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Research Article

Series Solution of Fractional Differential Equations Describing Physical Systems

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Abstract. The aim of this paper is to extend the iterative method based on the DGJM method of solving functional equations, to solve the fractional differential equations, where the order of derivative is taken in Caputo's sense. The iterative procedure is explained and demonstrated by solving non-linear time fractional partial differential equations like Heat equation, Burger's equation, Fokker Planck equation, Korteweg-de Vries (KdV) equation and Klien-Gordon equation. The scheme of iteration is also extended to solve the system of Drinfeld-Sokolov-Wilson equations and coupled Jaulent-Miodek equations. Graphs are used to depict the accuracy of the method and absolute errors between exact and approximate solutions are tabulated to ensure that the proposed scheme is both computationally intriguing and simple to implement.

Keywords. Non-linear fractional differential equations, Series solution, System of fractional differential equations, Decomposition technique

Mathematics Subject Classification (2020). 34A08, 35C10, 35R11

1. Introduction

Fractional calculus has been introduced to describe traditional differentiation and integration of an arbitrary order. The term fractional derivative was first coined by Johann Bernoulli and Benjamin in the early 16th century and its theory has been developed by Lagrange and Laplace in 18th century. N. H. Abel was the one who implemented the theory of fractional calculus to solve integral equations (Valério *et al*. [26]). Meanwhile, G. F. B. Riemann established fractional integration. The properties obtained by them created curiosity among the mathematicians and made them to turn their sights on fractional derivatives. The non-local property of fractional calculus makes it an essential component in providing accurate modelling of physical phenomena and thus find its applications in science and engineering field like elasticity, biomathematics, bioengineering, plasma physics, control systems, fluid and quantum mechanics, optics and so on (Jadhav *et al*. [11], and Sun *et al*. [25]). Consequently, the advancement of methods for solving those fractional equations has become essential and thus various analytical and numerical techniques for finding solutions of such equations have been proposed in literature. The fractional Laplace transform method (Liang *et al*. [17]), Green's function method of fractional integration (Odibat and Momani [22]), matrix method (Shloof *et al*. [24]), the method of orthogonal polynomials (Pu and Fasondini [23]), Adomian decomposition method (Mohammed *et al*. [20]), homotopy perturbation method (HPM) (Javeed *et al*. [12]), variational iteration method (VPM) (Elbeleze *et al*. [7]), homotopy analysis method (Ganjiani [9]) are worth to be mentioned. Moreover, hybrid techniques combining decomposition method and iterative method along with integral transforms like Laplace (LDM) (Bhargava *et al*. [3]), Sumudu transform method (Alomari [1]), Elzaki transform method (Kadhim and Gateataher [14]), Kamal transform method (Johansyah *et al*. [13]), Mohand transforms (Dubey *et al*. [6]) have also been developed. Among the existing methods, the series approximation based methods are considered to be the simplest method for solving fractional differential equations as it avoids many complications such as assumptions and restrictions on variables, choice of multipliers and convolution of functions in obtaining the solutions.

In this paper, we utilize an iterative method of finding the approximate series solution of non-linear time fractional differential equations and systems, where the scheme of evaluation is based on the reduction of the given fractional differential equation to a functional equation of the form $u = f + L(u) + N(u)$, by using the properties of fractional derivatives and integrations, then the reduced equation is approximated by a series of functions generated using DGJM (Daftardar-Gejji and Jafari [5]) iterative scheme. Solutions are given in the form of infinite series, and if the exact solution exists, the acquired series may converge to its closed form. In numerical applications, the truncated series can be applied to concrete problems for which the exact solution is not known.

The rest of the paper is structured as follows: The preliminary components on fractional derivatives are presented in Section 2 followed by an iterative scheme of finding approximate solution of non-linear time fractional differential equations and the system in Section 3. In Section 4, series solution of certain nonlinear differential equations and its systems, of fractional order, are illustrated along with graphs to present the strength and efficiency of the proposed scheme. Finally, conclusions are given in Section 5.

2. Preliminaries

Definition 2.1. The Caputo time-fractional derivative (Balachandran [2]) of order $0 < \alpha \le 1$ taken over a real-valued function $\Phi(\widetilde{x},\widetilde{t})$ is defined as

$$\frac{\partial^\alpha \Phi}{\partial \widetilde{t}^\alpha} = I_t^{1-\alpha}\left(\frac{\partial \Phi}{\partial \widetilde{t}}\right).$$

Definition 2.2. The Riemann-Liouville time fractional integral (Balachandran [2]) of order $\alpha > 0$ of a real-valued function $\Phi(\widetilde{x},\widetilde{t})$ is defined as

$$\frac{\partial^{-\alpha} \Phi}{\partial \widetilde{t}^{-\alpha}} = I_t^\alpha \Phi(\widetilde{x},\widetilde{t}) = \frac{1}{\Gamma(\alpha)}\int_0^t (\widetilde{t}-\widetilde{s})^{(\alpha-1)}\Phi(\widetilde{x},\widetilde{t})\, d\widetilde{s}.$$

Definition 2.3. The Mittag-Leffler function (Li and Hu [16]) of order $\alpha > 0$ of a complex valued function w is defined as

$$E_\alpha(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(k\alpha+1)}, \quad \mathrm{Re}(\alpha) > 0.$$

Theorem 2.1. *If g is a continuously differentiable function, $g \in C^n[a,b]$, $n \in \mathrm{N}$ and if α denotes the order of fractional derivative, then for $n-1 < alpha < n$,* (Mistry [18])

$$I_t^\alpha \left[\frac{d^\alpha g(t)}{dt^\alpha} \right] = g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} t^k.$$

3. Methodology

Let us consider the non-linear time fractional differential equation of the form

$$\frac{\partial^\alpha \Phi}{\partial \tilde{t}^\alpha} + L[\Phi(\tilde{x},\tilde{t})] + N[\Phi(\tilde{x},\tilde{t})] = H(\tilde{x},\tilde{t}), \quad 0 < \alpha \le 1, \tilde{x} \in \mathrm{R}, \tilde{t} > 0, \tag{3.1}$$

$$\Phi(\tilde{x},0) = g(\tilde{x}), \tag{3.2}$$

where L and N denotes the linear and non-linear operators respectively and α denotes the fractional order of derivative taken in Caputo's sense.

To solve eqn. (3.1)-(3.2), we propose an iterative scheme based on a new algorithm of DGJ method (Kumar *et al.* [15]) for solving functional equation $u = f + N(u)$.

To get the functional equation form of eqn. (3.1), we apply $\frac{\partial^{-\alpha}}{\partial \tilde{t}^{-\alpha}}$ on both sides and thus, by using Definition 2.1 and by substituting initial condition (3.2) we get

$$\Phi(\tilde{x},\tilde{t}) = h(\tilde{x},\tilde{t}) + L^*[\Phi(\tilde{x},\tilde{t})] + N^*[\Phi(\tilde{x},\tilde{t})]. \tag{3.3}$$

Here, $h(\tilde{x},\tilde{t})$ represents the homogeneous terms obtained after integration and substitution of initial conditions given and

$$L^*[\Phi(\tilde{x},\tilde{t})] = I_t^\alpha L[\Phi(\tilde{x},\tilde{t})], \quad N^*[\Phi(\tilde{x},\tilde{t})] = I_t^\alpha N[\Phi(\tilde{x},\tilde{t})].$$

For instance, let us assume that $\Phi(\tilde{x},\tilde{t})$ in eqn. (3.3) can be approximated by an infinite series

$$\Phi(\tilde{x},\tilde{t}) = \sum_{r=0}^{\infty} \Phi_r(\tilde{x},\tilde{t}), \tag{3.4}$$

then from eqn. (3.3) and (3.4), we get

$$\left.\begin{aligned}
&\Phi_0(\tilde{x},\tilde{t}) = h(\tilde{x},\tilde{t}),\\
&\sum_{r=0}^{1} \Phi_r(\tilde{x},\tilde{t}) = h(\tilde{x},\tilde{t}) + L^*[\Phi_0(\tilde{x},\tilde{t})] + N^*[\Phi_0(\tilde{x},\tilde{t})],\\
&\sum_{r=0}^{2} \Phi_r(\tilde{x},\tilde{t}) = h(\tilde{x},\tilde{t}) + L^*[(\Phi_0(\tilde{x},\tilde{t}) + \Phi_1(\tilde{x},\tilde{t})] + N^*[(\Phi_0(\tilde{x},\tilde{t}) + \Phi_1(\tilde{x},\tilde{t})],\\
&\qquad\qquad \vdots\\
&\sum_{r=0}^{k} \Phi_r(\tilde{x},\tilde{t}) = h(\tilde{x},\tilde{t}) + L^*[(\Phi_0(\tilde{x},\tilde{t}) + \Phi_1(\tilde{x},\tilde{t}) + \dots \Phi_{k-1}(\tilde{x},\tilde{t})]\\
&\qquad\qquad + N^*[(\Phi_0(\tilde{x},\tilde{t}) + \Phi_1(\tilde{x},\tilde{t}) + \dots \Phi_{k-1}(\tilde{x},\tilde{t})],\\
&\qquad\qquad \vdots
\end{aligned}\right\} \tag{3.5}$$

