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Research Article

On Average Hub Number of a Graph

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Abstract. The idea of local and average hub numbers is explored as an expansion of the hub number in graphs, a connectivity measure that holds significance in transportation networks. In this analysis, we investigate the characterization of graphs by examining the local and average hub numbers and study them for graph classes namely trees and thorn graphs. Additionally, we determine the precise values of the average hub number for certain graph operations and discuss the bounds of Nordhaus-Gaddum type inequalities.

Keywords. Hub set, Average hub number

Mathematics Subject Classification (2020). 05C05, 05C40, 05C69

1. Introduction

The hub set, also known as the weak connecting set (Newman-Wolfe *et al*. [13]) in earlier studies, has been a subject of exploration since 1988 when researchers delved into network schemes that catered to specific communication needs. The concept of hub number, which represents a transportation problem, was first introduced in 2006 as a visual tool for analysis and study of networks (Walsh [14]), after which numerous studies have delved into the fascinating correlation between connected domination number and connected hub number of graphs (Grauman *et al*. [4], Johnson *et al*. [7], Lin *et al*. [8], Liu *et al*. [9], Liu *et al*. [10]). The findings of these studies shed light on the intricate relationship between hub number and connected domination number of graphs, providing valuable insights into their underlying structures. Moreover, the continuous advancement and development of hub numbers by mathematicians worldwide have greatly contributed to enhancing the efficiency and effectiveness of communication and transportation networks (Basavanagoud *et al.* [2], Cuaresma Jr. and Paluga [3], Mathad and Puneeth [11], Mathad *et al.* [12]). With the increasing focus on hub sets, researchers have been drawn to the idea of utilizing them to enhance connectivity and facilitate efficient routing networks (Hamburger *et al.* [5], Newman-Wolfe *et al.* [13]). Thus, this concept continues to be a crucial aspect of graph theory, offering valuable insights into optimizing communication networks.

In the realm of communication networks, a graph is used as a natural and effective way to represent their topology. It consists of interconnected vertices and edges that depict the relationships between different elements. This paper focuses on studying simple finite undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$ and is defined as the number of edges incident with v in G. For the vertices $u,v \in V(G)$, the distance between u and v is denoted by $d(u,v)$ and is defined as the length of the shortest path connecting u and v in G. The eccentricity of a vertex v is denoted by $e(v)$ and is maximum of $d(u,v)$, for all $u \in V(G)$. The *diameter* of G denoted by $d(G)$ or d is defined as $d(G) = \max\{e(v) : v \in V(G)\}$. A bipartite graph G is a graph whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex V_1 to a vertex of V_2. If each vertex of V_1 is joined to every vertex of V_2, then G is called a complete bipartite graph. If V_1 and V_2 have m and n vertices respectively, then in a complete bipartite graph we write $G = K_{m,n}$. A star is a complete bipartite graph $K_{1,n}$.

The *union* $G_1 \cup G_2$ of disjoint graphs G_1 and G_2 is the graph having vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The *join* $G_1 + G_2$ is the graph consisting of $G_1 \cup G_2$ with all edges joining all vertices of V_1 with all vertices of V_2. The *corona* $G_1 \circ G_2$ is the graph obtained from the graphs G_1 and G_2 by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining each vertex of the ith copy of G_2 named (G_2, i), with the ith vertex of G_1 by an edge. The cartesian product of disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph G whose vertex set is $V_1 \times V_2$. Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V_1 \times V_2$, u and v are adjacent in $G_1 \Box G_2$ whenever $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1 (Harary [6]).

Definition 1.1 ([14])**.** Suppose that $H \subseteq V(G)$ and u and v be any two vertices. The H-path between u and v is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge uv or a single vertex v if $u = v$, call such an H-path trivial.)

Definition 1.2 ([14])**.** A hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by an H-path. The hub number of G, denoted $h(G)$, is the minimum size of a hub set in G. Here, any minimum sized hub set are represented by $h(G)$-set. Further, a vertex $v \in V(G)$ is a hub vertex of G if $\{v\}$ is a hub set of G.

