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Review Article

Study of Mappings and Metric Spaces in Fixed Point Theory: A Review

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Abstract. Fixed point theory is a rich, fascinating, and intriguing subject of mathematics. Though it is a completely established field of research, but very fresh. It has been a hot topic of research since its inception. In this work, we present an overview of the major branches of fixed-point theory along with the key results. The focus of this article is to investigate different types of mappings and their variants. Furthermore, a survey of several extensions of metric space has also been carried out.

Keywords. Fixed points, Mappings, Metric spaces

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

Fixed point theory is a branch of mathematics that studies the existence of fixed points to be used for mapping and their properties. It plays a pivotal role in both theoretical mathematics and applied sciences. It works as a bridge between topology and analysis. It is a useful tool in non-linear analysis with the concept of the existence and uniqueness of solutions for non-linear equations. Moreover, it is a mathematical field that studies solutions to equations of the form $T(x)=x$, where T is a given function or mapping. A fixed point of a function is a point that remains unchanged when the function is applied to it. In other words, a point x in X (non-empty set) is a fixed point of T if $T(x)=x$.

Successive approximation was the origin of fixed-point theory in the 19th century. It evolves with solutions of differential equations by using the method of successive approximations as stated by J. Liouville in 1837 [39] and E. Picard in 1890 [53]. Picard's result establishes solutions for non-linear equations.

"Let $T : [a,b] \to \mathbb{R}$ be a continuous and differentiable function. If there exists a positive number $k < 1$ such that $|T'(x)| < k$, for all $x \in (a,b)$, then the sequence $\{x_n\}$ in (a,b) defined by

$$x_{n+1} = Tx_n, \quad \text{for all } n \ge 0, \tag{1.1}$$

converges to a solution of the equation defined by $T(x) = x$. The iterative sequence defined by (1.1) is known as the Picard iterative sequence" (Picard [53]).

Classification of Fixed-Point Theory is considered as:

- Topological Fixed-Point Theory
- Metric Fixed-Point Theory
- Algebraic Fixed-Point Theory

1.1 Topological Fixed-Point Theory/Brouwer's Fixed Point Theorem

In 1912, Brouwer [13] proposed the classical fixed-point theorem. It is a fundamental consequence of both topology and fixed-point theory. It states that any continuous map defined on the closed unit all of $\mathbb{R}^n$ has a fixed point. It involves the study of spaces with fixed point properties. It does not offer any details about where the fixed points are located. This theorem does not provide information on the uniqueness of the solution, it only guarantees that a solution exists. It is not true in infinite dimensional spaces. Further, in 1930, Schauder [55] presented the first fixed point theorem in an infinite dimensional Banach space. It is a bridge between compact and non-compact sets. Releasing the compactness constraint is a straightforward way to prove the theorem. In 1935, Tychonoff [59] generalized the Schauder theorem to locally convex topological vector spaces for continuous maps. The theorem states that even in non-compact spaces, continuous mappings can have fixed points provided the domain is compact and convex.

1.2 Analytical/Metric Fixed Point Theory/Banach's Fixed Point Theorem

In 1922, Banach [9] stated that any contraction map defined on complete metric space has a unique fixed point. It is the study of fixed points in relevance to the underlying map and the space under investigation. Banach's significant achievement was the *Contraction Principle*, which filled a previous gap in Brouwer's existing principle for fixed points to locate the solution and its uniqueness. There is a revolution in metric fixed point theory after the escalation of the contraction principle. The Banach contraction principle is significant in that it provides the existence, uniqueness, and sequence of successive approximations that converge to a solution to the problem. The key characteristic of the result is that it ensures existence, uniqueness, and determination.

A wide range of classes of mappings for fixed point theorems have been developed, for example, contraction mappings, contractive mappings and their types, pair of mappings, etc.

"The *Contraction Map* as within the metric can be stated that let T be a self-map and if there exists a real number $0 \le \alpha < 1$, such that $d(Tx,Ty) \le d(x,y)$, for all $x,y \in X$" (Banach [9]).