Let us represent the series approximation with first $(k+1)$ terms by v_k, that is,

$$v_k = \sum_{r=0}^{k} \Phi_r(\widetilde{x},\widetilde{t}). \tag{3.6}$$

Hence, by combining eqn. (3.5) and (3.6), we get the simplest iterative scheme of finding solution as follows:

$$\left.\begin{aligned} v_0 &= \Phi_0(\widetilde{x},\widetilde{t}) = h(\widetilde{x},\widetilde{t}), \\ v_k &= \sum_{r=0}^{k} \Phi_r(\widetilde{x},\widetilde{t}) = h(\widetilde{x},\widetilde{t}) + L^*[v_{k-1}] + N^*[v_{k-1}], \quad \text{for } k \geq 1. \end{aligned}\right\} \tag{3.7}$$

As, $k \to \infty$, $v_k \to \Phi(\widetilde{x},\widetilde{t})$, which is the required solution of equation (3.1).

Theorem 3.1 (Condition for convergence)**.** *If* L^*, N^* *given in eqn.* (3.3) *are continuously differentiable functions defined on the Banach space* $\mathcal{B}$*, whose derivatives are bounded by the values*

$$\|DL^*\| = \max_{\|\Phi\|=1} \|DL^*(\Phi(\widetilde{x},\widetilde{t}))\| \leq M_L,$$

$$\|DN^*\| = \max_{\|\Phi\|=1} \|DN^*(\Phi(\widetilde{x},\widetilde{t}))\| \leq M_N,$$

then the sequence of iterated values v_k *in* (3.7) *converges uniformly to the solution function* $\Phi(\widetilde{x},\widetilde{t})$*, as* $k \to \infty$*, whenever* $0 < M = (M_L + M_N) < 1$.

Proof. Consider,

$$\begin{aligned} \|v_k\| &= \left\| v_0 + \sum_{n=0}^{k-1} (v_{n+1} - v_n) \right\| \\ &\leq \|v_0\| + \sum_{n=0}^{k-1} \|(v_{n+1} - v_n)\|. \end{aligned}$$

By mean value theorem for Banach spaces $\mathcal{B}$ (Ciarlet [4]),

$$\begin{aligned} \|v_{n+1} - v_n\| &= \|(L^*[v_n] - L^*[v_{n-1}]) + (N^*[v_n] - N^*[v_{n-1}])\| \\ &\leq \|DL^*\| \, \|v_n - v_{n-1}\| + \|DN^*\| \, \|v_n - v_{n-1}\| \\ &\leq (M_L + M_N)\|v_n - v_{n-1}\|, \quad \forall\ n = 0, 1, \cdots, k-1. \end{aligned}$$

Let $M = (M_L + M_N)$ with $0 < M < 1$.
Then,

$$\begin{aligned} \|v_{n+1} - v_n\| &\leq M\|v_n - v_{n-1}\| \\ &\leq M^n \|v_1 - v_0\|, \end{aligned}$$

$$\implies \qquad \|v_k\| \leq \|v_0\| + \sum_{n=0}^{k-1} M^n \|v_1 - v_0\|$$

As $k \to \infty$, the series $\sum_{n=0}^{\infty} M^n \|v_1 - v_0\|$ converges and by Weistrass M-test, the iterated values v_k in (3.7) converges uniformly to the solution $\Phi(\widetilde{x},\widetilde{t})$ given in (3.3).

Note. The condition mentioned in Theorem 3.1 is only sufficient for the convergence of the method.

3.1 For the System of Fractional Differential Equations

Suppose that, we have a system of m fractional differential equations

$$F_i(\Phi_i(\tilde{x},\tilde{t})) = H_i(\tilde{x},\tilde{t}), \quad 1 \le i \le m. \tag{3.8}$$

To find the set of approximate solutions $\{\Phi_i\}$ for the system (3.8), the iterative scheme (3.7) will be modified as:

$$\left.\begin{aligned} v_{i0} &= \Phi_{i0}(\tilde{x},\tilde{t}) = h_i(\tilde{x},\tilde{t}), \\ v_{ik} &= \sum_{r=0}^{k} \Phi_{ir}(\tilde{x},\tilde{t}) = h_i(\tilde{x},\tilde{t}) + L_i^*[v_{ik-1}] + N_i^*[v_{ik-1}], \quad \text{for } k \ge 1, \end{aligned}\right\} \tag{3.9}$$

where $h_i(\tilde{x},\tilde{t})$, N_i^* and L_i^* denotes the homogeneous term, non-linear and linear differential operators, respectively, obtained from the functional equation form

$$\Phi_i(\tilde{x},\tilde{t}) = h_i(\tilde{x},\tilde{t}) + L_i^*[\Phi_1,\Phi_2,\cdots,\Phi_m] + N_i^*[\Phi_1,\Phi_2,\cdots,\Phi_m], \tag{3.10}$$

corresponding to the ith differential equation of the given system (3.8).

4. Applications

This section covers examples that demonstrate the method of solving nonlinear time fractional differential equations using iterative scheme proposed in Section 3.

4.1 Fractional Partial Differential Equations

Example 4.1. Consider the time fractional Heat equation,

$$\frac{\partial^a u}{\partial t^a} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < a \le 1, 0 < x < \pi, \tag{4.1}$$

$$u(x,0) = \sin(x). \tag{4.2}$$

Integrating both sides of eqn. (4.1) with respect to t and substituting (4.2), we get

$$u = \sin(x) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{(a-1)} \frac{\partial^2 u}{\partial x^2}\, ds.$$

Using iterative scheme (3.7),

$$\begin{aligned} v_0 &= u_0(x,t) = \sin(x), \\ v_1 &= u_0(x,t) + u_1(x,t) = v_0 + L^*[v_0] = \sin(x) - \frac{t^a \sin(x)}{\Gamma(a+1)} = \sin(x)\left(1 - \frac{t^a}{\Gamma(a+1)}\right), \\ v_2 &= u_0(x,t) + u_1(x,t) + u_2(x,t) = v_0 + L^*[v_1] = \sin(x) - \frac{t^a \sin(x)}{\Gamma(a+1)} + \frac{t^{2a}\sin(x)}{\Gamma(2a+1)} \\ &= \sin(x)\left(1 - \frac{t^a}{\Gamma(a+1)} + \frac{t^{2a}}{\Gamma(2a+1)}\right), \\ &\vdots \end{aligned}$$

As $k \to \infty$, the series v_k converges to the infinite series $\sin(x)\left(\sum_{k=0}^{\infty} \frac{(-t)^{ka}}{\Gamma(ka+1)}\right)$. Hence, the closed-form solution of given differential equation (4.1)-(4.2) is $u(x,t) = \sin(x)E_a(-t)$, where $E_a(\cdot)$ denotes the Mittag-Leffler function of order a.