Definition 1.3. A hub set H of G is connected if $\langle H \rangle$ is connected. The connected hub number of G, denoted by $h_c(G)$, is the minimum cardinality of a connected hub set in G.

The average hub number of a graph is determined by the introduction of the concept of local hub number, which provides more accurate representation of the interconnectedness within the network as follows.

Definition 1.4. The local hub set of a graph G relative to a vertex $v \in V(G)$ is a minimum hub set containing v and is denoted by $h_v(G)$-set. The cardinality of local hub set relative to v is called local hub number of G relative to v and it is denoted by $h_v(G)$.

Definition 1.5. The average hub number of G, denoted by $h_{av}(G)$ is defined as

$$h_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} h_v(G).$$

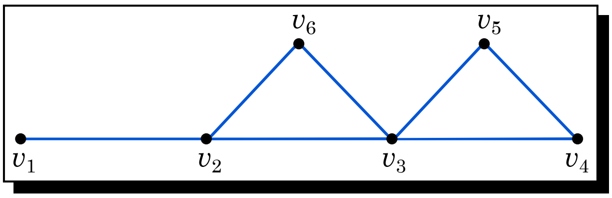

Figure 1. Graph G

Table 1. Graph G

Vertex v	Local hub set relative to v
v_1	$\{v_1,v_2,v_3\}$, $\{v_1,v_3,v_6\}$, $\{v_1,v_3,v_4\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_4,v_6\}$, $\{v_1,v_3,v_5\}$
v_2	$\{v_2,v_3\}$
v_3	$\{v_2,v_3\}$
v_4	$\{v_2,v_3,v_4\}$, $\{v_1,v_3,v_4\}$, $\{v_1,v_4,v_6\}$, $\{v_2,v_4,v_6\}$
v_5	$\{v_2,v_3,v_5\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_2,v_4,v_5\}$
v_6	$\{v_1,v_4,v_6\}$, $\{v_1,v_3,v_6\}$, $\{v_2,v_4,v_6\}$, $\{v_2,v_3,v_6\}$

For example, consider the graph G as shown in Figure 1. The local hub sets of G relative to each vertex v is given in Table 1. Here, $h_{v_2}(G) = 2 = h_{v_3}(G)$ and $h_{v_1}(G) = h_{v_4}(G) = h_{v_5}(G) = h_{v_6}(G) = 3$. Then $h_{av}(G) = \frac{1}{6}(2 \cdot 2 + 4 \cdot 3) = \frac{8}{3}$.

It is important to consider the average hub number while analysing graphs rather than just the hub number as the average hub number provides more comprehensive understanding of the connectivity of graphs and network structure. This allows us to assess effectiveness of information flow within the network in a better way. Therefore, focusing on the average hub number is crucial for a thorough analysis of graph properties.

In order to validate our findings, it will be beneficial to utilize the following established results.

Theorem 1.6 ([14])**.** *Let T be a tree with n vertices and l leaves. Then $h(G) = n - l$.*

Lemma 1.7 ([14])**.** *Let $d(G)$ denote the diameter of G. Then $h(G) \geq d(G)-1$, and the inequality is sharp.*

Theorem 1.8 ([14])**.** *If G is connected graph of order n, then $h(G) \leq n-\Delta(G)$, and the inequality is sharp.*

Theorem 1.9 ([3])**.** *Let $3 \leq p \leq n$, where $p,n \in \mathbb{Z}$. Then $h(K_p \Box K_n) = p$.*

Theorem 1.10 ([3])**.** *Let $n \geq 4$. If $p=2$ or $p=3$, then $h(P_p \Box P_n) = n$.*

2. Local Hub Number of Graphs

Proposition 2.1. *Let G be any graph of order n, then for each vertex $v \in V(G)$, $h_v(G) \leq h(G)+1$ and equality holds if and only if $G = K_n$.*

