Communications in Mathematics and Applications
Vol. 15, No. 2, pp. 865–873, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i2.2715

http://www.rgnpublications.com

Research Article

A Markovian Queue With Feedback and Second Optional Working Vacation

K. Santhi*[1], S. Pazhani Bala Murugan[2] and T. Senthamizhselvi[3]

[1] *Department of Mathematics, PS PT MGR Government Arts and Science College, Sirkali 609108, Tamilnadu, India*
[2] *Department of Mathematics, Annamalai University, Annamalainagar 608002, Tamilnadu, India*
[3] *Department of Mathematics, Aries Arts and Science College-(W), Karunkuzhi, Vadalur, Tamilnadu, India*
***Corresponding author:** santhimano3169@gmail.com

Received: May 15, 2024 **Accepted:** July 8, 2024

Abstract. A Markovian queue with feedback and second optional working vacation is analysed in this study. Rather of fully suspending service during a vacation, the server operates with varying service times. After the completion of the regular working vacation, the server may opt for second optional working vacation. After getting the service the customer who is unsatisfied with incomplete, partial or unsatisfactory service may opt the feedback with probability $\bar{r}$ or leave the system with the probability r. We use the matrix-geometric approach to find the essential and acceptable conditions for the system to be secure. After deriving the stationary probability distribution, various performance indicators are calculated. Some numerical examples are illustrated to show the model's stability.

Keywords. Second optional working vacation, Feedback, Matrix-geometric method

Mathematics Subject Classification (2020). 60K25, 90B22

1. Introduction

Wallace [11] used a Markov chain with a tridiagonal generator to study the *Quasi Birth-Death progress* (QBD) in queueing theory. In cases where an explicit solution to queuing problems cannot be achieved, congestion situations can be analyzed numerically; for this piece, the matrix geometric approach is perfect. Latouche and Ramaswamy [2] suggested the matrix geometric approach to the QBD procedure. An $M/G/1$ queue with an optional second vacation have been carried out by Choudhury [1]. Feedback Queueing System in an $M/G/1$ queue with various parameters were discussed by Manoharan and Sasi [4]. Manoharan *et al*. [5] discussed a Markovian retrial queue with working vacation under N-control pattern. Santhi and Murugan [7] analysed $M/G/1$ single and bulk input feedback queues with working vacation. One might refer to Tian and Zhang [10] for more information on this notable queueing system with attendant vacation. Working Vacation (WV) is a contemporary vacation policy that Servi and Finn [8] developed. The attendant's level of service is lower during WV than it is during the engaged period. Tian *et al*. [9] considered $M/M/1/SWV$ queue. Two vacation strategies were considered in the analysis of the $M/M/1$ queue by Ye and Liu [12].

The general structure of this article is as follows: in Section 2, we define the infinitesimal generator and present the model. In Section 3, the rate matrix (R) and stability condition are calculated. In Section 4, we derive the stagnant probability distribution using a matrix-analytic method. We compute execution mensuration in Section 5, and Section 6 grants numerical computations to show how firm the model is. In Section 7, the conclusion is given.

2. QBD Process Model

This article examines an $M/M/1$ queue with feedback and a second optional working vacation. The inter-arrival times of customers are exponentially distributed with rate λ. The service discipline is FCFS (First Come First Serve). At the times of the regular busy period and working vacation period, the service times are exponentially distributed with rate μ_b and μ_v, respectively. In this article, we are considering the feedback from the first customer and the service rate is taken as $r\mu_v$ and $r\mu_b$, respectively. At service completion time, the server takes a vacation, with the duration being exponentially distributed. This occurs each time the system runs out of resources. The working vacation completion rate of first regular working vacation is assumed to be θ_1 and the second optional working vacation is assumed to be θ_2. After the completion of first regular working vacation, the server may choose the second optional working vacation with probability $p\theta_1$ or he may choose the regular busy period with probability $q\theta_1$. When compared to the regular service period, the level of service during the working vacation period is lower. There is no correlation between the working vacation periods, service periods, and inter-arrival times.