Example 4.2. Consider the nonlinear time fractional Focker-Planck equation,

$$\frac{\partial^a u}{\partial t^a} - \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) = 0, \quad t > 0, x \in \mathrm{R}, 0 < a \le 1, \tag{4.3}$$

$$u(x,0) = x^2. \tag{4.4}$$

Integrating both sides of eqn. (4.3) with respect to t and substituting (4.4) we get,

$$u = x^2 + \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\frac{\partial}{\partial x}\left(\frac{xu}{3}\right)\,ds + \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\left(\frac{\partial^2 u^2}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{4u^2}{x}\right)\right)\,ds.$$

Using iterative scheme (3.7),

$$v_0 = u_0(x,t) = x^2,$$

$$v_1 = v_0 + L^*[v_0] + N^*[v_0] = x^2 + \frac{x^2 t^a}{\Gamma(a+1)},$$

$$v_2 = v_0 + L^*[v_1] + N^*[v_1] = x^2 + \frac{x^2 t^a}{\Gamma(a+1)} + \frac{x^2 t^{2a}}{\Gamma(2a+1)},$$

$$v_3 = v_0 + L^*[v_2] + N^*[v_2] = x^2 + \frac{x^2 t^a}{\Gamma(a+1)} + \frac{x^2 t^{2a}}{\Gamma(2a+1)} + \frac{x^2 t^{3a}}{\Gamma(3a+1)},$$

$$\vdots$$

This series v_k converges to the infinite series $x^2\left(\sum_{k=0}^{\infty}\frac{t^{ka}}{\Gamma(ka+1)}\right)$, which is similar to the one given in Example 4.3 of the article [19], where Mofarreh *et al.* use HPM combined with Elzaki transform to solve Focker-Planck equation.

Hence, the solution of eqn. (4.3)-(4.4) is $u(x,t) = x^2 E_a(t)$. A graphical simulation of solution curves obtained for different order of derivatives $a = \{0.25, 0.5, 0.75, 1\}$ is given in Figure 1. The figure shows a natural flow of surface that converge to the exact solution with an increase in order of derivative, demonstrating the efficiency of our method.

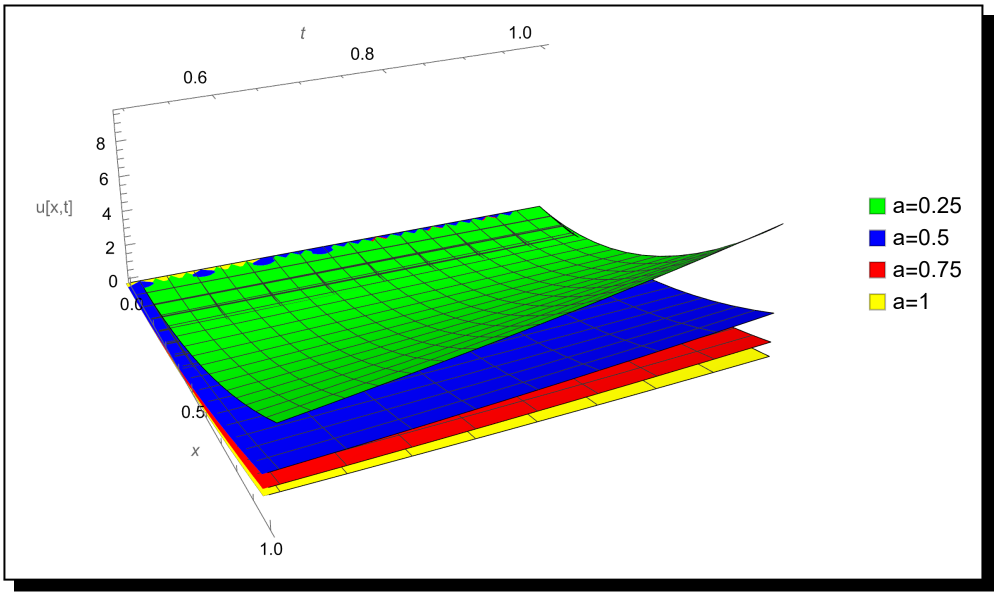


Figure 1. Approximate solution for different order of derivatives of the eqn. (4.3)-(4.4)

Example 4.3. Consider the nonlinear time fractional Burger's equation,

$$\frac{\partial^a u}{\partial t^a} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0, \quad 0 < a \le 1, \tag{4.5}$$

$$u(x,0) = x. \tag{4.6}$$

The exact solution of eqn.(4.5)-(4.6) for $a=1$ is $u=\frac{x}{1+t}$.
Integrating both sides of eqn. (4.5) with respect to t and substituting (4.6) we get,

$$u=x+\frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\frac{\partial^2 u}{\partial x^2}\,ds-\frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}u\frac{\partial u}{\partial x}\,ds.$$

Using iterative scheme (3.7),

$$\begin{aligned}
v_0&=u_0(x,t)=x,\\
v_1&=v_0+L^*[v_0]+N^*[v_0]=x-\frac{xt^a}{\Gamma(a+1)},\\
v_2&=v_0+L^*[v_1]+N^*[v_1]=x-\frac{xt^a}{\Gamma(a+1)}+\frac{2xt^{2a}}{\Gamma(2a+1)}-\frac{4^a xt^{3a}\Gamma\left(a+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(a+1)\Gamma(3a+1)},\\
v_3&=v_0+L^*[v_2]+N^*[v_2]\\
&=x-\frac{xt^a}{\Gamma(a+1)}+\frac{2xt^{2a}}{\Gamma(2a+1)}-\frac{2xt^{3a}\left(2a\Gamma(a)^2+\Gamma(2a)\right)}{3\Gamma(3a)\Gamma(a+1)^2}+\frac{xt^{4a}\left(2\Gamma(2a)^2+3\Gamma(a)\Gamma(3a)\right)}{2\Gamma(2a)\Gamma(4a)\Gamma(a+1)^2}+\cdots,\\
&\ \ \vdots
\end{aligned}$$

The obtained result is exactly same as the one proposed by Horan *et al.* [10] using LDM, and for $a=1$, we have

$$\begin{aligned}
v_0&=x,\\
v_1&=x(1-t),\\
v_2&=x-tx+t^2x-\frac{t^3x}{3}=x(1-t+t^2+\cdots),\\
v_3&=x-tx+t^2x-t^3x+\frac{2t^4x}{3}-\frac{t^5x}{3}+\frac{t^6x}{9}-\frac{t^7x}{63}=x(1-t+t^2-t^3+\cdots).
\end{aligned}$$

Hence for $a=1$, we have, $u(x,t)=\lim_{k\to\infty}v_k=\frac{x}{1+t}$, which is the exact solution of eqn. (4.5)-(4.6). Graphical simulation of approximate solution v_3 is plotted for different order of derivatives $a=\{0.25,0.5,0.75,1\}$ in Figure 2. Moreover, the flow of curve at a fixed position $x=5$ is also depicted between the time interval $0<t<10$ for different order of derivatives a.

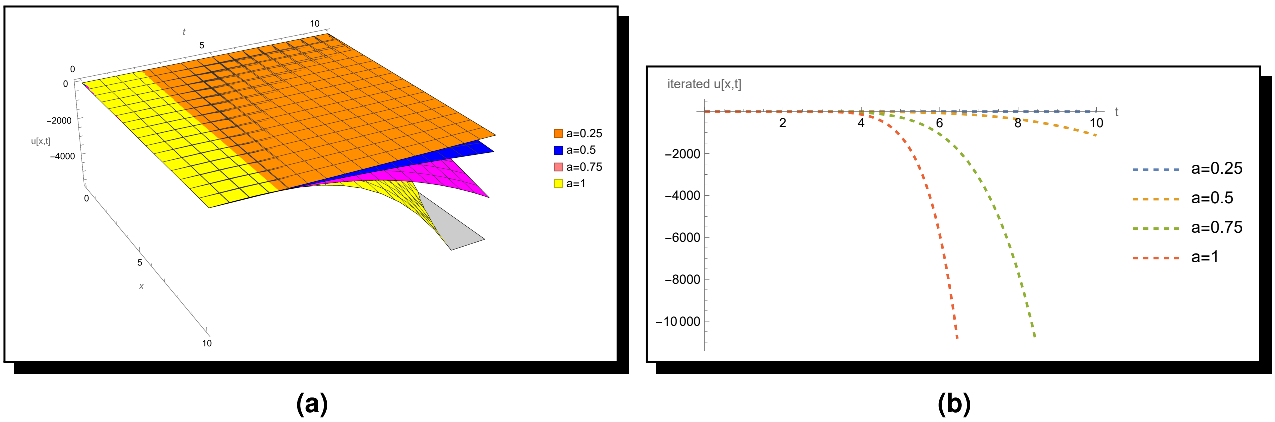


Figure 2. Approximate solution (a) 3D, (b) 2D for different order of derivatives of the eqn. (4.5)-(4.6)