Proof. For each $v \in V(G)$, either $h_v(G) = h(G)$ or $h_v(G) = h(G)+1$. Hence, $h_v(G) \leq h(G)+1$. Since for each $v \in V(G)$, $h_v(G) = h(G)+1$ if and only if none of the vertex in $V(G)$ belongs to any $h(G)$-set. This happens only when the minimum hub set of G is empty. Thus, $h(G) = 0$, which is possible only in the case of complete graphs. □

Corollary 2.2. *Let G be a graph of order n, then the following holds.*

(i) *For any $v \in V(G)$, $1 \leq h_v(G) \leq n$.*

(ii) *If u is a universal vertex of G, then $h_u(G) = 1$.*

(iii) *If u is an isolated vertex of G, then $h_u(G) = h(G)$.*

Theorem 2.3. *If G of order $n > 3$ is isomorphic to path P_n, cycle C_n or complete multipartite graph $K_{k_1,k_2,\dots,k_p}$ with $p \geq 2$, $k_i > 2$ for every $1 \leq i \leq p$, then $h_v(G) = h(G)$, for all $v \in V(G)$.*

Proof. Let G be a graph of order $n > 3$ with $V(G) = \{v_1, v_2, \dots, v_n\}$. The proof is attained in following cases.

Case (i): If $G \cong P_n$ then $h(G) = n-2$. For each vertex $v_i \in V(G)$, the set $S = V(G) \setminus \{v_k, v_l : k,l \neq i\}$ is an $h(G)$-sets. Thus, $h_v(G) = n-2$, for every $v \in V(G)$.

Case (ii): If $G \cong C_n$ then $h(G) = n-3$. Using similar argument as in *Case* (i), we get the required result.

Case (iii): If $G \cong K_{k_1,k_2\dots,k_p}$ where $p \geq 2$, $k_i > 2$ for every $1 \leq i \leq p$ then $h(G) = 2$. Here any $h(G)$-set consists of one random vertex selected from any two arbitrary distinct partite sets and hence $h_v(G) = 2 = h(G)$, for every $v \in V(G)$. □

From above result we obtain the following result by taking G to be a path of order $k+2$ or a cycle of order $k+3$, for any positive integer k.

Corollary 2.4. *For any given positive integer k, there exists a graph G such that for every vertex v in G, $h_v(G) = k = h(G)$.*

Theorem 2.5. *For any connected graph G of order n, if $h_v(G) = h(G)$ for each v in $V(G)$ then either G has no universal vertex or G has exactly $(n-2)$ universal vertices.*

Proof. Let G be a connected graph of order n such that $h_v(G) = h(G)$, for each v in $V(G)$. Then from Proposition 2.1, G is a non-complete graph. Now, if G has no universal vertex then there is nothing to prove. Suppose that G has k, $(1 \le k < n)$ number of universal vertices, then $h(G) = 1$. If $1 \le k \le n-3$, then there are more than two non-universal vertices in G. Thus, there exists atleast one non-universal vertex w in G with $h_w(G) = 2 = h(G)+1$, a contradiction to the hypothesis. Since the number of universal vertices $k \ne n-1$, it follows that G has exactly $(n-2)$ universal vertices. □

Corollary 2.6. *Let G be any graph of order n containing exactly $(n-2)$ universal vertices. Then for every vertex v in $V(G)$, $h_v(G) = h(G)$.*

Proof. Let G be any graph having exactly two non-universal vertices say, x and y in G both of whose degree is $n-2$, that is, x and y are the only two non-adjacent vertices of G. Thus $h_x(G) = 1 = h(G)$, as every pair of vertices not containing x are adjacent in G. On similar lines, $h_y(G) = 1 = h(G)$. Further, since remaining vertices are of full degree, we have $h_v(G) = 1 = h(G)$, for every $v \in V(G)$. □

Theorem 2.7. *If T is a tree of order n with $l > 2$ leaves and $\Delta(T) < n-1$ then for every vertex $v \in V(T)$, $h_v(T) = n-l$.*