Let $H(t)$ represent the server's condition at time t and $Q(t)$ represent the total number of consumers in the system at that moment. The server could be in any of the following three events:

$$H(t) = \begin{cases} 1 \text{ — on occasion of '}t\text{' if the attendant is in usual working vacation period,} \\ 2 \text{ — on occasion of '}t\text{' if the attendant is in second optional working vacation period,} \\ 3 \text{ — on occasion of '}t\text{' if the attendant is in a usual active period.} \end{cases}$$

Then $\{(Q(t),H(t)),t \geq 0\}$ is a Markov progress with $\Omega = \{(0,j) \cup (n,j) : n \geq 0, j = 1,2,3\}$. The events of the atomic generator can be explained as follows:

$$\widetilde{Q} = \begin{bmatrix} B_{00} & A & & & & & & & \\ C_{00} & B & A & & & & & & \\ & C & B & A & & & & & \\ & & C & B & A & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & C & B & A & & & \\ & & & & C & B & A & & \\ & & & & & C & B & A & \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} -(\lambda + r\mu_v + q\theta_1 + p\theta_1) & p\theta_1 & q\theta_1 \\ 0 & -(\lambda + r\mu_v + \theta_2) & \theta_2 \\ 0 & 0 & -(\lambda + r\mu_b) \end{bmatrix},$$

$$C = \begin{bmatrix} r\mu_v & 0 & 0 \\ 0 & r\mu_v & 0 \\ 0 & 0 & r\mu_b \end{bmatrix},$$

$$B_{00} = \begin{bmatrix} -(\lambda + q\theta_1 + p\theta_1) & p\theta_1 & q\theta_1 \\ 0 & -(\lambda + \theta_2) & \theta_2 \\ 0 & 0 & -\lambda \end{bmatrix}, \quad C_{00} = \begin{bmatrix} r\mu_v & 0 & 0 \\ 0 & r\mu_v & 0 \\ r\mu_b & 0 & 0 \end{bmatrix}.$$

Due to matrix $\widetilde{Q}$ the solid shape, the progress $\{(Q(t),H(t));t \geq 0\}$ is referred to as a *QBD* progress.

3. The Model's Stability Condition and *R*

Theorem 3.1. *The QBD progress* $\{(Q(t),H(t));t \geq 0\}$ *is a positive recurrent if and only if* $\rho \left(= \frac{\lambda}{\mu_b}\right) < 1$.

Proof. Consider

$$D = A + B + C = \begin{bmatrix} -(q\theta_1 + p\theta_1) & p\theta_1 & q\theta_1 \\ 0 & -\theta_2 & \theta_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because matrix A is observable, Theorem 7.3.1 in [2] provides a prerequisite for the *QBD* process's positive recurrence. Following row and column permutations, it indicates that the *QBD* is positive recurring

$$\Longleftrightarrow \pi \begin{bmatrix} r\mu_v & 0 \\ 0 & r\mu_b \end{bmatrix} e > \pi \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} e.$$

In this case, all element in $e = 1$ and π represents the system's exotic result, $\pi \begin{bmatrix} -\theta_2 & \theta_2 \\ 0 & 0 \end{bmatrix} = 0$, $\pi e = 1$. If and only if, following certain algebraic manipulations, $\rho < 1$, the *QBD* process is positive recurring. □

Theorem 3.2. *The minimal non-negative solution to the matrix quadratic equation $R^2C+RB+A=0$ if $\rho<1$ is*

$$R=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix},$$

where

$$r_{11}=\frac{(\lambda+r\mu_v+q\theta_1+p\theta_1)-\sqrt{(\lambda+r\mu_v+q\theta_1+p\theta_1)^2-4(r\mu_v)\lambda}}{2(r\mu_v)},$$

$$r_{12}=\frac{r_1p\theta_1}{(\lambda+r\mu_v+\theta_2)-(r\mu_v)(r_1+r_2)},$$

$$r_{13}=\frac{r_3r_4(r\mu_b)+r_1q\theta_1+\theta_2r_3}{(\lambda+r\mu_b)-(r\mu_b)(r_1+\rho)},$$

$$r_{22}=\frac{(\lambda+r\mu_v+\theta_2)-\sqrt{(\lambda+r\mu_v+\theta_2)^2-4r\mu_v\lambda}}{2r\mu_v},$$

$$r_{23}=\frac{r_2\theta_2}{(\lambda+r\mu_b)-[(r\mu_b)r_2+\rho]},$$

$$r_{33}=\frac{(\lambda+r\mu_b)-\sqrt{(\lambda+r\mu_b)^2-4(r\mu_b)\lambda}}{2(r\mu_b)}.$$

Proof. Let $R=\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$.