Example 4.4. Consider the time fractional Klien-Gordan equation,

$$\frac{\partial^a u}{\partial t^a}-\frac{\partial^2 u}{\partial x^2}+u=2\sin(x),\quad 1<a\le 2, \tag{4.7}$$

$$u(x,0) = \sin(x), \quad u_t(x,0) = 1. \tag{4.8}$$

The exact solution of eqn. (4.7)-(4.8) for $a = 1$ is $u = \sin(x) + \sin(t)$.
Integrating both sides of eqn. (4.7) with respect to t and substituting (4.8) we get,

$$u = \sin(x) + t + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left[2\sin(x) + \frac{\partial^2 u}{\partial x^2} - u \right] ds.$$

Using iterative scheme (3.7),

$$v_0 = u_0(x,t) = t + \sin(x),$$

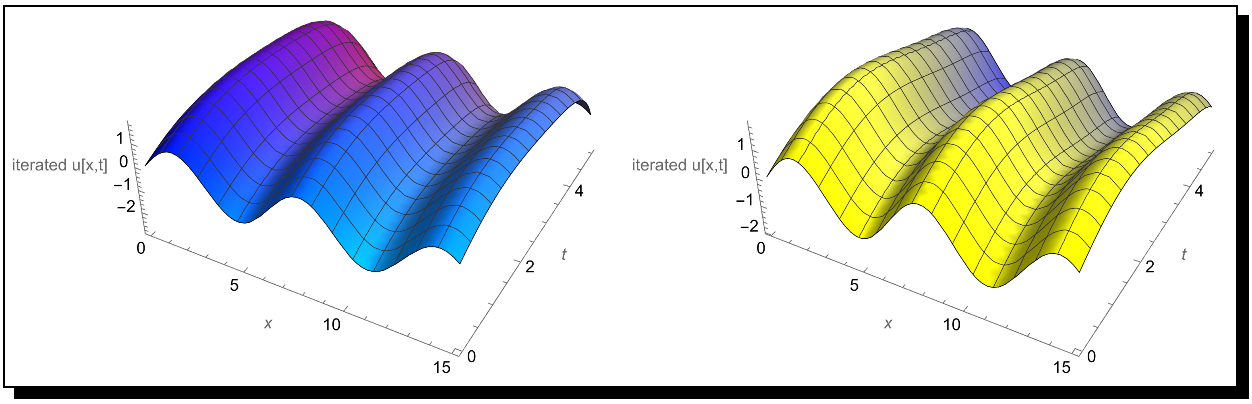
$$v_1 = v_0 + L^*[v_0] = t - \frac{t^{a+1}}{\Gamma(a+2)} + \sin(x),$$

$$v_2 = v_0 + L^*[v_1] = t - \frac{t^{a+1}}{\Gamma(a+2)} + \frac{t^{2a+1}}{\Gamma(2a+2)} + \sin(x),$$

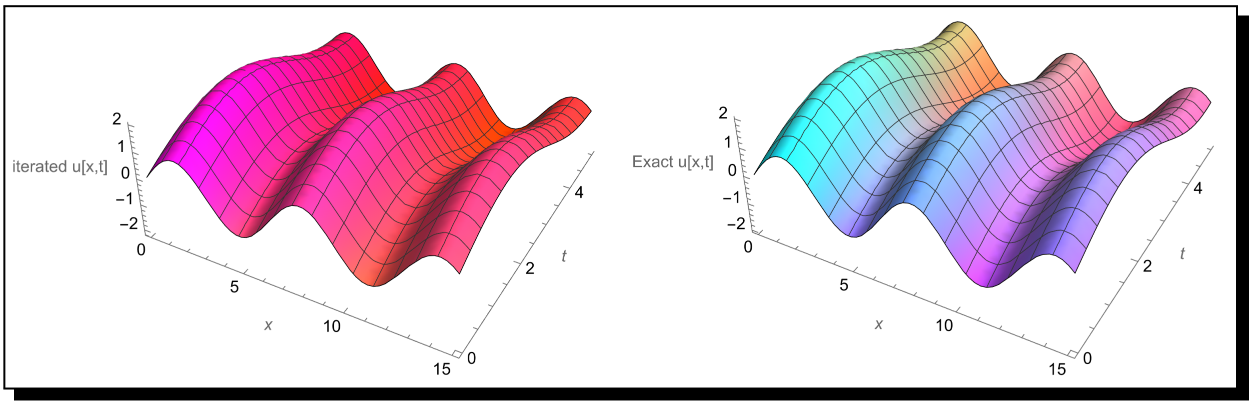
$$v_3 = v_0 + L^*[v_2] = t - \frac{t^{a+1}}{\Gamma(a+2)} + \frac{t^{2a+1}}{\Gamma(2a+2)} - \frac{t^{3a+1}}{\Gamma(3a+2)} + \frac{t^{4a+1}}{\Gamma(4a+2)} - \frac{t^{5a+1}}{\Gamma(5a+2)} + \sin(x),$$

$$\vdots$$

We see that for $a = 2$, the series v_k converges to the exact solution $u(x,t) = \sin(x) + \sin(t)$, as $k \to \infty$ and closely match with the one obtained by Mohyud-Din and Yildirim [21] via VIM.



(a) Approximate solution for $a = 1.5$ and $a = 1.75$



(b) Approximate and the exact solution for $a = 2$

Figure 3. Graphical simulation of Approximate solution and exact solution for the eqn. (4.7)-(4.8)

Graphical simulation of approximate solution v_3 for different order of derivatives $a = \{1.5, 1.75, 2\}$ and the exact solution for $a = 2$ is depicted in Figure 3. From figure, we find that as the order of derivative increases, the solution set tends to have the same structure as the exact one, which demonstrates the effectiveness of the proposed technique.

Example 4.5. Consider the nonlinear time fractional Korteweg-de-Vries (KdV) equation,

$$\frac{\partial^a u}{\partial t^a} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0, \quad 0 < a \le 1, \tag{4.9}$$

$$u(x,0) = 6x. \tag{4.10}$$

The exact solution of eqn. (4.9)-(4.10) for $a = 1$ is $u = \frac{6x}{1-36t}$.

Integrating both sides of eqn. (4.9) with respect to t and substituting (4.10) we get,

$$u = x - \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\frac{\partial^3 u}{\partial x^3}ds + \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\left[6u\frac{\partial u}{\partial x}\right]ds.$$