Proof. From Theorem 1.8, we sense that for each non-pendant vertex of a tree T, this result holds as they form an $h(T)$-set. Interestingly, for each pendant vertex of T, $h_v(T) = h(T)$. This happens as for each pendant vertex $v \in V(T)$, the $h_v(G)$-set is formed by the union of $\{v\}$ and set of all the non-pendant vertices other than the support of v. □

If $\Delta(T) = n-1$ in Theorem 2.7, then $T = K_{1,n-1}$, in which case for every vertex $v \in V(G)$, $h_v(T) \ge n-l$.

3. Average Hub Number of Graphs

From Definition 1.5 and Proposition 2.1, it is clear that $h(G) \le h_{av}(G) \le h(G)+1$ which attains its lower bound if and only if every vertex v of G belongs to an $h(G)$-set and upper bound if and only if G is K_n. Consequently, $h_{av}(K_n) = 1$, while from Theorem 2.3, we have the following proposition.

Proposition 3.1. (i) *For $n \ge 3$, $h_{av}(C_n) = n-3$.*

(ii) *For $n \ge 2$, $h_{av}(P_n) = n-2$.*

(iii) *For $m,n \ge 2$,*
$$h_{av}(K_{m,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n = 2; \\ \frac{2(n+1)}{n+2}, & \text{if } m = 2 \text{ and } n \ge 3; \\ 2, & \text{if } m \ge 3 \text{ and } n \ge 3. \end{cases}$$

In general, the result of complete bipartite graph holds true for any complete multi-partite graph as well with each partite set containing more than two vertices. Now, since the necessary and sufficient condition for $h(G) = h_{av}(G)$ is that $h_v(G) = h(G)$, for all $v \in V(G)$, the following results follow from Theorem 2.5.

Theorem 3.2. *For any connected graph G of order n, if $h_{av}(G) = h(G)$ then either G has no universal vertex or G has exactly $(n-2)$ universal vertices.*

Corollary 3.3. *If a graph G of order n has $k \geq 1$ universal vertices with $k \neq n-2$, then $h_{av}(G) \neq h(G)$.*

Theorem 3.4. *For any graph G of order n, $h_{av}(G) \leq h(G) + 1 - \frac{h(G)}{n}$ with equality if and only if G has a unique $h(G)$-set.*

Proof. Let G be a graph of order n with the vertex set $\{v_1, v_2, \dots v_n\}$. Note that, if v belongs to an $h(G)$-set, then $h_v(G) = h(G)$ otherwise, $h_v(G) = h(G) + 1$. By definition, we have the below equations which are equivalent if and only if G has a unique $h(G)$-set,

$$\begin{aligned} h_{av}(G) &= \frac{1}{n}\sum_{i=1}^{n} h_{v_i}(G) \\ &= \frac{1}{n}\left[\sum_{i=1}^{h(G)} h_{v_i}(G) + \sum_{i=h(G)+1}^{n} h_{v_i}(G)\right] \\ &= \frac{1}{n}[h(G)^2 + (n-h(G))(h(G)+1)] \\ &= h(G) + 1 - \frac{h(G)}{n}. \end{aligned}$$

On the other hand, if G does not have a unique $h(G)$-set then definitely the local hub number of each vertex is less than or equal to the local hub number value in the case of G having a unique $h(G)$-set. Hence, $h_{av}(G) \leq h(G) + 1 - \frac{h(G)}{n}$. □

As a consequence of the aforementioned result, for any star and wheel graphs with $n > 4$, we get $h_{av}(K_{1,n-1}) = h_{av}(W_n) = 2 - \frac{1}{n}$.