From the matrix C, matrix B and matrix A, we substitute R into $R^2C+RB+A=0$, we get

$$r_{11}^2r\mu_v-r_{11}(\lambda+r\mu_v+q\theta_1+p\theta_1)+\lambda=0,$$

$$(r_{11}r_{12}+r_{12}r_{22})r\mu_v+[r_{11}p\theta_1-r_{12}(\lambda+r\mu_v+\theta_2)]=0,$$

$$(r_{11}r_{13}+r_{12}r_{23}+r_{13}r_{33})r\mu_b+[r_{11}q\theta_1+\theta_2r_{12}-r_{13}(\lambda+r\mu_b)]=0,$$

$$r_{22}^2(r\mu_v)-r_{22}(\lambda+r\mu_v+\theta_2)+\lambda=0,$$

$$(r_{22}r_{23}+r_{23}r_{33})(r\mu_b)+r_{22}\theta_2-r_{23}(\lambda+r\mu_b)=0,$$

$$r_{33}^2(r\mu_b)-r_{33}(\lambda+r\mu_b)+\lambda=0.$$

After performing several computations on the aforementioned system of equations, we attain R_{11}, R_{12}, R_{13}, R_{22}, R_{23} and R_{33}.

4. Stagnant Probability Distribution

Probability distribution that is stagnant for the progress $\{(Q(t),H(t));t\ge 0\}$ should be assigned to (Q,H) if $\rho<1$. Act as a representative,

$$\pi_n=(\pi_{n1},\pi_{n2},\pi_{n3}),\quad n\ge 0,$$

$$\pi_{nj}=P\{Q=n,H=j\}$$

$$=\lim_{t\to\infty}P\{Q(t)=n,H(t)=j\},\quad (n,j)\in\Omega.$$

Theorem 4.1. *If $\rho < 1$, the stagnant probability distribution of (Q,H) is indicated by*

$$\pi_{n1} = r_1^n \pi_{01}, \quad n \ge 1, \tag{4.1}$$

$$\pi_{n2} = r_3 \sum_{j=0}^{n-1} r_1^{n-1-j} r_2^j \pi_{01} + r_2^n \frac{p\theta_1 + r_3 r\mu_v}{\lambda + \theta_2 - r_2 r\mu_v} \pi_{01}, \quad n \ge 1, \tag{4.2}$$

$$\begin{aligned}\pi_{n3} &= r_5 \sum_{j=1}^{k} r_1^{k-j} \rho^{j-1} + r_3 r_4 \sum_{n=2}^{k} \rho^{n-2} \sum_{j=1}^{k-n+1} r_1^{k-n+1-j} r_2^{j-1} \pi_{01} \\ &\quad + r_4 \sum_{j=0}^{n-1} r_2^{n-1-j} \rho^j \frac{p\theta_1 + r_3 r\mu_v}{\lambda + \theta_2 - r_2 r\mu_v} \pi_{01} \\ &\quad + \rho^n \frac{(\lambda + \theta_2 - r_2 r\mu_v) q\theta_1 + \theta_2 p\theta_1 + \theta_2 r_3 r\mu_v}{\lambda(\lambda + \theta_2 - r_2 r\mu_v)} \pi_{01}, \quad n \ge 1\end{aligned} \tag{4.3}$$

and

$$\pi_{02} = \pi_{01} \left[\frac{p\theta_1 + r_3 r\mu_v}{(\lambda + \theta_2 - r_2 r\mu_v)} \right],$$

$$\pi_{03} = \pi_{01} \left[\frac{(\lambda + \theta_2 - r_2 r\mu_v) q\theta_1 + \theta_2 p\theta_1 + \theta_2 r_3 r\mu_v}{\lambda(\lambda + \theta_2 - r_2 r\mu_v)} \right].$$

At last, one can utilize the normalization condition to find π_{01}.