It has been already proved that the contraction map is continuous but the converse is not true. Contraction mappings are a key topic in dynamic programming with dynamic system analysis and control theory. There are various types of contraction mappings, such as the ones listed below.

"Let (X,d) be a metric space. Then a mapping $T:X\to X$ is called Lipschitz mapping if there exists a real number $\alpha>0$ such that $d(Tx,Ty)\le \alpha d(x,y)$, for all $x,y\in X$" (Banach [9]).

- Contractive mapping is applicable if $d(Tx,Ty)<d(x,y)$ and $x\neq y$.
- For $\alpha\le 1$, the mapping is non-expansive.
- For $\alpha>1$, the mapping is expansive.

1.3 Algebraic/Discrete Fixed-Point Theory/Tarski's Fixed-Point Theorem

It was given by Tarski [58] in 1955 that every order-preserving map (increasing mapping) defined on a complete lattice has a fixed point. It necessitates the investigation of the order-preserving map. This theorem was applied and extended in the theory of real functions, Boolean algebras, and other domains.

2. Various Contractions and Types of Contractive Mappings

The *Banach Contraction Principle* has limits because it only applies to continuous mappings. So, to overcome this, Kannan [29] has introduced a novel contraction, known as *Kannan Contraction*, which is distinct from the Banach Contraction Principle. The map does not have to be continuous. It describes the metric completeness, as follows:

Definition 2.1 ([29])**.** Let T be a self-map on a complete metric space X such that

$$d(Tx,Ty)\le \alpha[d(x,Tx)+d(y,Ty)];\alpha\in\left(0,\frac{1}{2}\right),\quad \text{for all } x,y\in X.$$

In 1969, Meir-Keeler [42] defined a new contraction as stated below:

Definition 2.2 ([42])**.** For each $\varepsilon>0$, there exist a $\delta>0$ such that

$$\varepsilon\le d(x,y)<\varepsilon+\delta \implies d(Tx,Ty)<\varepsilon,\quad \text{for all } x,y\in X.$$

It is a generalization of Edelstein's contraction. It works on completeness property rather than compactness.

In 1972, Bianchini [11] introduced a new contraction that is relevant to the Kannan contraction.

Definition 2.3 ([11])**.** There exists $\alpha\in[0,1)$ such that for all $x,y\in X$,

$$d(Tx,Ty)\le \alpha\max[d(x,Tx)+d(y,Ty)].$$

It replaces Kannan contraction with a new contractive condition of max.

In 1975, Das and Gupta [15] proposed the rational-type contractive condition:

Definition 2.4 ([15])**.** For any $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$, we get for all $x,y\in X$,

$$d(Tx,Ty)\le \alpha\frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)}+\beta d(x,y),$$

where T is continuous.

It is an extension of the Banach contract principle. Here, continuity of T is flexible and $\alpha+\beta<1$ is a restrictive condition.

In 1976, Caristi [14] introduced a contraction which is stated below:

Definition 2.5 ([14]). There exists a lower semi-continuous function $\varphi : X \to [0,\infty)$ such that

$$d(x,Tx) \le \varphi(x) - \varphi(Tx).$$

It involves sequence of mappings. It emphasizes on metric completeness. It provides the existence of bounded solution of functional equations. It is the modification of ε-variational principle of Ekeland [17].

In 1977, Jaggi [21] introduced a new rational type contractive condition stated as:

Definition 2.6 ([21]). For continuous self-map T on X, there exists $\alpha, \beta \in [0,1)$, such that $\alpha+\beta<1$, for all $x, y \in X$, $x \neq y$, we have

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x,y).$$

It is independent of Dass and Gupta rational contraction. It refers to the extension of Banach's Fixed Point Theorem, which deals with continuous maps.

In 1997, Alber and Guerre-Delabriere [3] introduced a new type of contraction called "Weak contractions".

Definition 2.7 ([3]). If there exists a map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$ such that $d(Tx,Ty) \le d(x,y) - \varphi(d(x,y))$, for all $x,y \in X$, then the map is called φ -weak contraction.

In 2008, Dutta and Chodhury [16] proved a theorem using altering distance function and weak contraction to satisfy the following:

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)),$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and monotonic non-decreasing function with $\psi(t) = 0 = \varphi(t)$ if and only if $t=0$.