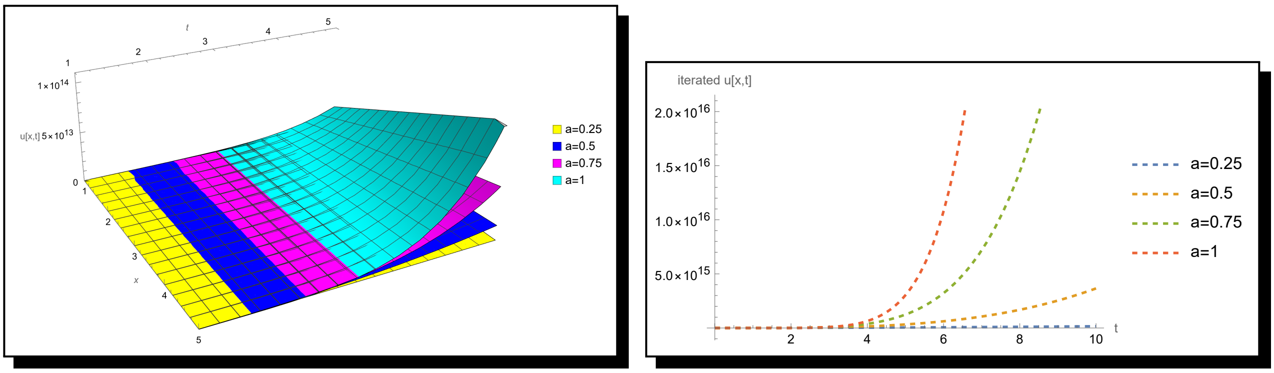
Using iterative scheme (3.7),

$$\begin{aligned}
v_0 &= u_0(x,t) = 6x,\\
v_1 &= v_0 + L^*[v_0] + N^*[v_0] = 6x + \frac{216xt^a}{\Gamma(a+1)},\\
v_2 &= v_0 + L^*[v_1] + N^*[v_1] = 6x + \frac{216xt^a}{\Gamma(a+1)} + \frac{15552xt^{2a}}{\Gamma(2a+1)} + \frac{2187\ 2^{2a+7}xt^{3a}\Gamma\left(a+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(a+1)\Gamma(3a+1)},\\
v_3 &= v_0 + L^*[v_2] + N^*[v_2]\\
&= 6x + \frac{216xt^a}{\Gamma(a+1)} + \frac{15552xt^{2a}}{\Gamma(2a+1)} + 93312xt^{3a}\left(\frac{4^a\Gamma\left(a+\frac{1}{2}\right)}{\sqrt{\pi}a^2\Gamma(a)\Gamma(3a)} + \frac{12}{\Gamma(3a+1)}\right)\\
&\quad + \frac{10077696xt^{4a}(\Gamma(2a+1)^2 + 2\Gamma(a+1)\Gamma(3a+1))}{a\Gamma(2a)\Gamma(a+1)^2\Gamma(4a+1)}\\
&\quad + \frac{59049\ 2^{2a+13}xt^{5a}\left(34^a\Gamma(3a)\Gamma\left(2a+\frac{1}{2}\right)\Gamma(a+1)^2 + 2\Gamma(4a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(2a+1)\right)}{\sqrt{\pi}\Gamma(3a)\Gamma(a+1)^2\Gamma(2a+1)\Gamma(5a+1)}\\
&\quad + \frac{14511882240xt^{6a}\Gamma(5a)}{a^3\Gamma(a)^2\Gamma(3a)\Gamma(6a)} + \frac{4782969\ 2^{10a+15}xt^{7a}\Gamma\left(a+\frac{1}{2}\right)^2\Gamma\left(3a+\frac{1}{2}\right)}{\pi^{\frac{3}{2}}a^3\Gamma(a)^2\Gamma(3a)\Gamma(7a+1)},\\
&\vdots
\end{aligned}$$

For $a = 1$, we have

$$\begin{aligned}
v_0 &= 6x,\\
v_1 &= 6x + 216tx = 6x(1+36t),\\
v_2 &= 6x + 216tx + 7776t^2x + 279936t^3x = 6x(1+36t+(36t)^2+\cdots),\\
v_3 &= 6x + 216tx + 7776t^2x + 279936t^3x + 6718464t^4x + 120932352t^5x + 1451188224t^6x\\
&= 6x(1+36t+(36t)^2+(36t)^3+\cdots).
\end{aligned}$$

Hence for $a = 1$, we have, $u(x,t) = \lim_{k\to\infty} v_k = \frac{6x}{1-36t}$, which is same as the exact solution of eqn. (4.9)-(4.10). Graphical simulation of approximate solution v_3 is plotted for different order of derivatives $a = \{0.25, 0.5, 0.75, 1\}$ in Figure 4. The 2D plot of solution curve in time $0 < t < 10$ at fixed $x = 5$ for different values of a shows that the curve approaches to its exact structure with increase in the order of derivative.



(a) (b)

Figure 4. Approximate solution (a) 3D, (b) 2D for different order of derivatives of the eqn. (4.9)-(4.10)

4.2 System of Fractional Differential Equations

Example 4.6. Consider the system of Drinfeld-Sokolov-Wilson equation,

$$\left.\begin{aligned} &\frac{\partial^a \phi_1}{\partial t^a} + 3\phi_2 \frac{\partial \phi_2}{\partial x} = 0, && 0 < a \le 1,\\ &\frac{\partial^a \phi_2}{\partial t^a} + 2\frac{\partial^3 \phi_2}{\partial x^3} + \phi_2 \frac{\partial \phi_1}{\partial x} + 2\phi_1 \frac{\partial \phi_2}{\partial x} = 0, && 0 < a \le 1, \end{aligned}\right\} \tag{4.11}$$

$$\left.\begin{aligned} &\phi_1(x,0) = \tfrac{3c}{2}\operatorname{sech}^2\left(\sqrt{\tfrac{c}{2}}x\right),\\ &\phi_2(x,0) = c\operatorname{sech}\left(\sqrt{\tfrac{c}{2}}x\right). \end{aligned}\right\} \tag{4.12}$$

The exact solution of system (4.11)-(4.12) for $a = 1$ is

$$\phi_1(x,t) = \frac{3c}{2}\operatorname{sech}^2\left(\sqrt{\frac{c}{2}}(x-ct)\right),$$

$$\phi_2(x,t) = c\operatorname{sech}\left(\sqrt{\frac{c}{2}}(x-ct)\right).$$

On integrating both sides of eqn. (4.11) with respect to t and using initial conditions (4.12), we get

$$\phi_1(x,t) = \frac{3c}{2}\operatorname{sech}^2\left(\sqrt{\frac{c}{2}}x\right) - \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\left[3\phi_2\frac{\partial \phi_2}{\partial x}\right] ds,$$

$$\begin{aligned}\phi_2(x,t) = c\operatorname{sech}\left(\sqrt{\frac{c}{2}}x\right) &- \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\left[2\frac{\partial^3 \phi_2}{\partial x^3}\right] ds\\ &- \frac{1}{\Gamma(a)}\int_0^t (t-s)^{a-1}\left[\phi_2\frac{\partial \phi_1}{\partial x} + 2\phi_1\frac{\partial \phi_2}{\partial x}\right] ds.\end{aligned}$$

Using iterative scheme (3.9),

$$v_{10} = \phi_{10}(x,t) = h_1^*(x,t) = \frac{3c}{2}\operatorname{sech}^2\left(\sqrt{\frac{c}{2}}x\right),$$

$$v_{20} = \phi_{20}(x,t) = h_2^*(x,t) = c\operatorname{sech}\left(\sqrt{\frac{c}{2}}x\right),$$

$$v_{11} = \phi_{10} + \phi_{11} = h_1^* + L_1^*(v_{10}, v_{20}) + N_1^*(v_{10}, v_{20}) = \frac{3c}{2}\operatorname{sech}^2\left(\sqrt{\frac{c}{2}}x\right) + \frac{3c^{\frac{5}{2}} t^a \tanh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\operatorname{sech}^2\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)}{\sqrt{2}\Gamma(a+1)},$$

$$v_{21}=\phi_{20}+\phi_{21}=h_2^*+L_2^*(v_{10},v_{20})+N_2^*(v_{10},v_{20})=c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}x\right)+\frac{c^{\frac{5}{2}}t^a\tanh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\mathrm{sech}\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)}{\sqrt{2}\Gamma(a+1)},$$

$$v_{12}=\phi_{10}+\phi_{11}+\phi_{12}=h_1^*+L_1^*(v_{11},v_{21})+N_1^*(v_{11},v_{21})=\frac{3c}{2}\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}x\right)+\frac{3c^{\frac{5}{2}}t^a\tanh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\mathrm{sech}^2\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)}{\sqrt{2}\Gamma(a+1)}$$

$$+\frac{3}{2}c^4t^{2a}\,\mathrm{sech}^4\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\left(\frac{2^{2a-\frac{3}{2}}c^{\frac{3}{2}}t^a\Gamma\left(a+\frac{1}{2}\right)\left(\sinh(\sqrt{2}\sqrt{cx})-4\tanh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\right)}{\sqrt{\pi}\Gamma(a+1)\Gamma(3a+1)}+\frac{\cosh(\sqrt{2}\sqrt{cx})-2}{\Gamma(2a+1)}\right),$$

$$v_{22}=\phi_{20}+\phi_{21}+\phi_{22}=h_2^*+L_2^*(v_{11},v_{21})+N_2^*(v_{11},v_{21})=c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}x\right)+\frac{c^{\frac{5}{2}}t^a\tanh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\mathrm{sech}\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)}{\sqrt{2}\Gamma(a+1)}$$

$$+\frac{c^4t^{2a}\,\mathrm{sech}^3\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\left(\frac{3\sqrt{2}c^{\frac{3}{2}}t^a\Gamma(2a+1)^2\left(\sinh\left(\frac{3\sqrt{cx}}{\sqrt{2}}\right)-6\sinh\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)\right)\mathrm{sech}^3\left(\frac{\sqrt{cx}}{\sqrt{2}}\right)}{\Gamma(a+1)^2\Gamma(3a+1)}+\cosh\left(\sqrt{2}\sqrt{cx}\right)-3\right)}{4\Gamma(2a+1)},$$

$$\vdots$$

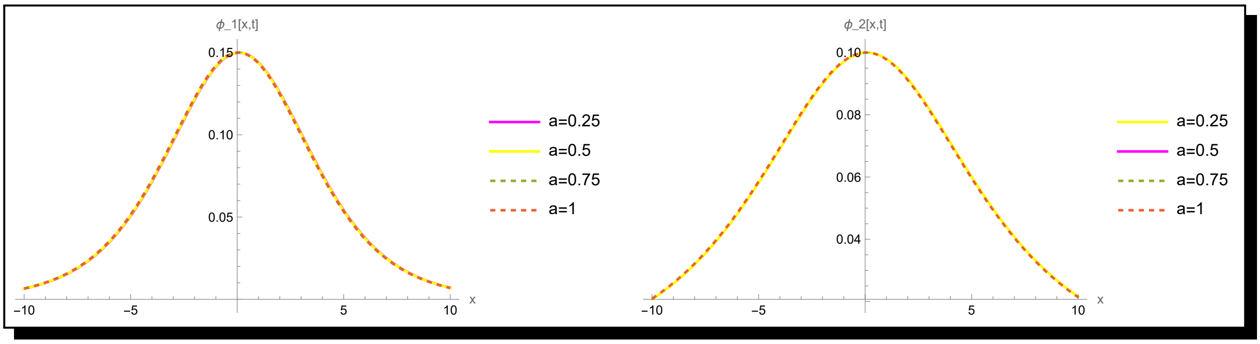


Figure 5. Approximate solution at $t=0.5, c=0.1$ and for different values of a for the eqn. (4.11)-(4.12)