Corollary 3.5. *If G is connected graph of order n then $h_{av}(G) \leq \frac{1}{n}[n^2 - (n-1)\Delta(G)]$, and the inequality is sharp.*

Proof. From Theorem 2.1 and Proposition 1.8, we have

$$\begin{aligned} h_{av}(G) &\leq h(G) + 1 - \frac{h(G)}{n} \\ &\leq \frac{1}{n}[(n-1)h(G) + n] \\ &\leq \frac{1}{n}[(n-1)(n-\Delta(G)) + n] \\ &\leq \frac{1}{n}[n^2 - (n-1)\Delta(G)]. \end{aligned}$$

The equality holds in the case of graphs $K_{1,n-1}$ and W_n. □

Corollary 3.6. *Let G be a graph of order n with unique $h(G)$-set. Then there exists an integer $t > n$ such that $h_{av}(G) = \frac{t}{n}$ if and only if $h(G) = \frac{t-n}{n-1}$.*

Proof. Since G is a graph with unique $h(G)$-set, the following are equivalent,

$$\begin{aligned} & h_{av}(G) = \frac{t}{n} \\ \Longleftrightarrow \quad & h(G) + 1 - \frac{h(G)}{n} = \frac{t}{n} \\ \Longleftrightarrow \quad & \frac{(n-1)}{n} h(G) + 1 = \frac{t}{n} \\ \Longleftrightarrow \quad & (n-1)h(G) + n = t \\ \Longleftrightarrow \quad & h(G) = \frac{t-n}{n-1} \end{aligned}$$

□

For illustration, consider a graph $G = K_1 \cup K_{1,2}$ for which $h(G) = 2$, we have $t = (3)(2) + 4 = 10$. Thus, $h_{av}(G) = \frac{10}{4} = \frac{5}{2}$.

In this context, next we derive the generalised values of average hub number for specific types of graph structures, specifically trees and thorn graphs.

Proposition 3.7. *If T is a tree of order n with $l > 2$ leaves and $\Delta(T) < n-1$ then*

$$h_{av}(T) = h(T) = n - l.$$

Proof. The proof directly follows from Theorem 2.7. □

Definition 3.8 ([1]). Let $p_1, p_2, \dots, p_n$ be non-negative integers and G be a graph of order n. The thorn graph G^* of the graph G, with parameters $p_1, p_2, \dots, p_n$, is obtained by attaching p_i new vertices of degree 1 to the vertex u_i of the graph G, $i = 1, 2, \dots, n$.

Theorem 3.9. *Let G be a non-trivial connected graph of order n and its corresponding thorn graph be G^* of order k. Let $t < n$, then*

$$h_{av}(G^*) = \begin{cases} n, & \text{if } p_i = 1, 1 \le i \le n; \\ \frac{n(k-1)}{k} + 1, & \text{if } p_i > 1, 1 \le i \le n; \\ n + 1 - \frac{t+n}{k}, & \text{if } p_i = 1, 1 \le i \le t \text{ and } p_i = 1, t+1 \le i \le n. \end{cases}$$

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and the proof is obtained from the following cases.

Case (i): If $p_i = 1$, $1 \le i \le n$, then $k = 2n$ and $h(G) = n$.

Let $V(G^*) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. Since $\{u_1, u_2, \dots, u_n\}$ and $\{u_1, u_2, \dots, u_{i-1}, v_i, u_{i+1}, \dots u_n\}$ are $h(G^*)$-sets, we have $h_v(G^*) = n$, for all $v \in V(G^*)$. Hence

$$h_{av}(G^*) = \frac{1}{|V(G^*)|} \sum_{v \in V(G^*)} h_v(G^*) = \frac{2n \cdot n}{2n} = n.$$