In 2012, Samet *et al.* [54] introduced the concept of α-ψ contractive mappings with α-admissible functions.

Definition 2.8 ([54]). Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$, then T is α-admissible if for all $x,y \in X$, $\alpha(x,y) \ge 1 \implies \alpha(Tx,Ty) \ge 1$.

Definition 2.9 ([54]). Let T be a self-map on metric space (X,d), then T is said to be α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)), \quad \text{for all } x,y \in X.$$

It is necessary for the condition that T is not continuous to be met.

In 2015, Khojasteh *et al.* [34] introduced a mapping using a simulation function called Z-contraction.

Definition 2.10 ([34])**.** Let $\tau : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be a mapping, then τ is called a simulation function if it satisfies the following conditions:

(a) $\tau(0,0) = 0$;

(b) $\tau(t,s) < s - t$, for all $t,s > 0$;

(c) If $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \tau(t_n,s_n) < 0$.

To find the specific fixed point, use the Picard sequence $\{T^n_{x_0}\}$ for every $x_0 \in x$. This map is both contractive and continuous.

In 2018, Karapınar [33] introduced interpolative fixed-point contraction.

Definition 2.11 ([33])**.** Let T be a self-map on a metric space (X,d). If there exists a constant $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \lambda[(d(x,Tx)]^{\alpha}[d(y,Ty)]^{1-\alpha}, \quad \text{for all } x,y \in x \text{ and } x \ne Tx.$$

It is an interpolation-based generalization of the Kannan-type contraction, also known as interpolative Kannan-type contraction.

In 2019, Mitrović *et al.* [44] introduced the (r,a)-weight type contraction.

Definition 2.12 ([44])**.** If there exists $\lambda \in [0,1)$ and such that $d(Tx,T) \le \lambda \mathcal{M}^r(Tx,y,a)$, where $r \ge 0$, $a = (a_1,a_2,a_3)$, $a_i \ge 0$, $i = [0,1)$ such that $a_1 + a_2 + a_3 = 1$ and

$$\mathcal{M}^r(Tx,y,a) = \begin{cases} [a_1(d(x,y))^r + a_2(d(x,Tx))^r + a_3(d(y,Ty))^r]^{\frac{1}{r}}, & r > 0, \\ (d(x,y))^{a_1}(d(x,Tx))^{a_2}(d(y,Ty))^{a_3}, & r = 0, \end{cases}$$

for all $x,y \in x\backslash \text{Fix}(t)$, where $\text{Fix}(t) = \{u \in X, Tu = u\}$.

It proposes a hybrid contraction that blends Reich and interpolative-type contractions, with a focus on complete b-metric spaces. In 2019, Karapınar and Fulga [31] conducted research on a new hybrid contraction that involves Jaggi [21] and interpolative type contractions, which they called the Jaggi type hybrid contraction. Since then, several other Jaggi-related contractions have been introduced. In 2022, Karapınar and Fulga [30] introduced Hybrid Jaggi-Meir-Keeler-type contractions. Additionally, Jiddah *et al.* [22] introduced a Jaggi-type hybrid G-contraction in the same year. In 2023, Petrov [52] introduced a new method of calculating the perimeter of triangles using contractions.

3. Various Types of Minimal Commutating Mappings

During the late 20th century, researchers investigated common fixed-point theorems for a particular type of mapping in metric spaces that satisfy contractive criteria. Many scholars have contributed interesting findings in this field. The fundamental objective of fixed-point theory is to establish a minimum set of pivotal fixed points that ensure the existence of a fixed point in mappings of a contractive nature. By applying various constraints on maps, scholars aim to relax the conditions of commuting, continuity, contractive /contraction, and range area containment.

In 1968, Gobel [20] discovered a Coincidence theorem, which marked a significant shift in fixed point theory.