Proof. Applying the method described in [11], we have

$$\begin{aligned}\pi_n &= (\pi_{01}, \pi_{02}, \pi_{03}), \\ &= \pi_n R^n \\ &= (\pi_{01}, \pi_{02}, \pi_{03}) R^n, \quad n \ge 1.\end{aligned}$$

For $n \ge 1$,

$$R^n = \begin{bmatrix} r_1^n & r_3 \sum_{j=0}^{n-1} r_1^{n-1-j} r_2^j & r_5 \sum_{j=1}^{k} r_1^{k-j} \rho^{j-1} + r_3 r_4 \sum_{n=2}^{k} \rho^{n-2} \sum_{j=1}^{k-n+1} r_1^{k-n+1-j} r_2^{j-1} \\ 0 & r_2^n & r_4 \sum_{j=0}^{n-1} r_2^{n-1-j} \rho^j \\ 0 & 0 & \rho^n \end{bmatrix}$$

Substituting R^n into the above equation, we get equations (4.1) to (4.3).
However, π_n satisfies the equation $\pi_n(B_{00} + RC_{00}) = 0$, where

$$B_{00} + RC_{00} = \begin{bmatrix} -(\lambda + p\theta_1 + q\theta_1) + r_1 r\mu_v + r_5 r\mu_b & p\theta_1) + r_3 r\mu_v & q\theta_1 \\ r_4 r\mu_b & -(\lambda + \theta_2) + r_2 r\mu_v & \theta_2 \\ \rho r\mu_b & 0 & -\lambda \end{bmatrix}.$$

The following equations are computed from $B_{00} + RC_{00}$,

$$[-(\lambda + p\theta_1 + q\theta_1) + r_1 r\mu_v + r_5 r\mu_b]\pi_{01} + r_4 r\mu_b \pi_{02} + \rho r\mu_b \pi_{03} = 0, \tag{4.4}$$

$$(p\theta_1 + r_3 r\mu_v)\pi_{01} + [-(\lambda + \theta_2) + r_2 r\mu_v]\pi_{02} = 0, \tag{4.5}$$

$$q\theta_1 \pi_{01} + \theta_2 \pi_{02} - \lambda \pi_{03} = 0. \tag{4.6}$$

Using normalizing condition, we get

$$\pi_{01}=\left[\frac{r_1}{1-r_1}+\frac{r_2}{1-r_2}\frac{p\theta_1+r_3r\mu_v}{\lambda+\theta_2-r_2r\mu_v}+\frac{r_2r_3}{(1-r_1)(1-r_2)}+\frac{r_5}{(1-r_1)(1-\rho)}\right.$$
$$+\frac{r_3r_4}{(1-r_1)(1-r_2)(1-\rho)}+\frac{r_4\rho}{(1-r_2)(1-\rho)}\frac{p\theta_1+r_3r\mu_v}{\lambda+\theta_2-r_2r\mu_v}$$
$$\left.+\frac{\rho}{1-\rho}\frac{(\lambda+\theta_2-r_2r\mu_v)q\theta_1+\theta_2p\theta_1+\theta_2r_3r\mu_v}{\lambda(\lambda+\theta_2-r_2r\mu_v)}\right]^{-1}.$$

Clearly, the event probability of the attendant is given by

$$P_1=P\{J=1\}=\sum_{n=1}^{\infty}\pi_{n1}=\frac{r_1}{(1-r_1)}\pi_{01},$$
$$P_2=P\{J=2\}=\sum_{n=0}^{\infty}\pi_{n2}=\frac{r_2}{(1-r_2)}\frac{p\theta_1+r_3r\mu_v}{\lambda+\theta_2-r_2r\mu_v}\pi_{01}+\frac{r_2r_3}{(1-r_1)(1-r_2)}\pi_{01},$$
$$P_3=P\{J=3\}=\sum_{n=0}^{\infty}\pi_{n3}=\frac{r_5}{(1-r_1)(1-\rho)}\pi_{01}+\frac{r_3r_4}{(1-r_1)(1-r_2)(1-\rho)}\pi_{01}$$
$$+\frac{r_4\rho}{(1-r_2)(1-\rho)}\frac{p\theta_1+r_3r\mu_v}{\lambda+\theta_2-r_2r\mu_v}\pi_{01}$$
$$+\frac{\rho}{1-\rho}\frac{(\lambda+\theta_2-r_2r\mu_v)q\theta_1+\theta_2p\theta_1+\theta_2r_3r\mu_v}{\lambda(\lambda+\theta_2-r_2r\mu_v)}\pi_{01}.$$