Case (ii): If $p_i > 1$, $1 \le i \le n$ then by the structure of G^* it can be easily analysed that G^* has a unique $h(G^*)$-set, namely $V(G)$. Then by Proposition 3.4 we obtain

$$h_{av}(G^*) = n + 1 - \frac{n}{k} = \frac{n(k-1)}{k} + 1.$$

Case (iii): If $p_i = 1$, $1 \le i \le t$ where $t < n$ and $p_i > 1$, $t+1 \le i \le n$, then $h(G^*) = |V(G)| = n$. Let $V(G^*) = \{u_1, u_2, \dots, u_t, \dots, u_n, v_1, v_2, \dots, v_t, \dots, v_{k-n}\}$, then $\{u_1, u_2, \dots, u_n\}$ is an $h(G^*)$-set. From Case (i), if $v \in \{u_1, \dots, u_n, v_1, \dots, v_t\}$ then $h_v(G^*) = n$ and if $v \in \{v_{t+1}, v_{t+2}, \dots, v_{k-n}\}$ then $h_v(G^*) = n+1$. Therefore,

$$h_{av}(G^*) = \frac{(n+t)n + (k-n-t)(n+1)}{k} = n + 1 - \frac{t+n}{k}$$ □

Theorem 3.10. *Let G_1 and G_2 be two disjoint graphs of order n_1 and n_2, respectively such that atleast one of them contains few universal vertices. If k_1 and k_2 are number of universal vertices in G_1 and G_2, respectively then*

$$h_{av}(G_1 + G_2) \le 2 - \frac{k_1 + k_2}{n_1 + n_2}.$$

Proof. As each vertex of G_1 is adjacent to G_2 and vice-versa in $G_1 + G_2$, we have $h_u(G_1 + G_2) = 1$, whenever u is an universal vertex of G_i, $i = 1, 2$. There are $k_1 + k_2$ number of such vertices in $V(G_1 + G_2)$. Now, for the remaining $n_1 + n_2 - k_1 - k_2$ vertices in $V(G_1 + G_2)$, we consider the following cases.

Case (i): Let $d(v) \le n_i - 2$, for $v \in V(G_i)$ such that $G_i - v = \langle K_{n_i - 1} \rangle$ then $h_v(G_1 + G_2) = 1$.

Case (ii): Let $d(v) = n_i - 2$, for $v \in V(G_i)$ with $G_i - v \neq \langle K_{n_i - 1} \rangle$. Now, if there exists exactly one vertex $w \in G_i - v$ such that w and v are not adjacent to each other but $w \sim w_i$ for each $w_i \in G_i - v$ then v forms a hub vertex and thus $h_v(G_1 + G_2) = 1$, else we need an additional vertex from G_j, $i \neq j$, $i, j = 1, 2$ to form a hub set. Thus, $h_v(G_1 + G_2) = 2$.

Since the statements in *Case* (i) and *Case* (ii) depend on the structure of graphs and cannot be generalised, from above arguments we have the following:

$$\begin{aligned} h_{av}(G_1 + G_2) &\le \frac{1}{n_1 + n_2}[(k_1 + k_2) \cdot 1 + (n_1 + n_2 - k_1 - k_2) \cdot 2] \\ &\le 2 - \frac{k_1 + k_2}{n_1 + n_2}. \end{aligned}$$ □

Theorem 3.11. *For any disjoint non-trivial connected graphs G and H of orders n and p, respectively, $h_{av}(G \circ H) = n + \frac{p}{p+1}$.*

Proof. Due to the structure of $G \circ H$, it has a unique $h(G \circ H)$-set namely $V(G)$ with $h(G \circ H) = n$. Since $|V(G \circ H)| = n(p+1)$, from Proposition 3.4, we have

$$h(G \circ H) = n + 1 - \frac{n}{n(p+1)} = n + \frac{p}{p+1}.$$ □

Theorem 3.12. *Let $3 \le p \le n$, where $p, n \in \mathbb{Z}$. Then $h_{av}(K_p \Box K_n) = p$.*

Proof. From Theorem 1.9, we know that the $h(K_p \Box K_n)$- set is $V(K_p) \times \{w\}$ for any arbitrary vertex w in $V(K_p \Box K_n)$. Thus, for each vertex $v \in V(K_p \Box K_n)$, we have $h_v(K_p \Box K_n) = p$. Therefore, $h_{av}(K_p \Box K_n) = p$. □

Theorem 3.13. *Let $n \ge 4$. If $p = 2$ or $p = 3$, then $h_{av}(P_p \Box P_n) = n$.*

Proof. Similar to the proof of above, using Theorem 1.10 we deduce the required result. □

After addressing these graph operations, we will now conclude by discussing the bounds for average hub number through Nordhaus-Gaddum type inequalities.