Theorem 3.1 ([20])**.** *Let (X,d) be a metric space and $\mathbb{A}$ be an arbitrary set. Let $S,T:\mathbb{A}\to X$ be two mappings such that $S(\mathbb{A})\subseteq T(\mathbb{A})$, where $T(\mathbb{A})$ is a complete subspace of X and $d(Sx,Ty)\leq kd(Tx,Ty)$, for all $x,y\in A$, $k\in[0,1)$. Then there exists a point $w\in\mathbb{A}$ such that $Sw=Tw$, that means S and T have a coincidence point.*

3.1 Commuting and Weak Commutative Mappings

In 1976 Jungck [24] discovered a fixed-point theorem for a pair of commuting mappings stated that any two pair S & T of self-map to a metric space (X,d) follows for all $x\in X$, $STx=TSx$. It has been developed using a constructive procedure of sequence of iterates. If $S(X)\subseteq T(X)$ then continuity of map is required for having unique fixed points. Further, Sessa discovered the notion of weak commutativity. It helps to strengthen standard fixed-point theorems. It is a point centred property. Commuting maps implies weakly commuting maps implies if two maps commute, they automatically have a weak commutation, however the contrary is not always true. The concept of R-weakly commuting mapping is proposed by Pant [50]. It concludes the result that maps are not necessarily continuous at the fixed point. R-weakly commuting map is the always Weakly commuting but converse is only true when constant R is always less than 1. The improvised notion of R-weakly commuting mappings called R-weakly commuting mappings of type (A_S) and type (A_T) came into existence by Pathak *et al.* [51]. R-weakly commuting and R-weak commuting mappings of type $(A_S)/(A_T)$ are equal at a coincidence point. These can be used to investigate mappings that are not compatible.

3.2 Compatible Mappings

Jungck [25] introduced the generalisation of commuting maps and weakly commutativity maps known as Compatible map. It satisfies contractive type condition as well as the continuation of at least one of the mappings. These generally satisfy inequality conditions than comparison to commuting maps. Weak Commuting map are always compatible. In continuation of these results, several authors had proposed various forms of compatible mappings called as minimal commuting mappings. In addition to compatible mapping various conditions of compatibility like as Compatibility of type (A), (B), (P), (C), (E), (R), (K) has been introduced by Kumar *et al.* [36]. In 1998, Jungck and Rhoades [26] proposed weak compatible mapping. These are also known as coincidently commuting maps since the pair commutes on the set of coincidence points. It is repetitive sequence of mappings. Generalization of weakly compatible mapping is proposed by Al-Thagafi and Shahzad [5] which is named as Occasionally weakly compatible mappings. As far as the continuity in mapping is concerned, the various types of compatible mappings are all the same.

3.3 Non-Compatible Mappings

In 2002, Aamri and El-Moutawakil [1] introduced the E.A property to enhance results in non-complete metric space and eliminates the continuity condition, thus presenting the idea of non-compatibility. A mapping that satisfies the E.A property does not have to continue the range space from one map to the other. Liu *et al.* [40] introduced the concept of Common E.A property, which includes E.A property for self-mappings pair. Additionally, Sintunavarat and Kumam [56] proposed the common limit range property (CLR), which can be compared to the (E.A) property. The CLR property does not impose any restrictions on the closed space or subspace.

4. Metric Space

In 1906, Maurice Fréchet [18] wrote his Ph.D. dissertation "Sur quelques points du calcul fonctionnel" which launched the study of metric spaces. He introduced the concept of distance function which deals with measurement/distance/metric between two points. However, Fréchet did not refer to it as a metric space. Hausdorff [37], one of the founding fathers of modern topology, is credited with the name. A diversified set of metric spaces provides a better grasp of optimization, the internet and networks, robotics, and a wide range of other difficulties. The term metric is derived from the word meter (measure— a collection of functions that generalize the concept of distance between any two items).

Definition 4.1 ([18])**.** Let X be a non-empty set. A metric on X is a function $d: X \times X \to R$ that satisfies the following requirements:

(1) $d(x,y) \geq 0$, for all $x,y \in X$. (Non-negativity)

(2) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$. (Identity)

(3) $d(x,y) = d(y,x)$, for all $x,y \in X$. (Symmetry)

(4) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in x$. (Triangular inequality)

In mathematics, the pair of values (X,d) is referred to as a metric space, where $d(x,y)$ represents the distance between x and y.