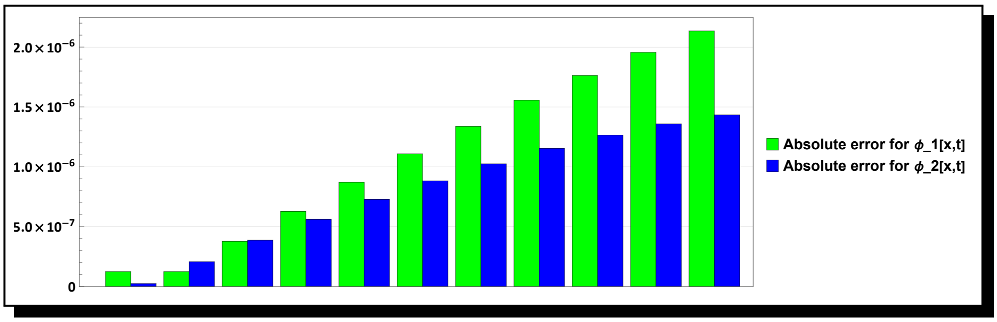
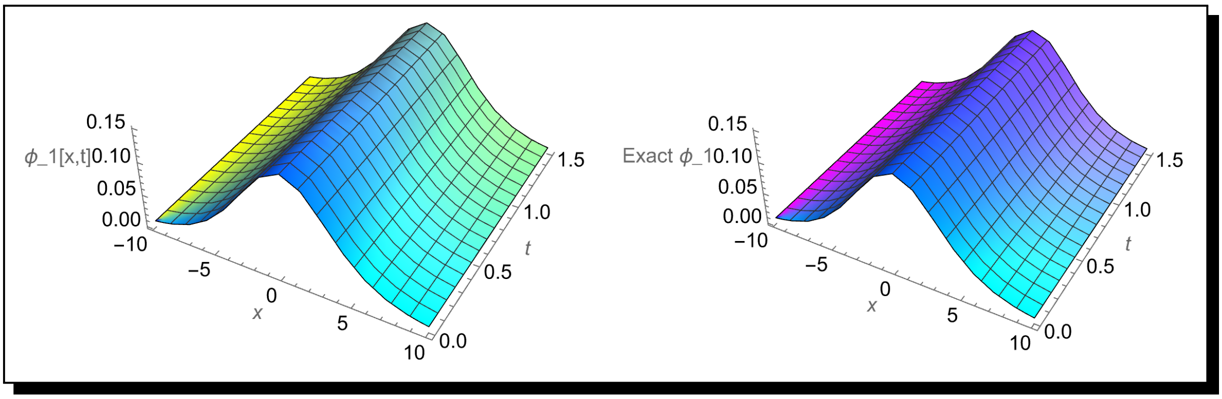


Figure 6. Absolute errors of ϕ_1 and ϕ_2 for $c=0.1$, $a=1$ of the eqn. (4.11)-(4.12)

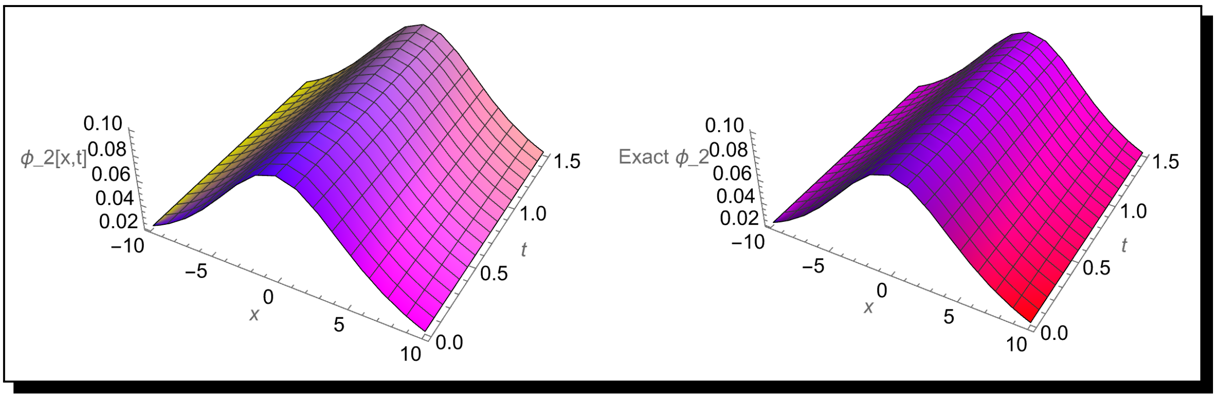
The approximate solution of the system and exact solutions are plotted in Figure 7 along with the absolute errors in Figure 6. Also, the solution curves at fixed time $t=0.5$ and for constant $c=0.1$ is plotted between $-10<x<10$ for different values of a in Figure 5. The exact and approximated values are listed in Table 1 and from the absolute error, we observe that the solution obtained with few iterative steps have closer approximation to the exact solution of the system. Furthermore, the same system is analysed by Ganie *et al*. [8] through Aboodh transform method. A close resemblance seen between the results obtained using the respective techniques shows the reliability of the proposed method.

Table 1. Solution obtained at time $t = 0.5$, $c = 0.1$, $a = 1$ for the eqn. (4.11)-(4.12)

x	$\phi_1(x,t)$	Exact ϕ_1	$\phi_2(x,t)$	Exact ϕ_2	Absolute error of ϕ_1	Absolute error of ϕ_2
0	0.1498310	0.1498310	0.0999438	0.0999438	0.0000000	0.0000000
0.1	0.1499810	0.1499810	0.0999935	0.0999938	0.0000000	0.0000003
0.2	0.1499820	0.1499810	0.0999934	0.0999938	0.0000010	0.0000004
0.3	0.1498320	0.1498310	0.0999432	0.0999438	0.0000010	0.0000006
0.4	0.1495330	0.1495320	0.0998432	0.0998440	0.0000010	0.0000008
0.5	0.1490860	0.1490850	0.0996936	0.0996945	0.0000010	0.0000009
0.6	0.1484930	0.1484910	0.0994949	0.0994959	0.0000020	0.0000010
0.7	0.1477550	0.1477540	0.0992473	0.0992485	0.0000010	0.0000012
0.8	0.1468770	0.1468750	0.0989517	0.0989530	0.0000020	0.0000013
0.9	0.1458610	0.1458590	0.0986087	0.0986100	0.0000020	0.0000013
1	0.1447110	0.1447090	0.0982191	0.0982205	0.0000020	0.0000014



(a) Approximate and the exact solution ϕ_1 for $c = 0.1$, $a = 1$



(b) Approximate and the exact solution ϕ_2 for $c = 0.1$, $a = 1$

Figure 7. Graphical simulation of (a) ϕ_1, and (b) ϕ_2 of the eqn. (4.11)-(4.12)

