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} 0_{n_1,n_1} & 1_{n_1,n_2} & 1_{n_1,n_3} \\ 1_{n_2,n_1} & 0_{n_2,n_2} & 1_{n_2,n_3} \\ 1_{n_3,n_1} & 1_{n_3,n_2} & (1-I)_{n_3,n_3} \end{bmatrix} \end{array}. \tag{2.13}$$

(b): The adjacency matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q}))$ is given in equation (2.13). Similar to the proof of Theorem 2.1(b), by performing elementary row transformations on $A(\Gamma'(\mathbb{Z}_{p^\alpha q}))$, the number of zero rows in the transformed matrix is $|X_1| + |X_2| - 2 = p^{\alpha-1}(p+q-2) - 2$.

The geometric multiplicity of an eigenvalue 0 is the nullity of $A(\Gamma'(\mathbb{Z}_{p^\alpha q}))$. Thus, the multiplicity of an eigenvalue 0 is $p^{\alpha-1}(p+q-2) - 2$. By performing elementary row operations on $A(\Gamma'(\mathbb{Z}_{p^\alpha q})) + I$, the number of zero rows in the transformed matrix is $|X_3| - 1 = p^{\alpha-1} - 2$. The geometric multiplicity of an eigenvalue -1 is the nullity of the matrix $A(\Gamma'(\mathbb{Z}_{p^\alpha q})) + I$. Therefore, -1 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{p^\alpha q}))$ with the multiplicity $p^{\alpha-1} - 2$.

Note that we express $\Gamma'(\mathbb{Z}_{p^\alpha q})$ as a generalized join of two graphs as depicted in Figure 4.

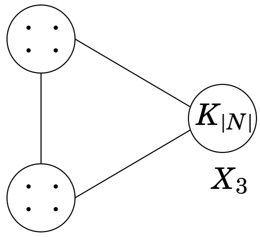


Figure 4. $\Gamma'(\mathbb{Z}_{p^\alpha q})$

Similar to Theorem 2.1(c), the remaining 3 eigenvalues are the eigenvalues of the matrix

$$M = \begin{bmatrix} 0 & \sqrt{n_1 n_2} & \sqrt{n_1 n_3} \\ \sqrt{n_1 n_2} & 0 & \sqrt{n_2 n_3} \\ \sqrt{n_1 n_3} & \sqrt{n_2 n_3} & n_3 - 1 \end{bmatrix}. \tag{2.14}$$

□

As an illustration, we find the eigenvalues of the adjacency matrix of $\Gamma'(\mathbb{Z}_{12})$.

Example 2.2. Let $p=2$, $q=3$, $\alpha=2$, $n=p^\alpha q=12$. Then $n_1=4$, $n_2=2$, $n_3=1$. The adjacency matrix of $\Gamma'(\mathbb{Z}_{12})$ is

$$A(\Gamma'(\mathbb{Z}_{12}))=\begin{bmatrix}0_{(4,4)} & 1_{(4,2)} & 1_{(4,1)}\\ 1_{(2,4)} & 0_{(2,2)} & 1_{(2,1)}\\ 1_{(1,4)} & 1_{(1,2)} & (1-I)_{(1,1)}\end{bmatrix}.$$

Here 0 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{12}))$ with multiplicity $p^{\alpha-1}(p+q-2)-2=4$ and -1 is an eigenvalue of $A(\Gamma'(\mathbb{Z}_{12}))$ with multiplicity $p^{\alpha-1}-2=0$. The remaining 3 eigenvalues of $A(\Gamma'(\mathbb{Z}_{12}))$ are the eigenvalues of the matrix M with the sum $p^{\alpha-1}-2=0$, where

$$M=\begin{bmatrix}0 & \sqrt{8} & 2\\ \sqrt{8} & 0 & \sqrt{2}\\ 2 & \sqrt{2} & 0\end{bmatrix}.$$

The characteristic polynomial of M is $x^3-14x-16$, and the eigenvalues of M are -2.918522599, -1.299664103, 4.218186702. Note that the sum of the three eigenvalues -2.918522599, -1.299664103, 4.218186702 is 0.

Hence the adjacency spectrum of $A(\Gamma'(\mathbb{Z}_{12}))$ is the multiset

$$\{0^{(4)}, -2.918522599^{(1)}, -1.299664103^{(1)}, 4.218186702^{(1)}\}. \tag{2.15}$$

We find the adjacency spectrum of $A(\Gamma'(\mathbb{Z}_{12}))$ directly using MAPLE software. Let $\{2,3,4,6,8,9,10\}$ be the vertex set of $\Gamma'(\mathbb{Z}_{12})$. The adjacency matrix of $\Gamma'(\mathbb{Z}_{12})$ with the row and column headings $2,4,8,10,3,9,6$ is

$$A(\Gamma'(\mathbb{Z}_{12}))=\begin{bmatrix}0&0&0&0&1&1&1\\0&0&0&0&1&1&1\\0&0&0&0&1&1&1\\0&0&0&0&1&1&1\\1&1&1&1&0&0&1\\1&1&1&1&0&0&1\\1&1&1&1&1&1&0\end{bmatrix}.$$

The characteristic polynomial of $A(\Gamma'(\mathbb{Z}_{12}))$ is $x^7-14x^5-16x^4$ and its eigenvalues are $0,0,0,0,-2.918522599,-1.299664103,4.218186702$, which are same as obtained in equation (2.15).

3. Some Basic Properties of the Generalized Zero-Divisor Graph $\Gamma'(\mathbb{Z}_n)$

First, we determine the clique number and the stability number of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ for distinct primes p,q, and integers $\alpha>1$, $\beta>1$.