Over the years, various versions of metric space have been introduced, some of which are discussed in our work.

Quasi Metric Space ([60])**.** In 1931, Wilson introduced the metric space without symmetric property, known as quasi-metric space.

Definition 4.2 ([60])**.** A quasi-metric function $d: X \times X \to [0,\infty)$ satisfies:

(i) $d(x,y) \geq 0$, for all $x,y \in X$.

(ii) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$.

(iii) $(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$.

The ordered pair (X,d) is called quasi-metric space.

Applications of quasi-metric spaces include Hamilton-Jacobi equations, iterated function systems, bioinformatics, Markov chains, fractal theory.

Probabilistic Metric Space ([43])**.** In 1942, Menger introduced the generalized probabilistic theory, where he replaced the number $d(p,q)$ with the real-valued function $Q_{p,q}(x)$ for any real number x. The theory stated that the distance between p and q is always less than x.

Definition 4.3 ([43])**.** A function $d : X \times X \to [0,\infty)$ is a probabilistic metric if it satisfies certain conditions.

(i) $Q_{p,q}(x) = 1$, for all $x > 0 \implies p = q$, for all $p,q \in X$.

(ii) $Q_{p,q}(x) = F_{p,q}(y)$, for all $p,q \in X$.

(iii) $Q_{p,r}(x) = 1$, $F_{q,r}(x) = 1$, for all $p,q,r \in X$.

(iv) $Q_{p,q}(x,y) = 1$, $x,y \in (0,\infty)$, for all $p,q \in X$.

Then the pair (X,Q) is called probabilistic metric space.

This concept is useful in solving the existence of random fixed-point theorems.

2-Metric Space ([19])**.** In 1963, Gähler introduced the concept of a 2-metric space. This space is a generalization of the ordinary distance function, which measures the distance between two points. However, in this case, it measures the distance between three points and provides a unique nonlinear structure. It is important to note that this space is not a continuous function of its variables.

Definition 4.4 ([19])**.** A function $d : X \times X \times X \to \mathbb{R}$ is said to be 2-metric if it satisfies the following:

(i) for every $x,y \in X$, there exists $z \in X$ such that $d(x,y,z) \neq 0$.

(ii) $d(x,y,z) = 0$ only if at least two of three points are the same.

(iii) $d(x,y,z) = d(x,z,y) = d(y,z,x) = d(y,x,z) = d(z,x,y) = d(z,y,x)$, for all $x,y,z \in X$.

(iv) $d(x,y,z) \le d(x,y,t) + d(y,z,t) + d(z,x,t)$, for all $x,y,z,t \in X$. (Rectangular inequality)

Then the ordered pair (X,d) is called 2-metric space.

Fuzzy Metric Space ([27])**.** In 1984, Kaleva and Seikkala introduced the concept of a fuzzy metric space, which is an extension of the probabilistic metric space. In a fuzzy metric space, the distance between two points is represented by a non-negative convex fuzzy upper semicontinuous number.

Definition 4.5 ([27])**.** Let M be a fuzzy set on $X \times X \times [0,\infty)$ and $*$ is a continuous t-norm, then the tuple $(X,M,*)$ is called fuzzy metric space if it satisfies:

(i) $M(x,y,0) = 0$,

(ii) $M(x,y,t) = 1$ if and only if $x = y$,

(iii) $M(x,y,t) = M(y,x,t)$,

(iv) $M(x,z,t) * M(z,y,s) \le M(x,y,t+s)$,

(v) $M(x,y,\bullet) : [0,\infty) \to [0,1]$ is left continuous, for all $x,y,z \in X$ and $s,t > 0$.

***b*-Metric Space** ([8])**.** In 1989, Backhtin introduced the concept of b-metric space. He proposed the relaxed triangle inequality.

Definition 4.6 ([8])**.** A function $d : X \times X \to [0,\infty)$ is a b-metric if it satisfies:

(i) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$,

(ii) $d(x,y) = d(y,x)$,

(iii) there exists $\beta \geq 1$ such that $d(x,y) \leq \beta[d(x,z)+d(z,y)]$, for all $x,y,z \in X$.
Then the ordered pair (X,d) is called b-metric space.

b-metric space is used to find the existence of unique solutions for a nonlinear fractional differential equation along with integral boundary conditions.