Example 4.7. Consider the coupled Jaulent-Miodek equations,

$$\left.\begin{aligned}&\frac{\partial^a\phi_1}{\partial t^a}+\frac{\partial^3\phi_1}{\partial x^3}+\frac{3}{2}\phi_2\frac{\partial^3\phi_2}{\partial x^3}+\frac{9}{2}\frac{\partial\phi_2}{\partial x}\frac{\partial^2\phi_2}{\partial x^2}-\frac{3}{2}\phi_2^2\frac{\partial\phi_1}{\partial x}-6\phi_1\phi_2\frac{\partial\phi_2}{\partial x}-6\phi_1\frac{\partial\phi_1}{\partial x}=0, && 0<a\le 1,\\ &\frac{\partial^a\phi_2}{\partial t^a}+\frac{\partial^3\phi_2}{\partial x^3}-6\phi_2\frac{\partial\phi_1}{\partial x}-6\phi_1\frac{\partial\phi_2}{\partial x}-\frac{15}{2}\phi_2^2\frac{\partial\phi_2}{\partial x}=0, && 0<a\le 1,\end{aligned}\right\} \tag{4.13}$$

$$\left.\begin{aligned}&\phi_1(x,0)=\frac{c^2}{8}\left(1-4\,\mathrm{sech}^2\left(\frac{cx}{2}\right)\right),\\ &\phi_2(x,0)=c\,\mathrm{sech}\left(\frac{cx}{2}\right).\end{aligned}\right\} \tag{4.14}$$

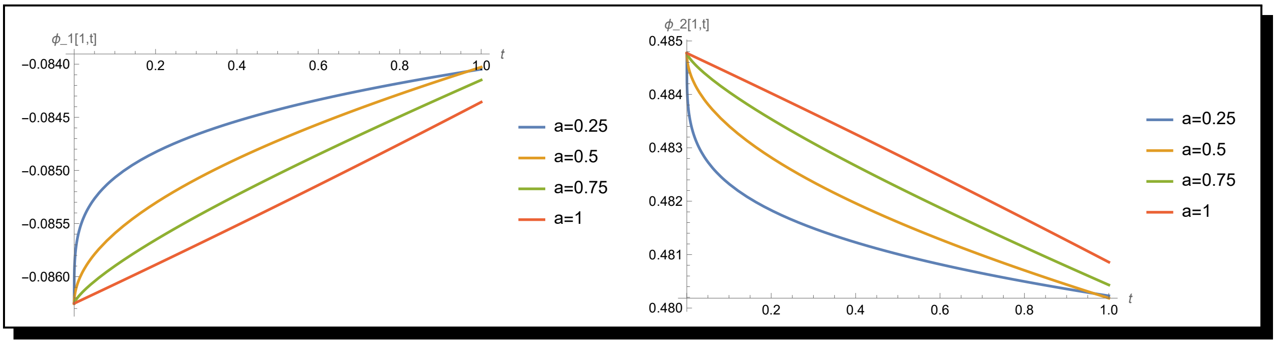


Figure 8. Approximate solution at $c=0.5$, $x=1$ and for different values of a for the eqn. (4.13)-(4.14)

The exact solution of system (4.13)-(4.14) for $a=1$ is

$$\phi_1(x,t)=\frac{c^2}{8}\left(1-4\,\mathrm{sech}^2\left(\frac{c}{2}\left(\frac{c^2t}{2}+x\right)\right)\right),$$

$$\phi_2(x,t)=c\,\mathrm{sech}\left(\frac{c}{2}\left(\frac{c^2t}{2}+x\right)\right).$$

On integrating both sides of eqn. (4.13) with respect to t and using initial conditions (4.14), we get

$$\begin{aligned}\phi_1(x,t)=&\frac{c^2}{8}\left(1-4\,\mathrm{sech}^2\left(\frac{cx}{2}\right)\right)-\frac{1}{\Gamma(a)}\int_0^t(t-s)^{a-1}\left[\frac{\partial^3\phi_1}{\partial x^3}\right]ds\\ &+\frac{1}{\Gamma(a)}\int_0^t(t-s)^{a-1}\left[\frac{3}{2}\phi_2^2\frac{\partial\phi_1}{\partial x}+6\phi_1\phi_2\frac{\partial\phi_2}{\partial x}+6\phi_1\frac{\partial\phi_1}{\partial x}-\frac{3}{2}\phi_2\frac{\partial^3\phi_2}{\partial x^3}-\frac{9}{2}\frac{\partial\phi_2}{\partial x}\frac{\partial^2\phi_2}{\partial x^2}\right]ds,\\ \phi_2(x,t)=&c\,\mathrm{sech}\left(\frac{cx}{2}\right)-\frac{1}{\Gamma(a)}\int_0^t(t-s)^{a-1}\left[\frac{\partial^3\phi_2}{\partial x^3}\right]ds\\ &+\frac{1}{\Gamma(a)}\int_0^t(t-s)^{a-1}\left[6\phi_2\frac{\partial\phi_1}{\partial x}+6\phi_1\frac{\partial\phi_2}{\partial x}+\frac{15}{2}\phi_2^2\frac{\partial\phi_2}{\partial x}\right]ds.\end{aligned}$$

Using iterative scheme (3.9),

$$v_{10}=\phi_{10}(x,t)=h_1^*(x,t)=\frac{c^2}{8}\left(1-4\,\mathrm{sech}^2\left(\frac{cx}{2}\right)\right),$$

$$v_{20}=\phi_{20}(x,t)=h_2^*(x,t)=c\,\mathrm{sech}\left(\frac{cx}{2}\right),$$

$$\begin{aligned}v_{11}&=\phi_{10}+\phi_{11}=h_1^*+L_1^*(v_{10},v_{20})+N_1^*(v_{10},v_{20})\\ &=\frac{c^2}{8}\left(1-4\,\mathrm{sech}^2\left(\frac{cx}{2}\right)\right)+\frac{c^5t^a\tanh\left(\frac{cx}{2}\right)\mathrm{sech}^2\left(\frac{cx}{2}\right)}{4\Gamma(a+1)},\end{aligned}$$

$$v_{21} = \phi_{20} + \phi_{21} = h_2^* + L_2^*(v_{10}, v_{20}) + N_2^*(v_{10}, v_{20}) = c\,\mathrm{sech}\left(\frac{cx}{2}\right) - \frac{c^4 t^a \tanh\left(\frac{cx}{2}\right)\mathrm{sech}\left(\frac{cx}{2}\right)}{4\Gamma(a+1)},$$

$$\begin{aligned}
v_{12} &= \phi_{10} + \phi_{11} + \phi_{12} = h_1^* + L_2^*(v_{11}, v_{21}) + N_2^*(v_{11}, v_{21}) \\
&= \frac{c^2}{8}\left(1 - 4\,\mathrm{sech}^2\left(\frac{cx}{2}\right)\right) + \frac{c^5 t^a \tanh\left(\frac{cx}{2}\right)\mathrm{sech}^2\left(\frac{cx}{2}\right)}{4\Gamma(a+1)} - \frac{c^8 t^{2a}(\cosh(cx) - 2)\,\mathrm{sech}^4\left(\frac{cx}{2}\right)}{16\Gamma(2a+1)} \\
&\quad + \frac{3c^{11} t^{3a}\Gamma(2a+1)(85\sinh(cx) - 20\sinh(2cx) + \sinh(3cx))\,\mathrm{sech}^8\left(\frac{cx}{2}\right)}{4096\Gamma(a+1)^2\Gamma(3a+1)} \\
&\quad - \frac{3c^{14} t^{4a}\Gamma(3a+1)(3\cosh(cx) - 8)\tanh^2\left(\frac{cx}{2}\right)\mathrm{sech}^6\left(\frac{cx}{2}\right)}{256\Gamma(a+1)^3\Gamma(4a+1)},
\end{aligned}$$