5. The Model's Execution Mensuration

We can calculate the probability when the server is doing no job, denoted by P_f is given by

$$P_f=P\{J=1\}+P\{J=2\}=P_1+P_2=1-P_b.$$

We can calculate the probability when the server is doing a job, denoted by P_b is given by

$$P_b=P\{J=3\}=P_3.$$

Let L be the number of client in the scheme. Then we can get

$$E[L]=\sum_{n=1}^{\infty}n(\pi_{n1}+\pi_{n2}+\pi_{n3})=\frac{r_1(1-r_1)^2(1-\rho)^2+r_2r_3(1-\rho)^2+r_5(1-r_2)^2+r_3r_4}{(1-r_1)^2(1-r_2)^2(1-\rho)^2}$$
$$+\frac{(r_2(1-\rho)^2+r_4\rho)}{(1-r_2)^2(1-\rho)^2}\frac{p\theta_1+r_3r\mu_v}{\lambda+\theta_2-r_2r\mu_v}$$
$$+\frac{\rho}{(1-\rho)^2}\frac{(\lambda+\theta_2-r_2r\mu_v)q\theta_1+\theta_2p\theta_1+\theta_2r_3r\mu_v}{\lambda(\lambda+\theta_2-r_2r\mu_v)}.$$

6. Numerical Results

By fixing the values of $r\mu_v=4.0$, $r\mu_b=3.0$, $p=0.3$, $q=0.7$, and based on permanence requirement, the estimates of $E[L]$ are computed and recorded in Table 1, and the related visual representations are displayed in Figure 1. This involves increasing the estimates of λ for $1.0, 1.2, 1.4, 1.6, 1.8, 2.0$ and increasing the estimates of θ_1 as $1.0, 2.0$ and 3.0. The graph suggests that $E[L]$ declines as predicted when λ increases.

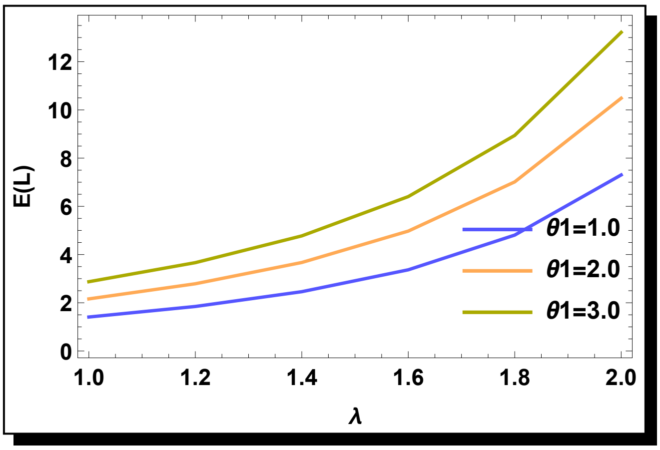

Figure 1. $E[L]$ vs λ

Table 1. $E[L]$ vs λ

λ	$\theta_1 = 1.0$	$\theta_1 = 2.0$	$\theta_1 = 3.0$
1.0	1.409210	2.160426	2.873665
1.2	1.846644	2.788139	3.664451
1.4	2.459527	3.670549	4.773940
1.6	3.367127	4.972250	6.403528
1.8	4.807285	7.013881	8.941960
2.0	7.302814	10.484334	13.216944

By fixing the values of $r\mu_v = 4.0$, $r\mu_b = 3.0$, $p = 0.3$, $q = 0.7$, and based on permanence requirement, the estimates of $E[L]$ are computed and recorded in Table 2, and the related visual representations are displayed in Figure 2. This involves increasing the estimates of λ for $1.0, 1.2, 1.4, 1.6, 1.8, 2.0$ and increasing the estimates of θ_2 as 2.5, 3.2 and 3.9. The graph suggests that $E[L]$ declines as predicted when λ increases.