Partial Metric Space ([41])**.** In 1994, Matthews introduced the notion of a partial metric, a type of metric space that relaxes the requirement of a non-zero self-distance. He also applied the Banach contraction principle to the definition of partial metric spaces.

Definition 4.7 ([41])**.** A function $d: X^2 \to \mathbb{R}^+$ is a partial metric on X if it satisfies:

(i) $d(x,x) = d(x,y) = d(y,y) \implies x = y$,

(ii) $d(x,x) \leq d(x,y)$,

(iii) $d(x,y) \leq d(y,x)$,

(iv) $d(x,y) \leq d(x,z)+d(z,y)-d(z,z)$, for all $x,y,z \in X$.

Then the ordered pair (X,d) is called partial metric space.

In partial metric spaces, the problem of fixed points with successive approximations is well-posed.

Rectangular Metric Space ([12])**.** In 2000, Branciari proposed the idea of a generalized rectangular metric space by replacing the triangle inequality with the quadrilateral inequality.

Definition 4.8 ([12])**.** A function $d: X^2 \to \mathbb{R}^+$ is a rectangular metric on X if it satisfies:

(i) $d(x,y) = 0 \implies x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z)+d(z,t)+d(t,y)$ [Quadrilateral inequality], for all $x,y,z,t \in X$.

Then the ordered pair (X,d) is called rectangular metric space.

The latest implementation of the triangle inequality now includes four points instead of the previous three-point implementation. This recent development offers a suitable solution for resolving fractional-order functional differential equations. This approach is expected to enhance the accuracy and efficiency of such problem-solving methods.

***G*-Metric Space** ([47])**.** In 2005, Mustafa and Sims introduced a new class of generalized metric spaces known as G-metric spaces.

Definition 4.9 ([47])**.** A function $G: X \times X \times X \to [0,\infty)$ is G-metric on X if it satisfies the following:

(i) $G(x,y,z) = 0$ if $x = y = z$,

(ii) $0 < G(x,x,y)$, for all $x,y \in X$ with $x \neq y$,

(iii) $G(x,x,y) \leq G(x,y,z)$, for all $x,y,z \in X$ with $y \neq z$,

(iv) $G(x,y,z) = G(x,z,y) = G(z,y,x) = G(y,x,z) = G(z,x,y) = G(y,z,x)$, for all $x,y,z \in x$,

(v) $G(x,y,z) \leq G(x,t,t)+G(t,y,z)$, for all $x,y,z,t \in x$.

Then G is called a generalized metric and the pair (X,G) is called a G-metric space.

In G-metric space, the distance between three variables is determined by a simple rule: It is zero if all three variables are identical, and positive if one of the variables differs from the other two. This metric plays a crucial role in the solution of integral equations, making it an essential concept for researchers and practitioners in various fields. By leveraging the G-metric space, it is possible to analyze and quantify the relationships between variables, which can help to inform decision-making processes and lead to more accurate predictions and outcomes.

Multiplicative Metric Space ([10])**.** In 2008, Bashirov *et al*. presented the concept of *multiplicative metric spaces* (MMS). This innovative approach replaces the traditional triangular inequality with a 'multiplicative triangle inequality'. The MMS is a generalization of the traditional metric space, where the usual additive structure is replaced by a multiplicative structure

Definition 4.10 ([10])**.** A function $d: X \times X \to [0,\infty)$ is said to be multiplicative metric if it satisfies:

(i) $d(x,y) \geq 1$, for all $x,y \in X$,

(ii) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$,

(iii) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(iv) $d(x,y) \leq d(x,z) \cdot d(z,y)$, for all $x,y,z \in X$.

Then the ordered pair (X,d) is called multiplicative metric space.

A mapping $T: X \times X$ is called multiplicative contraction mapping if real numbers $\alpha \in (0,1)$ such that for all $x,y \in X$, $d(Tx,Ty) \leq d(x,y)^{\alpha}$.