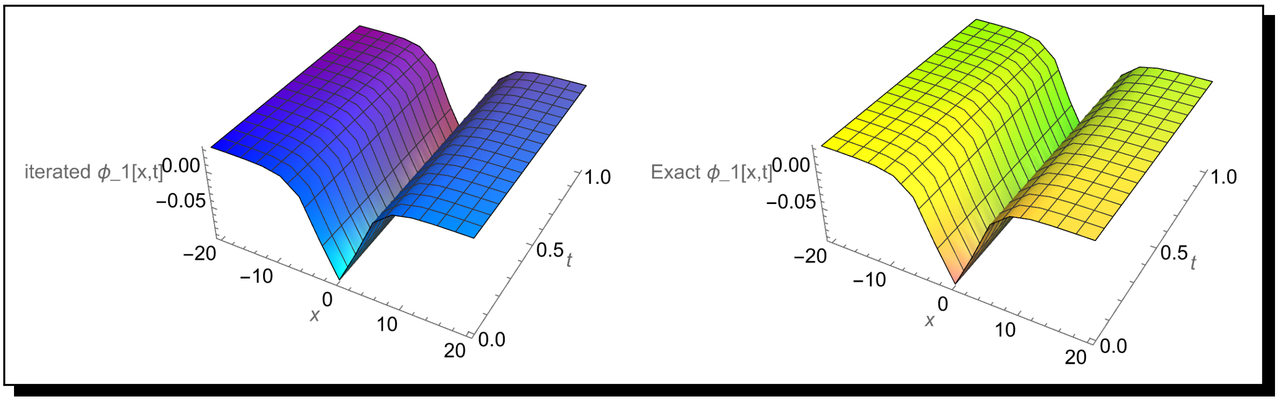
$$\begin{aligned}
v_{22} &= \phi_{20} + \phi_{21} + \phi_{22} = h_2^* + L_2^*(v_{11}, v_{21}) + N_2^*(v_{11}, v_{21}) \\
&= c\,\mathrm{sech}\left(\frac{cx}{2}\right) - \frac{c^4 t^a \tanh\left(\frac{cx}{2}\right)\mathrm{sech}\left(\frac{cx}{2}\right)}{4\Gamma(a+1)} + \frac{c^7 t^{2a}(\cosh(cx) - 3)\,\mathrm{sech}^3\left(\frac{cx}{2}\right)}{32\Gamma(2a+1)} \\
&\quad - \frac{3c^{10} t^{3a}\Gamma(2a+1)(3\cosh(cx) - 7)\tanh\left(\frac{cx}{2}\right)\mathrm{sech}^5\left(\frac{cx}{2}\right)}{128\Gamma(a+1)^2\Gamma(3a+1)} \\
&\quad + \frac{15c^{13} t^{4a}\Gamma(3a+1)(\cosh(cx) - 3)\tanh^2\left(\frac{cx}{2}\right)\mathrm{sech}^5\left(\frac{cx}{2}\right)}{512\Gamma(a+1)^3\Gamma(4a+1)},
\end{aligned}$$

$$\vdots$$

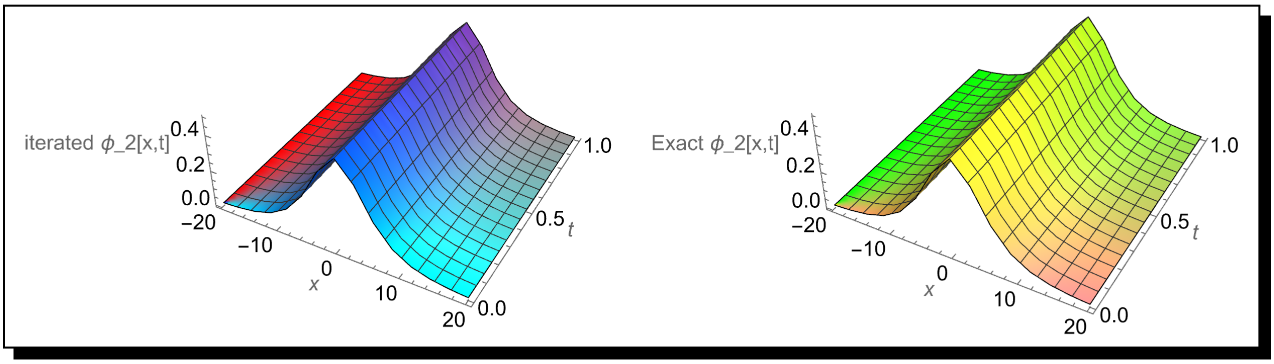
Table 2. Solution obtained for $c = 0.5$, $a = 1$ of the eqn. (4.13)-(4.14)

x	t	$\phi_1(x,t)$	Exact ϕ_1	$\phi_2(x,t)$	Exact ϕ_2	Absolute error of ϕ_1	Absolute error of ϕ_2
0.2	0.2	−0.0933553	−0.0933553	0.4992100	0.4992100	0.0000000	0.0000000
	0.4	−0.0932629	−0.0932630	0.4990250	0.4990250	0.0000001	0.0000000
	0.6	−0.0931608	−0.0931610	0.4988210	0.4988210	0.0000002	0.0000000
	0.8	−0.0930489	−0.0930495	0.4985970	0.4985970	0.0000006	0.0000000
	1	−0.0929273	−0.0929284	0.4983550	0.4983540	0.0000011	0.0000010
0.4	0.2	−0.0923494	−0.0923494	0.4971910	0.4971910	0.0000000	0.0000000
	0.4	−0.0921811	−0.0921812	0.4968530	0.4968530	0.0000001	0.0000000
	0.6	−0.0920033	−0.0920037	0.4964960	0.4964950	0.0000004	0.0000010
	0.8	−0.0918159	−0.0918170	0.4961200	0.4961190	0.0000011	0.0000010
	1	−0.0916190	−0.0916212	0.4957260	0.4957240	0.0000022	0.0000020
0.6	0.2	−0.0907472	−0.0907472	0.4939580	0.4939580	0.0000000	0.0000000
	0.4	−0.0905063	−0.0905065	0.4934700	0.4934700	0.0000002	0.0000000
	0.6	−0.0902563	−0.0902569	0.4929650	0.4929640	0.0000006	0.0000010
	0.8	−0.0899971	−0.0899987	0.4924410	0.4924400	0.0000016	0.0000010
	1	−0.0897288	−0.0897318	0.4919000	0.4918980	0.0000030	0.0000020

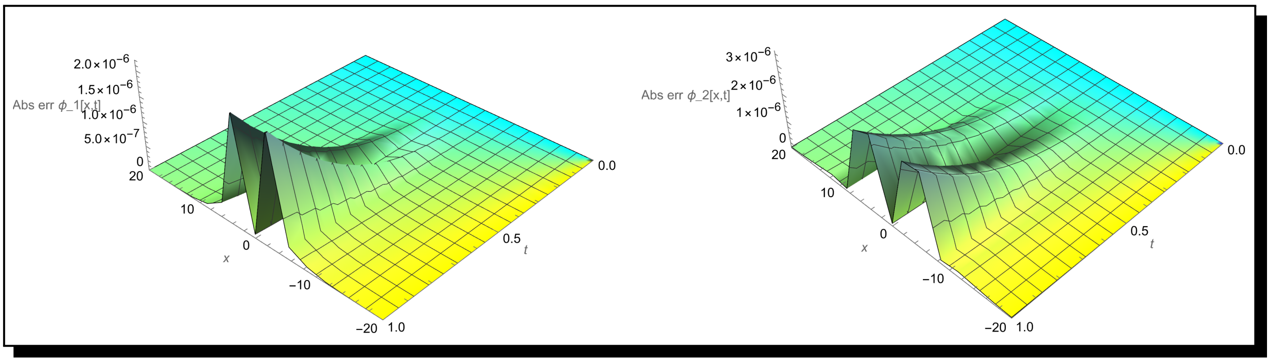
Figure 8 shows the solution curves for fixed $x = 1$, $c = 0.5$ in the time span $0 < t < 10$ with different order of derivative and the graphical simulation of approximate solutions $\phi_1(x,t)$ and $\phi_2(x,t)$, exact solution are depicted in Figure 9 along with absolute errors. The respective solution values are listed in the table 2 for different inputs (x,t). Moreover, the obtained results nearly correspond to the one obtained using q-homotopy analysis transformation method dealt by Vereesha [27], and our method involves less computational effort.



(a) Approximate and the exact solution ϕ_1 for $c = 0.5$, $a = 1$



(b) Approximate and the exact solution ϕ_2 for $c = 0.5$, $a = 1$



(c) Absolute errors of ϕ_1 and ϕ_2 for $c = 0.5$, $a = 1$

Figure 9. Graphical simulation of (a) ϕ_1, and (b) ϕ_2 of the eqn. (4.13)-(4.14)

5. Conclusion

An iterative technique for finding an approximate solution of fractional differential equations (FDEs) has been established and effectively applied to find the solution of non-linear time fractional partial differential equations and systems. Besides, being a direct approach to solve various FDEs, the scheme guarantees the accuracy of generating results, and only a few iterations are required to converge to the exact solution of the problems analysed.

Furthermore, by comparing the results obtained with those obtained using few methods present in the literature, it is clear that the proposed method is extremely efficient and easy to apply to many different types of FDEs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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