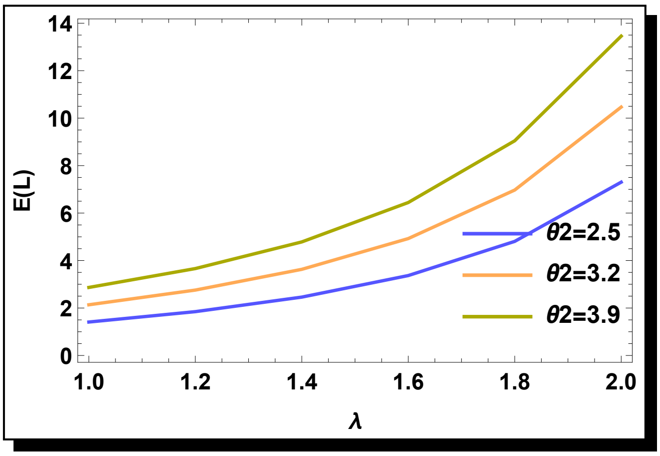

Figure 2. $E[L]$ vs λ

Table 2. $E[L]$ vs λ

λ	$\theta_2 = 2.5$	$\theta_2 = 3.2$	$\theta_2 = 3.9$
1.0	1.409210	2.136203	2.866434
1.2	1.846644	2.755327	3.661939
1.4	2.459527	3.629441	4.784763
1.6	3.367127	4.925736	6.444592
1.8	4.807285	6.970617	9.046922
2.0	7.302814	10.466565	13.456233

7. Conclusion

A Markovian queue with feedback and a second optional working vacation is assessed in this article. We computed the model's rate matrix and stability condition. Using the matrix-analytic method, we can derive the stagnant probability distribution. We derived some execution mensuration. We illustrated some numerical examples, and we verified the stability condition.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] G. Choudhury, An $M/G/1$ queue with an optional second vacation, *Information and Management Sciences* **17**(3) (2006), 19 – 30.

[2] G. Latouche and V. Ramaswamy, *Introduction to Matrix Analytic Methods in Stochastic Modeling*, ASA-SIAM Series on Statistics and Applied Mathematics, SIAM, USA, xiv + 334 pages, USA (1999), URL: https://epubs.siam.org/doi/book/10.1137/1.9780898719734.

[3] J. Li and N. Tian, The $M/M/1$ queue with working vacations and vacation interruptions, *Journal of Systems Science and Systems Engineering* **16** (2007), 121 – 127, DOI: 10.1007/s11518-006-5030-6.

[4] P. Manoharan and K. S. Sasi, An $M/G/1$ feedback queueing system with second optional service and with second optional vacation, *International Journal of Fuzzy Mathematical Archive* **8**(2) (2015), 101 – 113.

[5] P. Manoharan, S. P. B. Murugan and A. Sobanapriya, Analysis of a Markovian retrial queue with working vacation under N-control pattern, *Communications in Mathematics and Applications* **13**(3) (2015), 851 – 863, DOI: 10.26713/cma.v13i3.2065.

[6] M. R. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore, 352 pages (1981).

[7] K. Santhi and S. P. B. Murugan, An bulk input queueing system with feedback and single working vacations, *International Journal of Scientific Research and Management Studies* **1**(5)(2014), 168 – 176.

[8] L. D. Servi and S. G. Finn, $M/M/1$ queues with working vacations ($M/M/1/WV$), *Performance Evaluation* **50**(1) (2002), 41 – 52, DOI: 10.1016/S0166-5316(02)00057-3.

[9] N. Tian, N. Tian and X. Zhao, Wang Kaiyu, The M/M/1 queue with single working vacation, *International Journal of Information and Management Sciences* **19**(4) (2008), 621 – 634.

[10] N. Tian and Z. G. Zhang, *Vacation Queueing Models: Theory and Applications*, 1st edition, Springer-Verlag, New York, xii + 386 pages (2006), DOI: 10.1007/978-0-387-33723-4.

[11] V. L. Wallace, The solution of quasi birth and death processes arising from multiple access computer systems, Ph.D Dissertation, Systems Engineering Laboratory, University of Michigan, *Technical Report No. 07742-6-T*, (1969), URL: https://deepblue.lib.umich.edu/handle/2027.42/8180.

[12] Q. Ye and L. Liu, The analysis of the $M/M/1$ queue with two vacation policies ($M/M/1/SWV+MV$), *International Journal of Computer Mathematics* **94**(1) (2017), 115 – 134, DOI: 10.1080/00207160.2015.1091450.