Metric-Like Space (Dislocated Metric Space) ([38])**.** In 2000, Hitzler introduced the concept of metric-like space which allows for non-zero self-distance. At that time, during his work, he studied the metric-like space under the name of 'Dislocated metric space' and in 2012 Amini-Harandi [6] who reintroduced the dislocated metric space by a new name as metric-like space.

Definition 4.11 ([38])**.** A function $d: X \times X \to [0,\infty)$ is said to be Metric-like space if it satisfies:

(i) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$,

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$.

Then the ordered pair (X,d) is called metric-like space.

A metric-like space is a type of extension of a partial metric space, in which all axioms remain constant. This space is known for its utilization of Geraghty-type mappings [7].

***b*-Metric-Like Space** ([4])**.** In 2013, Alghamdi *et al*. introduced a new generalization of metric-like space and partial metric space, which is referred to as a b-metric-like space. This innovative concept offers a more flexible approach that allows for the relaxation of certain axioms of metric spaces. In contrast to partial metric spaces, b-metric-like spaces can handle cases where the distance between two points is not necessarily symmetric. This extension has attracted significant attention from researchers, as it has the potential to enrich the theory of metric spaces and improve our understanding of complex systems.

Definition 4.12 ([4]). A function $d : X \times X \to [0,\infty)$ is said to be b-metric-like space if it satisfies the following:

(i) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$,

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(iii) $d(x,y) \le \beta[d(x,z) + d(z,y)]$, for all $x,y,z \in X$ and $\beta \ge 1$.

Then the ordered pair (X,d) is called b-metric-like space.

b-metric-like functions do not need to be continuous in both variables and do not rely on a unique fixed point.

J-S Metric Space ([23]). In 2015, Jleli and Samet introduced a new metric space generalization called J-S metric space. It recovers a wide range of topological spaces, including standard metric spaces, b-metric spaces, and dislocated metric spaces.

Definition 4.13.1 ([23]). Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a given mapping. For every $x \in X$, let us define the set

$$C(d,X,x) = \{\{x_n\} \subset X : \lim_{n\to\infty} d(x_n,x) = 0\}.$$

Definition 4.13.2 ([23]). A function $d : X \times X \to [0,\infty)$ is said to be J-S metric if it satisfies the following:

(i) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$,

(ii) $d(x,y) = d(y,x)$,

(iii) there exists $\beta > 0$ such that

$$\{x_n\} \in C(d,X,x) \implies d(x,y) \le \beta \lim_{n\to\infty} d(x_n,y), \quad \text{for all } x,y \in X.$$

Then the ordered pair (X,d) is called J-S metric space.

If $C(d,X,x) = 0$ for every $x \in X$ then J-S metric space is defined when two basic property identity and symmetry satisfy.

Bipolar Metric Space ([48]). In 2016, Mutlu and Gürdal introduced the concept of a bipolar metric space as a form of partial distance. This particular type of distance measurement pertains to the distances between elements of two distinct sets as opposed to the distances between individual points within a single set. The abstraction of a bipolar metric space is a significant development in the field of mathematics and promises to have numerous practical applications across a range of industries.

Definition 4.14 ([48]). Let x and Đ are two non-empty sets and $d : X \times Đ \to [0,\infty)$ be a function satisfying the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$, where $(x,y) \in X \times Đ$;

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X \cap Đ$;

(iii) $d(x_1,y_2) \le d(x_1,y_1) + d(x_2,y_1) + d(x_2,y_2)$, for all $x_1,x_2 \in X$ and $y_1,y_2 \in Đ$.

Then d is referred to as the bipolar metric, and $(X,Đ,d)$ is the bipolar metric space.

Kishore *et al.* [35] demonstrated the existence and uniqueness of common coupled fixed-point outcomes in bipolar metric spaces for three covariant mappings. Mutlu *et al.* [49] generalized several coupled fixed-point theorems to bipolar metric spaces, and posits the extension of the Banach fixed-point theorem.

Extended b-Metric Space ([28]). In 2017, Kamran *et al*. introduced the concept of extended b-metric by weakening the triangular inequality.