Theorem 3.14. *If G and $\bar{G}$ are connected graphs of order n, then*

(i) $d(G)+d(\bar{G})-2 \le h_{av}(G)+h_{av}(\bar{G}) \le \frac{1}{n}[n^2+n(\delta-\Delta+2)+\Delta-\delta-1]$.

(ii) $(d(G)-1)(d(\bar{G})-1) \le h_{av}(G)\cdot h_{av}(\bar{G}) \le \frac{1}{n}[n^2(\delta+2)-n[\Delta(\delta+2)]+\Delta(\delta+1)]$.

Proof. For the lower bounds, we have $d(G)-1 \le h(G) \le h_{av}(G)$ from Lemma 1.7 and hence $d(G)+d(\bar{G})-2 \le h_{av}(G)+h_{av}(\bar{G})$ and $(d(G)-1)(d(\bar{G})-1) \le h_{av}(G)\cdot h_{av}(\bar{G})$. Here the bounds are attained for the self-complementary graph P_4 as $d(P_4)=3$ and $h_v(P_4)=2$, for every $v \in V(P_4)$. On the other hand, for upper bound we make use of Proposition 3.4 to get the required result as

$$h_{av}(G)+h_{av}(\bar{G}) \le h(G)+1-\frac{h(G)}{n}+h(\bar{G})+1-\frac{h(\bar{G})}{n} = \frac{n-1}{n}[h(G)+h(\bar{G})]+2.$$

Now, as both G and $\bar{G}$ are connected, from Theorem 1.8, we have

$$h(G) \le n-\Delta(G)$$

and

$$h(\bar{G}) \le n-\Delta(\bar{G}) = n-[(n-1)-\delta(G)] = 1+\delta(G).$$

Thus,

$$\begin{aligned} h_{av}(G)+h_{av}(\bar{G}) &\le \frac{n-1}{n}[n-\Delta(G)+\delta(G)+1]+2 \\ &= \frac{1}{n}[n^2+n(\delta-\Delta+2)+\Delta-\delta-1] \end{aligned}$$

and

$$\begin{aligned} h_{av}(G)\cdot h_{av}(\bar{G}) &\le \left(h(G)+1-\frac{h(G)}{n}\right)\left(h(\bar{G})+1-\frac{h(\bar{G})}{n}\right) \\ &= \frac{n-1}{n}\left[\left(\frac{n-1}{n}\right)h(G)\cdot h(\bar{G})+h(G)+h(\bar{G})\right]+1 \\ &\le \frac{n-1}{n}\left[\left(\frac{n-1}{n}\right)(n-\Delta(G))(1+\delta(G))+n-\Delta(G)+\delta(G)+1\right] \\ &= \frac{1}{n}[n^2(\delta+2)-n[\Delta(\delta+2)]+\Delta(\delta+1)]. \end{aligned}$$

Here the equality holds if and only if both G and $\bar{G}$ possess unique minimum hub sets. □

4. Conclusion

This study introduces novel parameters termed the *local hub number* and *average hub number* and investigates their properties in detail. By focusing on the average hub number, researchers can achieve a deeper understanding of the connectivity and structural characteristics of graphs, facilitating more comprehensive analyses of graph properties. Additionally, examining the behaviour of average hub numbers across various graph classes–such as cubic graphs, circulant graphs, and generalized Petersen graphs–provides valuable insights into their structural nuances. This exploration leads to the formulation of a necessary and sufficient condition for the average hub number to coincide with the hub number of a given graph. By elucidating this relationship, the study advances our understanding of network connectivity and centrality within the domain of graph theory.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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