Theorem 3.1. *Let p,q be distinct primes, and $\alpha>1$, $\beta>1$ be integers. Then the clique number of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is $\omega(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))=p^{\alpha-1}q^{\beta-1}+1$ and the stability number $\alpha(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))=q^{\beta-1}(q-1)(p^{\alpha-1}-1)+p^{\alpha-1}(p-1)(q^{\beta-1}-1)$.*

Proof. Let X_1,X_2,X_3,X_4,X_5 be as in the proof of Theorem 2.1. From Figure 3, the subgraph induced by the vertices in X_5 and any one vertex from each of X_1,X_4 correspond to a complete subgraph of maximum order in $A(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta}))$. Thus, the clique number of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is $\omega(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = |X_5| + 2 = p^{\alpha-1}q^{\beta-1} + 1$.

For any $a, b \in X_1 \cup X_2$, we have $ab \neq 0$. Therefore, no two vertices of $X_1 \cup X_2$ are adjacent in $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$. Thus, the stability number of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$ is

$$\begin{aligned}\alpha(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) &= |X_1| + |X_2| \\ &= q^{\beta-1}(q-1)(p^{\alpha-1}-1) + p^{\alpha-1}(p-1)(q^{\beta-1}-1).\end{aligned}$$

□

For $\alpha = 2$, $\beta = 2$ analogues to [11, Theorem 3.2], we get the clique number of $\Gamma'(\mathbb{Z}_{p^2q^2})$. From [11, Theorem 3.3], the stability number of $\Gamma(\mathbb{Z}_{p^2q^2})$ is

$$\alpha(\Gamma(\mathbb{Z}_{p^2q^2})) = p(q-1)(p+q-1).$$

By Theorem 3.1, for $\alpha = 2$, $\beta = 2$, the stability number of $\Gamma'(\mathbb{Z}_{p^2q^2})$ is

$$\alpha(\Gamma'(\mathbb{Z}_{p^2q^2})) = (p+q)(p-1)(q-1),$$

which is different from the stability number of $\Gamma(\mathbb{Z}_{p^2q^2})$. According to [11, Theorems 3.4 and 3.5], $\mathrm{gr}(\Gamma(\mathbb{Z}_{p^2q^2})) = 3$ and $\mathrm{diam}(\Gamma(\mathbb{Z}_{p^2q^2})) = 3$. The following result gives the diameter and the girth of $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$.

Theorem 3.2. *Let p, q be distinct primes and $\alpha > 1$, $\beta > 1$ be integers. Then,*

$$\mathrm{gr}(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = 3 \quad \textit{and} \quad \mathrm{diam}(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = 2.$$

Proof. From the proof of Theorem 2.1, we have $|X_5| = p^{\alpha-1}q^{\beta-1} - 1 \geq 3$. If $x, y, z \in X_5$, then $\{x, y, z\}$ form a clique in $\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})$. Thus, $\mathrm{gr}(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = 3$. If x, y are any two distinct nilpotent elements, then $d(x, y) = 1$. If either x or y is nilpotent, then $d(x, y) = 1$. If both x, y are non-nilpotent elements and if x, y are adjacent, then $d(x, y) = 1$. If both x, y are non-nilpotent elements and suppose they are not adjacent, then $x \leftrightarrow z \leftrightarrow y$ is a path, where z is a nilpotent element. Therefore, $d(x, y) = 2$. Thus, $\mathrm{diam}(\Gamma'(\mathbb{Z}_{p^\alpha q^\beta})) = 2$. □

In the following theorem, we determine the clique number and the stability number of $\Gamma'(\mathbb{Z}_{p^\alpha q})$, where p, q are distinct primes and α is a positive integer.

Theorem 3.3. *Let p, q be distinct primes and $\alpha > 1$ be an integer. Then the clique number of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ is $\omega(\Gamma'(\mathbb{Z}_{p^\alpha q})) = p^{\alpha-1} + 1$ and the stability number is*

$$\alpha(\Gamma'(\mathbb{Z}_{p^\alpha q})) = \max\{p^{\alpha-1}(q-1),\ p^{\alpha-1}(p-1)\}.$$

Proof. Assume that X_1, X_2, X_3 are as in the proof of Theorem 2.2. From Figure 4, the subgraph induced by the vertices in X_3 and any one vertex from each of X_1 and X_2 corresponds to a complete subgraph of maximum order of $A(\Gamma'(\mathbb{Z}_{p^\alpha q}))$. Thus, the clique number of $\Gamma'(\mathbb{Z}_{p^\alpha q})$ is $\omega(\Gamma'(\mathbb{Z}_{p^\alpha q})) = |X_3| + 2 = p^{\alpha-1} + 1$. Since no two vertices of X_1 are adjacent and no two vertices of X_2 are adjacent in $\Gamma'(\mathbb{Z}_{p^\alpha q})$, the stability number

$$\alpha(\Gamma'(\mathbb{Z}_{p^\alpha q})) = \max\{p^{\alpha-1}(q-1),\ p^{\alpha-1}(p-1)\}.$$

□

Theorem 3.4. *Let p, q be distinct primes and $\alpha > 1$ be an integer. Then*

$$\mathrm{gr}(\Gamma'(\mathbb{Z}_{p^\alpha q})) = 3 \quad \textit{and} \quad \mathrm{diam}(\Gamma'(\mathbb{Z}_{p^\alpha q})) = 2.$$

Proof. The proof is similar to the proof of Theorem 3.2. □

4. Conclusion

We have determined the adjacency matrix and the eigenvalues of the generalized zero-divisor graph of the ring $\mathbb{Z}_{p^\alpha q^\beta}$, for distinct primes p,q and positive integers α,β. Furthermore, we have obtained the clique number, stability number, diameter, and girth of the generalized zero-divisor graph of $\mathbb{Z}_{p^\alpha q^\beta}$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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