Definition 4.15 ([28]). Let X be a non-empty set and $\theta : X \times X \to [1,\infty)$. A function $d_\theta : X \times X \to (0,\infty)$ is said to be extended b-metric if it satisfies:

(i) $d_\theta(x,y) = 0 \implies x = y$, for all $x,y \in X$.

(ii) $d_\theta(x,y) = d_\theta(y,x)$, for all $x,y \in X$.

(iii) $d_\theta(x,y) \le \theta(x,y)\ [d_\theta(x,z) + d_\theta(z,y)]$, for all $x,y,z \in X$.

Then the ordered pair (X,d_θ) is called extended b-metric space.

In the context of b-metric spaces, it is worth noting that if $\theta(x,y) = s$ for $s \ge 1$, then said space can be obtained. This methodology is deemed suitable for resolving the issue of the existence of a unique solution in Fredholm integral equations

Controlled Metric Space ([45]). In 2018, Mlaiki *et al*. introduced a novel generalization of the metric space known as a controlled metric space. This space is a supplement of the extended b-metric space that involves a controlled function. The controlled metric space is a valuable mathematical tool used in various fields, including computer science, optimization theory, and machine learning. Its application in these areas is mainly due to its unique ability to provide a more robust framework for quantifying the distance between points.

Definition 4.16 ([45]). Let x be a non-empty set and $\alpha : X \times X \to [1,\infty)$. A function $d : X \times X \to (0,\infty)$ is said to be controlled metric if it satisfies:

(i) $d(x,y) = 0 \implies x = y$, for all $x,y \in X$.

(ii) $d(x,y) = d(y,x)$, for all $x,y \in X$.

(iii) $d(x,y) \le \alpha(x,z)d(x,z) + \alpha(z,y)d(z,y)$, for all $x,y,z \in X$.

Then the ordered pair (X,d) is controlled metric space.

Every b-metric space is a controlled metric-type space.

In 2018, Abdeljawad *et al.* [2] proposed a modification to controlled metric-type spaces using two control functions, $\alpha(x,y)$ and $\mu(x,y)$, on the right-hand side of the b-triangle inequality. This modification is known as a double controlled metric space, which satisfies the condition $d(x,y) \le \alpha(x,z)d(x,z) + \mu(z,y)(z,y)$, for all $x,y,z \in X$. In 2019, Souayah and Mrad [57] proposed a broader idea of controlled partial metric-type spaces. Finally, in 2021, Mlaiki *et al.* [46] introduced a complex-valued triple-controlled metric space.

Super Metric Space ([32]). In 2022, Karapınar and Khojasteh introduced a novel extension of a metric space, which they named "super metric space". Their work is poised to make significant contributions to the field of mathematics, particularly in the domain of metric spaces. By creating this new type of metric space, Karapınar and Khojasteh have opened up new avenues for research and development, and their work is likely to spur further exploration into the properties and applications of super metric spaces. This development not only highlights the innovative contributions of Karapınar and Khojasteh, but also underscores the importance of continued research and development in the field of mathematics.

Definition 4.17 ([32])**.** Let X be a nonempty set. A function $m: X \times X \to [0,\infty)$ is super-metric if it satisfies:

(i) $m(x,y) = 0 \implies x = y$, for all $x, y \in X$,

(ii) $m(x,y) = m(y,x)$, for all $x, y \in X$,

(iii) There exists $s \geq 1$ such that for all $y \in X$ there exist distinct sequences $\{x_n\}, \{y_n\} \subset X$ with $m(x_n, y_n) \to 0$ as $n \to \infty$, such that

$$\limsup_{n\to\infty} m(y_n, y) \leq s \limsup_{n\to\infty} m(x_n, y),$$

then, we call (X, m) a super metric space.

5. Conclusion

The present paper endeavors to provide a concise overview of fixed-point theory. This theory encompasses the study of various mappings, their variations, and metric spaces, which hold a significant position in the development of non-linear analysis. By explicating the essential aspects of fixed-point theory, we aim to facilitate a better understanding of the concept, and its implications in contemporary research and development.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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